PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 10, October 2011, Pages 3451–3460
S 0002-9939(2011)10906-7
Article electronically published on March 9, 2011

DJKM ALGEBRAS I: THEIR UNIVERSAL CENTRAL EXTENSION

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(Communicated by Gail R. Letzter)

Abstract. The purpose of this paper is to explicitly describe in terms of generators and relations the universal central extension of the infinite dimensional Lie algebra, $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1},u|u^2=(t^2-b^2)(t^2-c^2)]$, appearing in the work of Date, Jimbo, Kashiwara and Miwa in their study of integrable systems arising from the Landau-Lifshitz differential equation.

1. Introduction

In this paper the authors explicitly describe in terms of generators and relations and three families of polynomials, the universal central extension of an algebra appearing in the work of Date, Jimbo, Kashiwara and Miwa (see [DJKM83, DJKM85]), where they study integrable systems arising from the Landau-Lifshitz differential equation. Two of these families of polynomials are described below in terms of elliptic integrals and the other family is a variant of certain ultraspherical polynomials. The authors Date, Jimbo, Kashiwara and Miwa solved the Landau-Lifshitz equation using methods developed in some of their previous work on affine Lie algebras. The hierarchy of this equation is written in terms of free fermions on an elliptic curve. The infinite dimensional Lie algebra mentioned above is shown to act on solutions of the Landau-Lifshitz equation as infinitesimal Bäcklund transformations where they derive an N-soliton formula. These authors arrive at an algebra that is a one dimensional central extension of $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1},u|u^2=(t^2-b^2)(t^2-c^2)]$ where $b\neq\pm c$ are complex constants and $\mathfrak{g}$ is a simple finite dimensional Lie algebra defined over the complex numbers. Below we explicitly describe its four dimensional universal central extension. Modulo the center, this algebra is a particular example of a Krichever-Novikov current algebra (see ([KN87b], [KN87a], [KN89]). A fair amount of interesting and fundamental work has be done by Krichever, Novikov,

Received by the editors September 5, 2010.

2010 *Mathematics Subject Classification.* Primary 17B65, 17B67; Secondary 81R10.

Key words and phrases. Krichever-Novikov algebras, Landau-Lifshitz differential equation, Date-Jimbo-Miwa-Kashiwara algebras, universal central extension, ultraspherical polynomials, elliptic integrals.

The first author is grateful to the Fapesp (processo 2009/17533-6) and the University of São Paulo for their support and hospitality during his visit to São Paulo. The first author was also partially supported by a research and development grant from the College of Charleston.

The second author was partially supported by Fapesp (processo 2005/60337-2) and CNPq (processo 301743/2007-0).

Schlichenmaier, and Sheinman on the representation theory of certain one dimensional central extensions of these latter current algebras and of analogues of the Virasoso algebra. In particular, Wess-Zumino-Witten-Novikov theory and analogues of the Knizhnik-Zamolodchikov equations are developed for these algebras (see the survey article [She05] and, for example, [SS99], [SS99] ,[She03], [Sch03a], [Sch03b], and [SS98]).

M. Bremner on the other hand has explicitly described in terms of generators, relations and certain families of polynomials (ultraspherical and Pollaczek), the structure constants for the universal central extension of algebras of the form $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1},u|u^2=p(t)]$ where $p(t)=t^2-2bt+1$ and $p(t)=t^3-2bt^2+t$ (see [Bre95, Bre94]). He determined more generally the dimension of the universal central extension for affine Lie algebras of the form $\mathfrak{g}\otimes R$ where R is the ring of regular functions defined on an algebraic curve with any number of points removed. He obtained this using C. Kassel's result ([Kas84]) where one knows that the center is isomorphic as a vector space to Ω^1_R/dR (the *space of Kähler differentials of R modulo exact forms*). We will review this material below as needed.

In our previous work (see [Cox08, BCF09]), the authors used Bremner's aforementioned description to obtain certain free-field realizations of the four point and elliptic affine algebras depending on a parameter $r=0,1$ that correspond to two different normal orderings. These later realizations are analogues of Wakimoto-type realizations which have been used by Schechtman and Varchenko and various other authors in the affine setting to pin down integral solutions to the Knizhnik-Zamolodchikov differential equations (see for example [ATY91], [Kur91], [EFK98], [SV90]). Such realizations have also been used in the study of the Drinfeld-Sokolov reduction in the setting of W-algebras and in E. Frenkel's and B. Feigin's description of the center of the completed enveloping algebra of an affine Lie algebra (see [FFR94], [Fre05], and [FF92]). In future work the authors plan to use results of this paper to describe free-field realizations of the universal central extension of the algebras of Date, Jimbo, Kashiwara and Miwa (which, since this is a mouth full, will be called DJKM algebras).

