MODIFICATIONS/CORRECTIONS TO SECTIONS 6 AND 7 OF "REPRESENTATIONS OF QUANTUM GROUPS DEFINED OVER COMMUTATIVE RINGS"

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6. Invariants of Liftings

6.1. Here we consider the ambient setting where ${}_R\mathbf{U}_i$ is a Hopf subalgebra of another quantum group ${}_R\mathbf{U}$ and the bilinear pairing ϕ has $\boldsymbol{\rho}$-invariance not merely $\boldsymbol{\rho}|_{{}_R\mathbf{U}_i}$-invariance. Let $\boldsymbol{\rho}$ be the involutive anti-automorphism of ${}_R\mathbf{U}$ as defined in (1.3). Assume the restriction of $\boldsymbol{\rho}$ to ${}_R\mathbf{U}_i$ equals the anti-automorphism given in (4.1). Also as introduced in (1.9), for ${}_R\mathbf{U}$-modules A and B, let $\mathbb{P}_{\boldsymbol{\rho}}(A,B)$ denote the space of R-bilinear maps $\phi : A\times B\to R$ which are $\boldsymbol{\rho}$-invariant.

Theorem 6.1.1. *Let A and B be objects in $\mathcal{C}_R$ and suppose $\phi\in\mathbb{P}_{\boldsymbol{\rho}}(A,B)$. Then the lifted pairing ϕ_{F_i} is ${}_R\mathbf{U}$-invariant as well ; i.e. $\phi_Y\in\mathbb{P}_{\boldsymbol{\rho}}(A_{F_i},B_{F_i})$.*

This proof is an adaptation of the proof of $\mathfrak{g}$-invariance in [E2] given in the Lie algebra setting. There are some delicate points in the extension to the quantum setting. The presence of the R-matrix, ${}_fR$ is the main complication as we shall see. We begin with several results on vector valued pairings and the resulting two notions of liftings in this setting.

6.2. Recall a $\mathbf{U}$-module M is said to be *integrable* if for any $m\in M$ and all $i\in I$, there exists a positive integer N such that $E_i^{(n)}m=0=F_i^{(n)}m$ for all $n\geq N$, and $M=\oplus_{\lambda\in X}M^\lambda$ where for any $\mu\in Y,\lambda\in X$ and $m\in M^\lambda$ one has $K_\mu m=v^{\langle\mu,\lambda\rangle}m$. Let $\mathbf{U}_0^\times$ denote the set of units of $\mathbf{U}_0$ and let $f:X\times X\to\mathbf{U}_0^\times$ be a function such that

$$f(\zeta+\nu,\zeta'+\nu')=f(\zeta,\zeta')v^{-\sum\nu_i\langle i,\zeta'\rangle(i\cdot i/2)-\sum\nu_i'\langle i,\zeta\rangle(i\cdot i/2)-\nu\cdot\nu'}\tilde{K}_\nu \tag{6.2.1}$$

for all $\zeta,\zeta'\in X$ and all $\nu,\nu'\in X$ (see [L, 32.1.3] or [Ja, 3.15]). Here $\tilde{K}_\nu=\prod_i K_{(i\cdot i/2)\nu_i i}$. (This corrects the formula 6.2.1 in the earlier version of [CE].)

Theorem 6.2.1. *[L, 32.1.5], or [Ja, 3.14] If $\mathcal{E}$ is an integrable ${}_R\mathbf{U}$ module and $A\in\mathcal{C}_R$, then for each f satisfying (3.1.1), there exists an isomorphism ${}_f\mathcal{R}:A\otimes\mathcal{E}\to\mathcal{E}\otimes A$.*

The map $\tau : A\otimes B\to B\otimes A$ for any two modules A and B denotes the twist map $\tau(a\otimes b)=b\otimes a$. Define $\prod_f :\in \mathrm{End}\,_R({}_R\mathcal{E}\otimes{}_R\mathcal{F}\otimes_R M)$ by $\prod_f(e\otimes e'\otimes m)=f(\lambda,\lambda')e\otimes e'\otimes m$ for $m\in M^{\lambda'}$ and $e\otimes e'\in(\mathcal{E}\otimes{}_R\mathcal{F})^\lambda$. Lastly we define $\chi\in\mathrm{End}\,_R({}_R\mathcal{E}\otimes{}_R\mathcal{F}\otimes_R M)$ by

$$\chi(e\otimes e'\otimes m)=\sum_\nu\sum_{b,b'\in\mathbf{B}_\nu}p_{b,b'}b^-(e\otimes e')\otimes b'^+m$$

where $p_{b,b'}=p_{b',b}\in R$, and $\mathbf{B}_\nu$ is a subset of $\mathfrak{f}$. Then ${}_f\mathcal{R}$ is defined to be equal to $\chi\circ\prod_f\circ\tau$. The proof that it is an $\mathbf{U}$-module homomorphism is almost exactly the same as in [L, 32.1.5] or [Ja, 3.14], which the exception that one must take into account that M is in the category $\mathcal{C}_R$ instead of ${}_R\mathcal{C}'$.