2. Universal central extensions of current algebras

Let R be a commutative algebra defined over $\mathbb{C}$. Consider the left R-module with action $f(g\otimes h)=fg\otimes h$ for $f,g,h\in R$ and let K be the submodule generated by the elements $1\otimes fg-f\otimes g-g\otimes f$. Then $\Omega^1_R=F/K$ is the module of Kähler differentials. The element $f\otimes g+K$ is traditionally denoted by fdg. The canonical map $d:R\to\Omega^1_R$ is denoted by $df=1\otimes f+K$. The *exact differentials* are the elements of the subspace dR. The coset of fdg modulo dR is denoted by $\overline{fdg}$. As C. Kassel showed, the universal central extension of the current algebra $\mathfrak{g}\otimes R$ where $\mathfrak{g}$ is a simple finite dimensional Lie algebra defined over $\mathbb{C}$, is the vector space $\hat{\mathfrak{g}}=(\mathfrak{g}\otimes R)\oplus\Omega^1_R/dR$ with Lie bracket given by

$$[x\otimes f,Y\otimes g]=[xy]\otimes fg+(x,y)\overline{fdg},[x\otimes f,\omega]=0,[\omega,\omega']=0,$$

where $x,y\in\mathfrak{g}$ and $\omega,\omega'\in\Omega^1_R/dR$, and (x,y) denotes the Killing form on $\mathfrak{g}$.

Consider the polynomial

$$p(t)=t^n+a_{n-1}t^{n-1}+\cdots+a_0,$$

where $a_i\in\mathbb{C}$ and $a_n=1$. Fundamental to the description of the universal central extension for $R=\mathbb{C}[t,t^{-1},u|u^2=p(t)]$ is the following:

Theorem 2.0.1 ([Bre94, Theorem 3.4])**.** *Let R be as above. The set*

$$\{\overline{t^{-1}\,dt}, \overline{t^{-1}u\,dt}, \dots, \overline{t^{-n}u\,dt}\}$$

forms a basis of Ω^1_R/dR (omitting $\overline{t^{-n}u\,dt}$ if $a_0 = 0$).

Set $u^m = p(t)$. Then $u\,d(u^m) = mu^m du$ and

$$\sum_{j=1}^{n} ja_j t^{j-1} u\,dt - m\left(\sum_{j=0}^{n} a_j t^j\,du\right) = 0$$

or

$$p'(t)udt - mp(t)du = 0.$$

Multiplying by t^i we get

$$\sum_{j=1}^{n} ja_j t^{i+j-1} u\,dt - m\left(\sum_{j=0}^{n} a_j t^{i+j}\,du\right) = 0. \tag{2.1}$$

Lemma 2.0.2. *If $u^m = p(t)$ and $R = \mathbb{C}[t, t^{-1}, u|u^m = p(t)]$, then in Ω^1_R/dR, one has*

$$((m+1)n + im)t^{n+i-1}u\,dt \equiv -\sum_{j=0}^{n-1}((m+1)j + mi)a_j t^{i+j-1}u\,dt \mod dR. \tag{2.2}$$

Proof. We have, expanding $d(t^{i+j}u)$,

$$(i+j)t^{i+j-1}u\,dt \equiv -t^{i+j}\,du \mod dR,$$

so that (2.1) implies

$$\sum_{j=0}^{n} ja_j t^{i+j-1}u\,dt + m\left(\sum_{j=0}^{n}(i+j)a_j t^{i+j-1}u\,dt\right) = 0 \mod dR \tag{2.3}$$

or

$$\sum_{j=0}^{n}((m+1)j + mi)a_j t^{i+j-1}u\,dt \equiv 0 \mod dR. \tag{2.4}$$

This gives (2.2). □

3. Description of the universal central extension of Date-Jimbo-Miwa-Kashiwara algebras

In the Date-Jimbo-Miwa-Kashiwara setting one takes $m = 2$ and $p(t) = (t^2 - a^2)(t^2 - b^2) = t^4 - (a^2 + b^2)t^2 + (ab)^2$ with $a \neq \pm b$ and neither a nor b is zero. We fix from here onward $R = \mathbb{C}[t, t^{-1}, u \,|\, u^2 = (t^2 - a^2)(t^2 - b^2)]$. As in this case $a_0 = (ab)^2$, $a_1 = 0$, $a_2 = -(a^2 + b^2)$, $a_3 = 0$ and $a_4 = 1$, then letting $k = i + 3$, the recursion relation in (2.2) looks like

$$(6 + 2k)\overline{t^k u\,dt} = -2(k-3)(ab)^2\overline{t^{k-4}u\,dt} + 2k(a^2 + b^2)\overline{t^{k-2}u\,dt}.$$

After a change of variables we may assume that $a^2b^2 = 1$. Then the recursion relation looks like

$$(6 + 2k)\overline{t^k u\,dt} = -2(k-3)\overline{t^{k-4}u\,dt} + 4kc\overline{t^{k-2}u\,dt}, \tag{3.1}$$

after setting $c = (a^2 + b^2)/2$, so that $p(t) = t^4 - 2ct^2 + 1$. Let $P_k := P_k(c)$ be the polynomial in c satisfying the recursion relation

$$(6 + 2k)P_k(c) = 4kcP_{k-2}(c) - 2(k - 3)P_{k-4}(c)$$

for $k \geq 0$. Then set

$$P(c, z) := \sum_{k\geq -4} P_k(c)z^{k+4} = \sum_{k\geq 0} P_{k-4}(c)z^k,$$

so that after some straightforward rearrangement of terms we have

$$\begin{aligned}
0 &= \sum_{k\geq 0}(6 + 2k)P_k(c)z^k - 4c\sum_{k\geq 0} kP_{k-2}(c)z^k + 2\sum_{k\geq 0}(k - 3)P_{k-4}(c)z^k \\
&= (-2z^{-4} + 8cz^{-2} - 6)P(c, z) + (2z^{-3} - 4cz^{-1} + 2z)\frac{d}{dz}P(c, z) \\
&\quad + (2z^{-4} - 8cz^{-2})P_{-4}(c) - 4cP_{-3}(c)z^{-1} - 2P_{-2}(c)z^{-2} - 4P_{-1}(c)z^{-1}.
\end{aligned}$$

We then multiply the above by z^4 to get

$$\begin{aligned}
0 &= (-2 + 8cz^2 - 6z^4)P(c, z) + (2z - 4cz^3 + 2z^5)\frac{d}{dz}P(c, z) \\
&\quad + (2 - 8cz^2)P_{-4}(c) - 4cP_{-3}(c)z^3 - 2P_{-2}(c)z^2 - 4P_{-1}(c)z^3.
\end{aligned}$$

Hence $P(c, z)$ must satisfy the differential equation

(3.2)
$$\frac{d}{dz}P(c, z) - \frac{3z^4 - 4cz^2 + 1}{z^5 - 2cz^3 + z}P(c, z) = \frac{2\left(P_{-1} + cP_{-3}\right)z^3 + P_{-2}z^2 + (4cz^2 - 1)P_{-4}}{z^5 - 2cz^3 + z}.$$

This has integrating factor

$$\begin{aligned}
\mu(z) &= \exp\int\left(\frac{-2\left(z^3 - cz\right)}{1 - 2cz^2 + z^4} - \frac{1}{z}\right)dz \\
&= \exp(-\frac{1}{2}\ln(1 - 2cz^2 + z^4) - \ln(z)) = \frac{1}{z\sqrt{1 - 2cz^2 + z^4}}.
\end{aligned}$$

3.1. **Elliptic Case 1.** If we take initial conditions $P_{-3}(c) = P_{-2}(c) = P_{-1}(c) = 0$ and $P_{-4}(c) = 1$, we arrive at a generating function

$$P_{-4}(c, z) := \sum_{k\geq -4} P_{-4,k}(c)z^{k+4} = \sum_{k\geq 0} P_{-4,k-4}(c)z^k,$$

defined in terms of an elliptic integral

$$P_{-4}(c, z) = z\sqrt{1 - 2cz^2 + z^4}\int \frac{4cz^2 - 1}{z^2(z^4 - 2cz^2 + 1)^{3/2}}\,dz.$$