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For ${}_R\mathbf{U}$-modules M,N and $\mathcal{F}$, let $\mathbb{P}(M,N)$ and $\mathbb{P}(M,N,\mathcal{F})$ denote the space of R-bilinear maps of $M\times N$ to R and $\mathcal{F}$ respectively, with the following invariance condition:

$$\sum x_{(1)}\cdot\phi(Sx_{(3)}\cdot a,\varrho(x_{(2)})b)=\mathbf{e}(x)\phi(a,b) \tag{6.2.2}$$

where $\Delta\otimes 1\circ\Delta(x)=\sum x_{(1)}\otimes x_{(2)}\otimes x_{(3)}$ and $\mathbf{e}:\mathbf{U}\to k(v)$ is the counit. If we let $\hom_{{}_R\mathbf{U}}(A,B)$ denote the set of module ${}_R\mathbf{U}$-module homomorphisms, then one can check on generators of ${}_R\mathbf{U}$ that $\mathbb{P}(M,N,\mathcal{F})\cong\hom_{{}_R\mathbf{U}}(M\otimes_R N^{\rho_1},{}_R\mathcal{F})$ (see [Ja, 3.10.6]. Formula (1.1.2) corrects an error in [CE, 6.2.2]. Let $\mathbb{P}(N)=\mathbb{P}(N,N)$ denote the R-module of invariant forms on N.

Theorem 6.2.2. *[L, 32.1.5] Fix an integer $d\geq 1$. If M is an integrable ${}_R\mathbf{U}$ module and $A\in\mathcal{C}_R$, then for each f satisfying (1) with $\mathrm{im}\,f\in\frac{1}{d}\mathbb{Z}$, there exists an isomorphism ${}_f\mathcal{R}:A\otimes M\to M\otimes A$.*

Let $\phi\in\mathbb{P}_{\boldsymbol{\rho}}(A,B,\mathcal{F})$. We define $\mathcal{K}$-valued bilinear mappings $\phi^r : {}_\mathcal{K}A\times({}_\mathcal{K}B\otimes{}_\mathcal{K}\mathcal{F})\to\mathcal{K}$ and $\phi^l:({}_\mathcal{K}A\otimes{}_\mathcal{K}\mathcal{F}^{\rho_1})\times{}_\mathcal{K}B\to\mathcal{K}$ by the formulas: for $e\in_\mathcal{K}\mathcal{F}$, $a\in A$ and $b\in B$

$$\phi^r(a,b\otimes e)=\sum\phi(a,b_i)(e_i),\qquad \phi^l(a\otimes e,b)=\sum\phi(a,b_i)(e_i) \tag{6.2.3}$$

where $\sum b_i\otimes e_i={}_f\mathcal{R}^{-1}(e\otimes b)$, ${}_f\mathcal{R}:B^{\rho_1}\otimes\mathcal{F}^{\rho_1}\to\mathcal{F}^{\rho_1}\otimes B^{\rho_1}$. Clearly ϕ^r (resp. ϕ^l) is determined by ϕ and vice versa. (Note that if $\mathcal{F}$ is a finite dimensional representation for $\mathbf{U}$ then the map $x\mapsto\phi(x)$ where $\phi(x)f=f(x)$ is a module isomorphism $\mathcal{F}\to(\mathcal{F}^\rho)^\rho$.)

Lemma 6.2.1. *The maps $\theta_r:\mathbb{P}_{\boldsymbol{\rho}}(A,B,\mathcal{F})\to\mathbb{P}_{\boldsymbol{\rho}}(A,B\otimes\mathcal{F})$ and $\theta_l:\mathbb{P}_{\boldsymbol{\rho}}(A,B,\mathcal{F})\to\mathbb{P}_{\boldsymbol{\rho}}(A\otimes\mathcal{F}^{\rho_1},B)$, given by $\theta_r\phi=\phi^r$ and $\theta_l\phi=\phi^l$, are isomorphisms.*

Proof. We need to note a number of small results before we can prove the lemma. One can check on generators that ${}^t\Delta\circ\rho_1=(\rho_1\otimes\rho_1)\circ\Delta$ where ${}^t\Delta$ denotes the comultiplication followed by the twist map. This implies that $(B\otimes\mathcal{F})^{\rho_1}\cong\mathcal{F}^{\rho_1}\otimes B^{\rho_1}$. Next we let C be another $\mathbf{U}$-module. Now since $m\circ(1\otimes\mathbf{e})\circ\Delta=\mathrm{id}$, where m is multiplication, the canonical map $\mathrm{Hom}\,(A,C^{\boldsymbol{\rho}})\to\mathrm{Hom}\,(A\otimes C^{\rho_1},k(v))$ is a $\mathbf{U}$-module isomorphism. Thus $\mathrm{Hom}\,_\mathbf{U}(A,C^{\boldsymbol{\rho}})\cong\mathrm{Hom}\,_\mathbf{U}(A\otimes C^{\rho_1},k(v))$. Second, it is straightforward to check $\mathrm{Hom}\,_\mathbf{U}(A\otimes C^{\rho_1},\mathcal{F})\cong\mathbb{P}(A,C,\mathcal{F})$, for any $\mathbf{U}$-module $\mathcal{F}$. One then combines these facts together to obtain isomorphisms;

$$\begin{aligned}\mathbb{P}_{\boldsymbol{\rho}}(A,B\otimes\mathcal{F})&\cong\mathrm{Hom}\,_\mathbf{U}(A\otimes(B\otimes\mathcal{F})^{\rho_1},k(v))\\&\cong\mathrm{Hom}\,_\mathbf{U}(A\otimes\mathcal{F}^{\rho_1}\otimes B^{\rho_1},k(v))\\&\cong\mathrm{Hom}\,_\mathbf{U}(A\otimes B^{\rho_1}\otimes\mathcal{F}^{\rho_1},k(v))\\&\cong\mathrm{Hom}\,_\mathbf{U}(A\otimes B^{\rho_1},\mathcal{F}^{\varrho})\\&\cong\mathbb{P}_{\boldsymbol{\rho}}(A,B,\mathcal{F})\end{aligned}$$

where the third isomorphism is induced by ${}_f\mathcal{R}$.

In addition we have

$$\begin{aligned}\mathbb{P}_{\boldsymbol{\rho}}(A\otimes\mathcal{F}^{\rho_1},B)&\cong\mathrm{Hom}\,_\mathbf{U}(A\otimes\mathcal{F}^{\rho_1}\otimes B^{\rho_1},\mathcal{K})\\&\cong\mathrm{Hom}\,_\mathbf{U}(A\otimes B^{\rho_1}\otimes\mathcal{F}^{\rho_1},\mathcal{K})\\&\cong\mathrm{Hom}\,_\mathbf{U}(A\otimes B^{\rho_1},\mathcal{F}^{\varrho})\ \cong\mathbb{P}_{\boldsymbol{\rho}}(A,B,\mathcal{F})\ ,\end{aligned}$$

where the second isomorphism is implemented by ${}_f\mathcal{R}$. We need to check that these isomorphisms are the inverses to θ_r and θ_l respectively. The first set of isomorphisms going upward gives

$$\begin{aligned}\phi &\mapsto (a \otimes \left(\sum b_i \otimes e_i\right) \mapsto \sum \phi(a, b_i)(e_i))\\ &\mapsto (a \otimes e \otimes b \mapsto \sum \phi(a, b_i)(e_i))\\ &\mapsto (a \otimes b \otimes e \mapsto \sum \phi(a, b_i)(e_i))\end{aligned}$$

□

6.3. Now define the right and left liftings ϕ_{rF_i} and $\phi_{lF_i} \in \mathbb{P}_{\boldsymbol{\rho}|_{R\mathbf{U}_i}}(A_{F_i}, B_{F_i}, {}_R\mathcal{F})$ by the identities:

$$(\phi_{rF_i})^r = (\phi^r)_{F_i}\ , \qquad (\phi_{lF_i})^l = (\phi^l)_{F_i} \tag{6.3.1}$$

Theorem 6.3.1. *For $\phi \in \mathbb{P}_{\boldsymbol{\rho}}(A, B, \mathcal{F})$ the right and left liftings of ϕ are equal; i.e. $\phi_{rF_i} = \phi_{lF_i}$.*

Proof. By (4.2) it is sufficient to prove the identity for ${}_{\mathcal{K}}\phi$ in place of ϕ. Let

$$\nu :_{\mathcal{K}} \mathcal{F}^{\boldsymbol{\rho}} \otimes_{\mathcal{K}} \mathcal{F}^{\rho_1} \to \mathcal{K}$$

denote the natural pairing of ${}_{\mathcal{K}}\mathcal{F}^{\rho_1}$ and ${}_{\mathcal{K}}\mathcal{F}^{\boldsymbol{\rho}}$. Choose a basis $\{x_i\}$ and dual basis $\{x_i^{\boldsymbol{\rho}}\}$ for ${}_{\mathcal{K}}\mathcal{F}^{\rho_1}$ and ${}_{\mathcal{K}}\mathcal{F}^{\boldsymbol{\rho}}$ and let

$$\iota : \mathcal{K} \to_{\mathcal{K}} \mathcal{F}^{\rho_1} \otimes_{\mathcal{K}} \mathcal{F}^{\boldsymbol{\rho}}, \qquad \iota(1) = \sum x_i \otimes x_i^{\boldsymbol{\rho}}$$

the map induced by the isomorphism

$${}_{\mathcal{K}}\mathcal{F}^{\rho_1} \otimes_{\mathcal{K}} \mathcal{F}^{\boldsymbol{\rho}} \cong \operatorname{Hom}(\mathcal{F}^{\rho_1}, \mathcal{F}^{\rho_1}) \tag{6.3.2}$$

(see [Ja, 3.9.(3) and 3.10.(3)]). Note the map $i = 1_A \otimes \iota$ is a $\mathbf{U}$-module map ${}_{\mathcal{K}}A \to_{\mathcal{K}} A \otimes_K \mathcal{F}^{\rho_1} \otimes_{\mathcal{K}} \mathcal{F}^{\boldsymbol{\rho}}$ given by $a \mapsto \sum a \otimes x_i \otimes x_i{}^{\boldsymbol{\rho}}$. Recall the twist $\tau : (B \otimes \mathcal{F})^{\rho_1} \to \mathcal{F}^{\rho_1} \otimes B^{\rho_1}$ is a module isomorphism. Now let

$$\phi_c^l = \phi^l \circ ((1_{A \otimes_{\mathcal{K}} \mathcal{F}^{\rho_1}} \otimes \nu \otimes 1_B) \circ (i \times \tau \circ (1_{(B \otimes \mathcal{F})^{\rho_1}})) \tag{6.3.3}$$

denote the contraction as a $\mathcal{K}$-valued pairing on ${}_{\mathcal{K}}A \times ({}_{\mathcal{K}}B \otimes_{\mathcal{K}} \mathcal{F})^{\rho_1}$. We claim: for all $y \in {}_{\mathcal{K}}\mathcal{F}, a \in {}_{\mathcal{K}}A$ and $b \in {}_{\mathcal{K}}B$

$$\phi^r(a, b \otimes e) = \sum \phi(a, b_i)(e_i) \ = \phi^l(a \otimes e, b) = \phi^l_c(a, b \otimes e) \tag{6.3.3}$$