One way to interpret the right-hand integral is to expand $(z^4 - 2cz^2 + 1)^{-3/2}$ as a Talyor series about $z = 0$ and then formally integrate term by term and multiply the result by the Taylor series of $z\sqrt{1 - 2cz^2 + z^4}$. More precisely, one integrates formally with zero constant term:

$$\int (4c - z^{-2})\sum_{n=0}^{\infty} Q_n^{(3/2)}(c)z^{2n}\,dz = \sum_{n=0}^{\infty}\frac{4cQ_n^{(3/2)}(c)}{2n + 1}z^{2n+1} - \sum_{n=0}^{\infty}\frac{Q_n^{(3/2)}(c)}{2n - 1}z^{2n-1},$$

where $Q_n^{(\lambda)}(c)$ is the n-th Gegenbauer polynomial. After multiplying this by

$$z\sqrt{1-2cz^2+z^4} = \sum_{n=0}^{\infty} Q_n^{(-1/2)}(c) z^{2n+1},$$

one arrives at the series $P_{-4}(c,z)$.

3.2. **Elliptic Case 2.** If we take initial conditions $P_{-4}(c) = P_{-3}(c) = P_{-1}(c) = 0$ and $P_{-2}(c) = 1$, we arrive at a generating function defined in terms of another elliptic integral:

$$P_{-2}(c,z) = z\sqrt{1-2cz^2+z^4}\int \frac{1}{(z^4-2cz^2+1)^{3/2}}\,dz.$$

3.3. **Gegenbauer Case 3.** If we take $P_{-1}(c) = 1$ and $P_{-2}(c) = P_{-3}(c) = P_{-4}(c) = 0$ and set

$$P_{-1}(c,z) = \sum_{n\geq 0} P_{-1,n-4} z^n,$$

then we get a solution which, after solving for the integration constant, can be turned into a power series solution,

$$\begin{aligned} P_{-1}(c,z) &= (z\sqrt{1-2cz^2+z^4})\left(\int \frac{2cz^3}{t\sqrt{1-2cz^2+z^4}(z^5-2cz^3+z)}\,dt + C\right) \\ &= \frac{z(c-z^3)}{c^2-1} - \frac{c}{c^2-1} z\sqrt{z^4-2cz^2+1} \\ &= \frac{1}{c^2-1}\left(cz - z^3 - cz\sqrt{z^4-2cz^2+1}\right) \\ &= \frac{1}{c^2-1}\left(cz - z^3 - \sum_{k=0}^{\infty} cQ_n^{(-1/2)}(c) z^{2n+1}\right) \\ &= \frac{1}{c^2-1}\left(cz - z^3 - cz + c^2z^3 - \sum_{k=2}^{\infty} cQ_n^{(-1/2)}(c) z^{2n+1}\right), \end{aligned}$$

where $Q_n^{(-1/2)}(c)$ is the n-th Gegenbauer polynomial. Hence

$$\begin{aligned} P_{-1,-4}(c) &= P_{-1,-3}(c) = P_{-1,-2}(c) = P_{-1,2m}(c) = 0, \\ P_{-1,-1}(c) &= 1, \\ P_{-1,2n-3}(c) &= \frac{-cQ_n(c)}{c^2-1}, \end{aligned}$$

for $m \geq 0$ and $n \geq 2$. The $Q_n^{(-1/2)}(c)$ are known to satisfy the second order differential equation

$$(1-c^2)\frac{d^2}{d^2c} Q_n^{(-1/2)}(c) + n(n-1) Q_n^{(-1/2)}(c) = 0$$

so that the $P_{-1,k} := P_{-1,k}(c)$ satisfy the second order differential equation

$$(c^4-c^2)\frac{d^2}{d^2c} P_{-1,2n-3} + 2c(c^2+1)\frac{d}{dc} P_{-1,2n-3} + (-c^2 n(n-1) - 2) P_{-1,2n-3} = 0$$

for $n \geq 2$.

3.4. **Gegenbauer Case 4.** Next we consider the initial conditions $P_{-1}(c) = 0 = P_{-2}(c) = P_{-4}(c) = 0$ with $P_{-3}(c) = 1$ and set

$$P_{-3}(c, z) = \sum_{n\geq 0} P_{-3,n-4}(c)z^n.$$

Then we get a power series solution

$$\begin{aligned}
P_{-3}(c,z) &= (z\sqrt{1-2cz^2+z^4})\left(\int \frac{2cz^3}{z\sqrt{1-2cz^2+z^4}(z^5-2cz^3+z)}\,dz + C\right)\\
&= \frac{cz(c-z^3)}{c^2-1} - \frac{1}{c^2-1}z\sqrt{z^4-2cz^2+1}\\
&= \frac{1}{c^2-1}\left(c^2z - cz^3 - z\sqrt{z^4-2cz^2+1}\right)\\
&= \frac{1}{c^2-1}\left(c^2z - cz^3 - \sum_{k=0}^{\infty} Q_n^{(-1/2)}(c)z^{2n+1}\right)\\
&= \frac{1}{c^2-1}\left(c^2z - cz^3 - z + cz^3 - \sum_{k=2}^{\infty} Q_n^{(-1/2)}(c)z^{2n+1}\right),
\end{aligned}$$

where $Q_n^{(-1/2)}(c)$ is the n-th Gegenbauer polynomial. Hence

$$\begin{aligned}
P_{-3,-4}(c) &= P_{-3,-2}(c) = P_{-3,-1}(c) = P_{-1,2m}(c) = 0,\\
P_{-3,-3}(c) &= 1,\\
P_{-3,2n-3}(c) &= \frac{-Q_n(c)}{c^2-1},
\end{aligned}$$

for $m \geq 0$ and $n \geq 2$, and hence

$$(c^2-1)\frac{d^2}{d^2c}P_{-3,2n-3} + 4c\frac{d}{dc}P_{-3,2n-3} - (n+1)(n-2)P_{-3,2n-3} = 0$$

for $n \geq 2$ and $P_{-1,2n-3} = cP_{-3,2n-3}$ for $n \geq 2$.

4. Main result

First we give an explicit description of the cocyles contributing to the *even* part of the DJKM algebra.

Proposition 4.0.1 (cf. [Bre94, Prop. 4.2])**.** *Set* $\omega_0 = \overline{t^{-1}\,dt}$. *For* $i, j \in \mathbb{Z}$ *one has*

$$t^i\,d(t^j) = j\delta_{i+j,0}\omega_0 \tag{4.1}$$

and

$$t^{i-1}u\,d(t^{j-1}u) = (\delta_{i+j,-2}(j+1) - 2cj\delta_{i+j,0} + (j-1)\delta_{i+j,2})\,\omega_0. \tag{4.2}$$

Proof. First observe that $2u\,du = d(u^2) = (4t^3 - 4ct)\,dt$. The second congruence then follows from

$$\begin{aligned}
t^{i-1}u\,d(t^{j-1}u) &= (j-1)t^{i+j-3}u^2\,dt + t^{i+j-2}u\,du\\
&= (j-1)t^{i+j-3}(t^4-2ct^2+1)\,dt + 2t^{i+j-2}(t^3-ct)\,dt\\
&= (j-1)(t^{i+j+1} - 2ct^{i+j-1} + t^{i+j-3})\,dt + 2(t^{i+j+1} - ct^{i+j-1})\,dt\\
&= (j+1)t^{i+j+1}\,dt - 2cjt^{i+j-1}\,dt + (j-1)t^{i+j-3}\,dt. \qquad \square
\end{aligned}$$

The map $\sigma : R \to R$ given by $\sigma(t) = t^{-1}$, $\sigma(u) = t^{-2}u$ is an algebra automorphism as $\sigma(u^2) = t^{-4}u^2 = 1 - 2ct^{-2} + t^{-4} = \sigma(1 - 2ct^2 + t^4)$. This descends to a linear map $\sigma : \Omega^1_R/dR$ where

$$\begin{aligned}
\sigma(\overline{t^{-1}\,dt}) &= -\overline{t^{-1}\,dt},\\
\sigma(\overline{t^{-1}u\,dt}) &= \overline{t(t^{-2}u)d(t^{-1})} = -\overline{t^{-3}u\,dt},\\
\sigma(\overline{t^{-2}u\,dt}) &= \overline{t^{2}(t^{-2}u)d(t^{-1})} = -\overline{t^{-2}u\,dt},\\
\sigma(\overline{t^{-3}u\,dt}) &= -\overline{t^{-1}u\,dt},\\
\sigma(\overline{t^{-4}u\,dt}) &= \overline{t^{4}(t^{-2}u)d(t^{-1})} = -\overline{u\,dt} = -\overline{t^{-4}u\,dt},
\end{aligned}$$

whereby the last identity follows from the recursion relation (3.1) with $k = 0$. Setting $\omega_{-k} = \overline{t^{-k}u\,dt}$, $k = 1, 2, 3, 4$, then $\sigma(\omega_{-1}) = -\omega_{-3}$, and $\sigma(\omega_{-l}) = -\omega_{-l}$ for $l = 2, 4$.