Fix an index j. Then

$$\phi^l(a \otimes x_j, b) \ = \phi^l \circ (1_{A \otimes_{\mathcal{K}} \mathcal{F}^{\rho_1}} \otimes \nu \otimes 1_B) \left(\sum_i a \otimes x_i \otimes x_i^{\boldsymbol{\rho}}, \tau(b \otimes x_j)\right) \ = \phi^l_c(a, b \otimes x_j).$$

Extending this linearly over the basis $\{x_j\}$ proves the claim.

Let $\varrho^\sharp = S \circ \rho_1$ and let $\psi : M \to (M^*)^*$ denote the $\mathbf{U}$-module map $\psi(m)(f) = f(K^{-1}m)$ (see [Ja, 3.9.(2)]) for any module M. Then the composition of this map ψ with the isomorphism

$$\mathcal{F}^* \otimes (\mathcal{F}^*)^* \to \operatorname{Hom}(\mathcal{F}^*, \mathcal{F}^*) \qquad \text{(see [Ja,3.10.(3)])}$$

gives rise to an embedding

$$\iota^\sharp : \mathcal{K} \to {}_{\mathcal{K}}\mathcal{F}^* \otimes {}_{\mathcal{K}}\mathcal{F}, \qquad \iota^\sharp(1) = \sum \xi_i^* \otimes \xi_i$$

Here ξ_i^* is dual to ξ under the map $\iota^\sharp$ i.e. $\xi_i^*(\psi(\xi_j)) = \delta_{ij}$. We can also define a contraction $\nu^\sharp : {}_{\mathcal{K}}\mathcal{F}^{\rho_1} \otimes {}_{\mathcal{K}}\mathcal{F}^{\rho^\sharp} \to \mathcal{K}$ where $\nu^\sharp(e \otimes \xi) = \xi(\psi(e))$.

We can define in a similar way the right contraction

$$\phi_c^r = \phi^r \circ \left(\left(1_A \otimes \nu^\sharp \otimes 1_{\mathcal{F}^{\rho_1} \otimes B^{\rho_1}}\right) \circ \left(1_{A \otimes_{\mathcal{K}} \mathcal{F}^{\rho_1}} \times (\tau \circ 1_B \otimes \iota^\sharp)\right)\right) \tag{6.3.4}$$

Then equations (6.3.3) and (6.3.4) we find that knowing any one of the maps ϕ, ϕ^r, ϕ^l, ϕ^r_c or ϕ^l_c determines all the others. Also by (6.3.3) for any ${}_{\mathcal{K}}\mathbf{U}$-invariant ${}_{\mathcal{K}}\mathcal{F}$-valued pairings ϕ and χ ; $\phi = \chi$ if and only if $\phi^l_c = \chi^r$. Thus to prove the theorem it is sufficient to establish the identity: $(\phi_{lF_i})^l_c = (\phi_{rF_i})^r$. Now applying localization to the definition of ϕ^l_c, its functoriality and (6.3.1) we obtain

$$(\phi^l_c)_{F_i} = \phi^l_{F_i} \circ ((1_{A_{F_i} \otimes_{\mathcal{K}} \mathcal{F}^{\rho_1}} \otimes \nu \otimes 1_{B_{F_i}}) \circ (i \times \tau \circ (1_{(B_{F_i} \otimes \mathcal{F})^{\rho_1}})) = (\phi_{lF_i})^l_c.$$

Combining this with the first identity in (6.3.3) and the result of localizing identity (6.3.4) we obtain;

$$(\phi_{rF_i})^r = (\phi^r)_{F_i} = (\phi^l_c)_{F_i} = (\phi_{lF_i})^l_c$$

which completes the proof of the theorem.

□

6.4. If M is a $\mathbf{U}$-module, then $\operatorname{Hom}_{\mathcal{K}}({}_{\mathcal{K}}M, \mathcal{K})$, becomes a $\mathbf{U}$-module by defining $u.\phi(x) = \phi(S'(u))$ for $u \in \mathbf{U}$, where $S' : \mathbf{U} \to \mathbf{U}$ is the *skew-antipode* given on generators by: $S'(E_i) = -E_i\tilde{K}_{-i}$, $S'(F_i) = -\tilde{K}_iF_i$, $S'(K_\mu) = K_{-\mu}$ (cf. [L, §3.3]). Let this $\mathbf{U}$-module be denoted by M'.

Suppose $\mathcal{F}$ is a finite dimensional $\mathbf{U}$-stable subspace of $\mathbf{U}$ under the adjoint action $\operatorname{ad} u(b) = \sum u_{(1)}bS(u_{(2)})$ and $\phi \in \mathbb{P}_{\boldsymbol{\rho}}(A, B)$. Define ${}_{\mathcal{K}}\mathcal{F}^{\boldsymbol{\rho}}$-valued pairings ϕ_l and ϕ_r by the formula: for $X \in \mathcal{F}, a \in A$ and $b \in B$

$$\phi_l(a, b)(X) = \phi(\boldsymbol{\rho}(X) \cdot a, b) \quad , \quad \phi_r(a, b)(X) = \phi(a, X \cdot b) \tag{6.4.1}$$

Since $\phi \in \mathbb{P}_{\boldsymbol{\rho}}(A, B)$, $\boldsymbol{\rho} \otimes \boldsymbol{\rho} \circ \Delta = \Delta \circ \boldsymbol{\rho}$ and $\mu \circ S \otimes 1 \circ \Delta = \mathbf{e}$,

$$\begin{aligned}(x * \phi_l(a, b))(e) &= \sum x_{(1)} * \phi_l(S(x_{(3)})a, \boldsymbol{\rho}(x_{(2)})b)(e)\\ &= \sum \phi(\boldsymbol{\rho}(\operatorname{ad} \boldsymbol{\rho}(x_{(1)})e)S(x_{(3)})a, \boldsymbol{\rho}(x_{(2)})b)\\ &= \sum \phi(\boldsymbol{\rho}(\boldsymbol{\rho}(x_{(1)})eS(\boldsymbol{\rho}(x_{(2)})))S(x_{(4)})a, \boldsymbol{\rho}(x_{(3)})b)\\ &= \sum \phi(\boldsymbol{\rho}(S(\boldsymbol{\rho}(x_{(2)}))\boldsymbol{\rho}(e)x_{(1)}S(x_{(4)})a, \boldsymbol{\rho}(x_{(3)})b)\\ &= \sum \phi(\boldsymbol{\rho}(e)x_{(1)}S(x_{(4)})a, S(\boldsymbol{\rho}(x_{(2)}))\boldsymbol{\rho}(x_{(3)})b)\\ &= \sum \phi(\boldsymbol{\rho}(e)x_{(1)}S(x_{(2)})a, b)\\ &= \mathbf{e}(x)\phi(\boldsymbol{\rho}(e)a, b) \ = \mathbf{e}(x)\phi_l(a, b)(e)\end{aligned}$$

for $x \in \mathbf{U}, e \in \mathcal{F}, a \in A$ and $b \in B$. This shows $\phi_l \in \mathbb{P}_{\boldsymbol{\rho}|_{R\mathbf{U}_i}}(A, B, \mathcal{F})$. Similarly the calculation

$$\begin{aligned}(x.\phi_r(a, b))(e) &= \sum x_{(1)} * \phi_r(S(x_{(3)})a, \boldsymbol{\rho}(x_{(2)})b)(e)\\ &= \sum \phi_r(S(x_{(3)})a, \boldsymbol{\rho}(x_{(2)})b)(\boldsymbol{\rho}(x_{(1)})e)\\ &= \sum \phi(S(x_{(3)})a, (\operatorname{ad} \boldsymbol{\rho}(x_{(1)})e)\boldsymbol{\rho}(x_{(2)})b)\\ &= \sum \phi(S(x_{(4)})a, \boldsymbol{\rho}(x_{(1)})eS(\boldsymbol{\rho}(x_{(2)}))\boldsymbol{\rho}(x_{(3)})b)\\ &= \sum \phi(S(x_{(2)})a, \boldsymbol{\rho}(x_{(1)})eb)\\ &= \mathbf{e}(x)\phi(a, eb) = \mathbf{e}(x)\phi_r(a, b)(e)\end{aligned}$$

shows $\phi_r \in \mathbb{P}_{\boldsymbol{\rho}|_{R}\mathbf{U}_i}(A, B, \mathcal{F})$.

Lemma 6.4.1. *For ϕ as above, $(\phi_{F_i})^l_l = (\phi^l_l)_{F_i}$, $(\phi_{F_i})^r_r = (\phi^r_r)_{F_i}$.*

Proof. Let π be the $\mathbf{U}$-module map , $\pi : \mathcal{F}^{\rho_1} \otimes B^{\rho_1} \to B^{\rho_1}$ given by $e \otimes b \mapsto \rho_1(e) \cdot b$ for $b \in B$ (here we view $\mathcal{F}^{\rho_1} \subset \mathbf{U}$. Then

$$\phi^l_l \circ 1_A \otimes {}_f\mathcal{R} = \phi \circ (1_A \times \pi).$$

By functoriality of localization , $(\phi^l_l)_{F_i} \circ 1_{A_{F_i}} \otimes {}_f\mathcal{R}_{F_i} = \phi_{F_i} \circ (1_{A_{F_i}} \times \pi_{F_i})$. The isomorphism (5.1) implies π_{F_i} has the same form as π ; i.e. $\pi_{F_i}(e \otimes b) = \rho_1(e) \cdot b$, for $e \in \mathcal{F}$ and $b \in B_{F_i}$. As we remarked earlier ${}_f\mathcal{R}_{F_i} = {}_f\mathcal{R}$, which proves the first identity. The second identity is proved in a similar fashion. □

6.5. **Proof of Theorem (6.1):** From [JL1,§ 6] $\mathbf{U}$ can be expressed as a localization of its ad-integrable part and is equal to $\mathbf{T}_<^{-1}I(\mathbf{U})[\mathbf{T}/\mathbf{T}_0]$ where $I(\mathbf{U})$ is a sum of integrable $\mathbf{U}$-stable subspaces, $\mathbf{T}_<$ is an Øre subset of $I(\mathbf{U})$ contained in the abelian group $\{K_\mu | \mu \in Y\}$ and $[\mathbf{T}/\mathbf{T}_0]$ is a finite set of elements in $\{K_\mu | \mu \in Y\}$. To prove (6.1) we must first show that ϕ_{F_i} is invariant on the locally finite part of $\mathbf{U}$; i.e. $(\phi_{F_i})_l = (\phi_{F_i})_r$. Combining the definition (6.3.1) and the preceding lemma gives $((\phi_l)_{lF_i})^l = (\phi^l_l)_{F_i} = (\phi_{F_i})^l_l$. This implies $(\phi_l)_{lF_i} = (\phi_{F_i})_l$ and by a similar argument $(\phi_r)_{rF_i} = (\phi_{F_i})_r$. Finally since ϕ is $\boldsymbol{\rho}$-invariant, $\phi_l = \phi_r$ and by (6.3) the left and right liftings agree giving $(\phi_l)_{lF_i} = (\phi_r)_{rF_i}$.

To complete the proof we must show ϕ_{F_i} is $\boldsymbol{\rho}$-invariant by elements from U^0. By Lemma 2.7 and 4.2 one may reduce to case that ${}_{\mathcal{K}}A$ and ${}_{\mathcal{K}}B$ are direct sums of Verma modules ${}_{\mathcal{K}}M_{m+\lambda}$ for some $\lambda \in X$. Since ${}_{\mathcal{K}}\phi_{F_i}$ is ${}_{\mathcal{K}}\phi$ localized at F_i distinct weight spaces are orthogonal. This is the desired invariance.

6.6. The significance of the invariance theorem for us comes from its implications regarding the Shapovalov forms on Verma modules: By [L, 3.2.5] we have a direct sum decomposition: $\mathbf{U}^0 \oplus (\mathbf{U}^-_+\mathbf{U} + \mathbf{U}\mathbf{U}^+_+)$, where $\mathbf{U}^\pm_+ = \mathbf{U}^\pm \cap \ker \mathbf{e}$. For $a \in \mathbf{U}^0$ and $\lambda \in X$ let $a(\lambda)$ denote the value obtained when each K_μ is replaced by $v^{\langle\lambda,\mu\rangle}$. If we let $\psi : \mathbf{U} \to \mathbf{U}^0$ denote the projection then the bilinear form ψ_λ on $\mathbf{U}$ given by $\psi_\lambda(a, b) = \psi(\boldsymbol{\rho}(a)b)(\lambda)$ factors to the *Shapovalov form* on M_λ which we also denote by ψ_λ.

Corollary 6.6.1. *Let $A = {}_RM_{m+\lambda}$ and let ϕ equal the Shapovalov form on A. Then $A_\pi \cong {}^TM_{m+\lambda}$ and ϕ_{F_i} is a scalar multiple of the pullback to A_{F_i} of the Shapovalov form on A_π.*

Proof. The space of ${}_{\mathcal{K}}\mathbf{U}$-invariant forms on A_π is a one dimensional $\mathcal{K}$-space. □

7. Induced Forms and Pairings

In this section we assume $\mathbf{U}$ is of finite type.

7.1. Let M and N be ${}_R\mathbf{U}$-modules and $\phi = \phi_{M,N}$ an R-valued invariant pairing of M and N. Let $\mathcal{E}$ and $\mathcal{F}$ be finite dimensional $\mathbf{U}$-module. We now consider a natural family of pairings of $M \otimes {}_R\mathcal{E}$ and $N \otimes {}_R\mathcal{F}$. Recalling the isomorphisms of Lemma (6.2) we obtain an isomorphism:

$$\Theta : \mathbb{P}(M \otimes_R \mathcal{E} \otimes {}_R\mathcal{F}^{\rho_1}, N) \to \mathbb{P}(M \otimes {}_R\mathcal{E}, N \otimes {}_R\mathcal{F}),$$

defined by $\Theta(\chi) = \theta_r \circ \theta_l^{-1}(\chi)$, for $\chi \in \mathbb{P}(M \otimes_R \mathcal{E} \otimes {}_R\mathcal{F}^{\rho_1}, N)$.

7.2. Let $\mathcal{E}$ and $\mathcal{F}$ be finite dimensional $\mathbf{U}$-modules and τ a $\mathbf{U}$-module homomorphism into $\mathbf{U}$; $\beta :_R \mathcal{E}^{\rho_1} \otimes_R \mathcal{F}^{\rho_1} \to {}_R\mathbf{U}$. Suppose ϕ is a pairing of M and N. Define $\psi_{\beta,\phi}$ to be the invariant pairing of $M \otimes_R \mathcal{E}^{\rho_1}$ and $N \otimes_R \mathcal{F}$ defined by the formula, for $e \in \mathcal{E}^{\rho_1}, f \in \mathcal{F}, m \in M$, and $n \in N$,

$$\psi_{\beta,\phi}(m \otimes e, n \otimes f) = \phi(m, \beta(e \otimes f)n). \tag{7.1.1}$$

Here $\mathcal{E}^{\rho_1}$ is a twist of the representation $\mathcal{E}$ by ρ_1. *(Should we have $\cdot$ denote the action due to ρ_1?)* We call the pairing $\psi_{\beta,\phi}$ the *pairing induced* by β and ϕ. In the cases when M, N and ϕ are fixed we write ψ_β in place of $\psi_{\beta,\phi}$ and say this pairing is induced by β.

7.3. As above let S denote the set of positive integral powers of $F = F_i$. There are times when the subscript notation for localization of $\mathbf{U}$ with respect to S is clumsy and easily confused with other required subscripts. When this occurs we write $loc_F M$ in place of M_F and loc_F for the localization functor in general.

Proposition 7.3.1. *With Θ defined as in (7.1), the following diagram is commutative.*

$$\begin{array}{ccc} \mathbb{P}_{\boldsymbol{\rho}}(M \otimes_R \mathcal{E} \otimes_R \mathcal{F}^{\rho_1}, N) & \xrightarrow{\Theta} & \mathbb{P}_{\boldsymbol{\rho}}(M \otimes_R \mathcal{E}, N \otimes_R \mathcal{F}) \\ loc_F \downarrow & & loc_F \downarrow \\ \mathbb{P}_{\boldsymbol{\rho}}(M_F \otimes_R \mathcal{E} \otimes_R \mathcal{F}^{\rho_1}, N_F) & \xrightarrow{\Theta} & \mathbb{P}_{\boldsymbol{\rho}}(M_F \otimes_R \mathcal{E}, N_F \otimes_R \mathcal{F}) \end{array}$$

Proof. Let C be a finite dimensional $\mathbf{U}$-module and A and B objects in $\mathcal{C}_R$. We have isomorphisms by (6.2): $\theta_r : \mathbb{P}_{\boldsymbol{\rho}}(A, B, C) \to \mathbb{P}_{\boldsymbol{\rho}}(A, B \otimes C)$, and $\theta_l : \mathbb{P}_{\boldsymbol{\rho}}(A, B, C) \to \mathbb{P}_{\boldsymbol{\rho}}(A \otimes C^{\rho_1}, B)$. The map Θ is $\theta_r\theta_l^{-1}$ with $B = N, C = \mathcal{F}$ and $A = M \otimes \mathcal{E}$. The theorem on the equality of the left and right liftings (6.5) can be rephrased : for $\phi \in \mathbb{P}_{\boldsymbol{\rho}}(A, B, C)$, $\theta_r^{-1} loc_{F_i} \theta_r(\phi) = \theta_l^{-1} loc_{F_i} \theta_l(\phi)$. Setting $\phi = \theta_l^{-1}(\gamma)$ and multiplying by θ_r we obtain: $\Theta \; loc_{F_i}(\gamma) = loc_{F_i} \; \Theta(\gamma)$. □

Lemma 7.3.1. *Let A, B, C and D be objects in $\mathcal{C}_R$ and δ and γ homomorphisms $\delta : A \to C$ and $\gamma : B \to D$. Set $F = F_i$ for $i \in I$ fixed. The following diagram is commutative.*

$$\begin{array}{ccc} \mathbb{P}_{\boldsymbol{\rho}}(C, D) & \xrightarrow{\delta\otimes\gamma} & \mathbb{P}_{\boldsymbol{\rho}}(A, B) \\ loc_F \downarrow & & loc_F \downarrow \\ \mathbb{P}_{\boldsymbol{\rho}}(C_F, D_F) & \xrightarrow{\delta_F\otimes\gamma_F} & \mathbb{P}_{\boldsymbol{\rho}}(A_F, B_F) \end{array}$$

Proof. Recall from (4.3) the definition of lifting and the surrounding notation. For all maps $\mathcal{S}$ and $\mathcal{T}$ of $M_{m+\epsilon\lambda}$ into A and B respectively and $\phi \in \mathbb{P}_{\boldsymbol{\rho}}(C, D)$, we have:

$$\begin{aligned} ((\delta \otimes \gamma)\phi)(\mathcal{S}w_{\epsilon\lambda,-n}, \mathcal{T}w_{\epsilon\lambda,-n}) &= \phi(\delta\mathcal{S}w_{\epsilon\lambda,-n}, \gamma\mathcal{T}w_{\epsilon\lambda,-n}) \\ &= [n-1]^!_i[\epsilon\iota\tilde{K}_i; n-1]_{(n-1)}\phi_F((\delta\mathcal{S})_F w_{\epsilon\lambda,n}, (\gamma\mathcal{T})_F w_{\epsilon\lambda,n}) \\ &= [n-1]^!_i[\epsilon\iota\tilde{K}_i; n-1]_{(n-1)}((\delta_F \otimes \gamma_F)\phi_F)(\mathcal{S}_F w_{\epsilon\lambda,n}, \mathcal{T}_F w_{\epsilon\lambda,n}) \; . \end{aligned}$$

But this is precisely the desired identity: $((\delta \otimes \gamma)\phi)_F = (\delta_F \otimes \gamma_F)\phi_F$. □

Proposition 7.3.2. *Localization and induction of pairings commute ; $loc_F \psi_{\beta,\phi} = \psi_{\beta, loc_F \phi}$.*

Proof. By Proposition (7.3.1) it is sufficient to prove the proposition for the case where $\mathcal{F}$ equals the trivial representation and β is a homomorphism of $\mathcal{E}$ into $\mathbf{U}$;i.e. $\psi'_{\beta,\phi}$ equals $\psi_{\beta,\phi}$. The induced pairing is determined by the maps, $(M \otimes_R \mathcal{E})^{\rho_1} \xrightarrow{\tau} {}_R\mathcal{E}^{\rho_1} \otimes M^{\rho_1} \xrightarrow{\beta\otimes 1} \mathbf{U} \otimes M^{\rho_1} \xrightarrow{a} M^{\rho_1}$, where a denotes the action of $\mathbf{U}$ on M^{ρ_1}. Since localization of a gives the action on $M_F^{\rho_1}$, we have:

$$(M_F \otimes_R \mathcal{E})^{\rho_1} \xrightarrow{\tau} {}_R\mathcal{E}^{\rho_1} \otimes M_F^{\rho_1} \xrightarrow{\beta\otimes 1} \mathbf{U} \otimes M_F^{\rho_1} \xrightarrow{a} M_F^{\rho_1} \; .$$

Let β' denote the composition of these maps. Then $\psi_{\beta,\phi} = \phi \circ (1 \times \beta')$ and so we conclude: $loc_F \psi_{\beta,\phi} = \phi_F \circ (1 \times \beta') = \psi_{\beta, loc_F \phi}$. □

7.4. We end this section by pointing out one important instance where all pairings are induced .

Proposition 7.4.1. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are finite dimensional $\mathbf{U}$-modules. Let $\lambda \in X$ and M be the $\mathbf{U}$-Verma module $M_{m+\lambda}$ and ϕ the Shapovalov form on M. Then every invariant pairing of $M \otimes_R \mathcal{E}$ and $M \otimes_R \mathcal{F}$ is induced by ϕ.*

Proof. From (4.2) it is sufficient to prove the theorem with R replaced by $\mathcal{K}$. Since $\mathbb{P}_{\boldsymbol{\rho}}({}_{\mathcal{K}}M \otimes {}_{\mathcal{K}}\mathcal{E}, {}_{\mathcal{K}}M \otimes {}_{\mathcal{K}}\mathcal{F}) \cong \mathbb{P}_{\boldsymbol{\rho}}({}_{\mathcal{K}}M \otimes {}_{\mathcal{K}}\mathcal{E} \otimes {}_{\mathcal{K}}\mathcal{F}^{\rho_1}, {}_{\mathcal{K}}M)$, the dimension of this space equals d_0 the dimension of the weight space for the trivial $\mathbf{U}^0$ module in $\mathcal{E} \otimes \mathcal{F}^{\rho_1}$.

Set $\mathbb{F}$ equal to the ad-finite submodule of End $({}_{\mathcal{K}}M_{m+\lambda})$. The work of Joseph and Letzter [JL2,§7] and [JL3,§4] proves that the ad-finite part of $\mathbf{U}$ maps surjectively onto $\mathbb{F}$ and $\mathbb{F} = \sum_{\delta, 1 \leq i \leq m_\delta} \mathbb{F}_{\delta,i}$ with m_δ equal to the dimension of the weight space for the trivial $\mathbf{U}^0$ submodule. Let $\mathcal{E} \otimes \mathcal{F}^{\rho_1} = \sum_{\delta, 1 \leq j \leq n_\delta} \mathcal{F}_{\delta,j}$ denote the corresponding isotypic decomposition $\mathcal{E} \otimes \mathcal{F}^{\rho_1}$. Since the Shapovalov form is nondegenerate on ${}_{\mathcal{K}}M_{m+\lambda}$, for any subspace $L \neq 0$ of $\mathbb{F}$, $\phi(L \cdot M, M) \neq 0$. For $1 \leq i \leq n_\delta$ and $1 \leq j \leq m_\delta$, let $\beta_{\delta,j,i}$ be any map of $\mathcal{E} \otimes \mathcal{F}^{\rho_1}$ to $\mathbf{U}$ inducing a map to $\mathbb{F}$ which is an isomorphism of $\mathcal{F}_{\delta,j}$ onto $\mathbb{F}_{\delta,i}$ and is zero on the other summands of $\mathcal{E} \otimes \mathcal{F}^{\rho_1}$. Now by the nondegeneracy of ϕ, the induced pairings $\chi_{\beta_{\delta,j,i},\phi}$ are linearly independent. So the dimension of the space $Ind({}_{\mathcal{K}}M \otimes {}_{\mathcal{K}}\mathcal{E}, {}_{\mathcal{K}}M \otimes {}_{\mathcal{K}}\mathcal{F})$ is greater than or equal to $\sum n_\delta \; m_\delta$ where the sum is taken over δ. However we know m_δ equals the dimension of the weight subspace for the trivial $\mathbf{U}^0$ submodule. So $d_0 = \sum n_\delta \; m_\delta$ and thus the dimension of the space of induced pairings is greater than or equal to the dimension of the space of all invariant pairings. □

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