Theorem 4.0.2. *Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over the complex numbers with the Killing form $(\,|\,)$ and define $\psi_{ij}(c) \in \Omega^1_R/dR$ by*

(4.3)

$$\psi_{ij}(c) = \begin{cases} \omega_{i+j-2} & \text{for } i+j = 1, 0, -1, -2,\\ P_{-3,i+j-2}(c)(\omega_{-3} + c\omega_{-1}) & \text{for } i+j = 2n-1 \geq 3,\ n \in \mathbb{Z},\\ P_{-3,i+j-2}(c)(c\omega_{-3} + \omega_{-1}) & \text{for } i+j = -2n+1 \leq -3, n \in \mathbb{Z},\\ P_{-4,|i+j|-2}(c)\omega_{-4} + P_{-2,|i+j|-2}(c)\omega_{-2} & \text{for } |i+j| = 2n \geq 2, n \in \mathbb{Z}. \end{cases}$$

The universal central extension of the Date-Jimbo-Kashiwara-Miwa algebra is the $\mathbb{Z}_2$-graded Lie algebra

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}^0 \oplus \widehat{\mathfrak{g}}^1,$$

where

$$\widehat{\mathfrak{g}}^0 = \big(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\big) \oplus \mathbb{C}\omega_0, \qquad \widehat{\mathfrak{g}}^1 = \big(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]u\big) \oplus \mathbb{C}\omega_{-4} \oplus \mathbb{C}\omega_{-3} \oplus \mathbb{C}\omega_{-2} \oplus \mathbb{C}\omega_{-1}$$

with bracket

$$\begin{aligned}
[x \otimes t^i, y \otimes t^j] &= [x, y] \otimes t^{i+j} + \delta_{i+j,0}\,j(x, y)\omega_0,\\
[x \otimes t^{i-1}u, y \otimes t^{j-1}u] &= [x, y] \otimes (t^{i+j+2} - 2ct^{i+j} + t^{i+j-2})\\
&\quad + \left(\delta_{i+j,-2}(j+1) - 2cj\delta_{i+j,0} + (j-1)\delta_{i+j,2}\right)(x, y)\omega_0,\\
[x \otimes t^{i-1}u, y \otimes t^j] &= [x, y]u \otimes t^{i+j-1} + j(x, y)\psi_{ij}(c).
\end{aligned}$$

Proof. The first two equalities follow from Proposition 4.0.1. For the last one we first observe that for $k = i + j - 2 \neq -3$,

$$\begin{aligned}
j\omega_{ij}(c) &= \overline{t^{i-1}u\,d(t^j)} = j\overline{t^{i+j-2}u\,dt}\\
&= j\left(\frac{-2(k-3)\overline{t^{k-4}u\,dt} + 4kc\overline{t^{k-2}u\,dt}}{6+2k}\right),
\end{aligned}$$

where the last equality is derived from (3.1). Then by setting $k = 0, 1, 2, 3, 4, 5$ in (3.1),

$$(6+2k)\overline{t^k u\,dt} = -2(k-3)\overline{t^{k-4}u\,dt} + 4kc\overline{t^{k-2}u\,dt}$$

gives us

$$\begin{aligned}
6\overline{u\,dt} &= 6\overline{t^{-4}u\,dt},\\
8\overline{tu\,dt} &= 4\overline{t^{-3}u\,dt} + 4c\overline{t^{-1}u\,dt},\\
10\overline{t^2u\,dt} &= 2\overline{t^{-2}u\,dt} + 8c\overline{u\,dt},\\
12\overline{t^3u\,dt} &= 12c\overline{tu\,dt},\\
14\overline{t^4u\,dt} &= -2\overline{u\,dt} + 16c\overline{t^2u\,dt},\\
16\overline{t^5u\,dt} &= -4\overline{tu\,dt} + 20c\overline{t^3u\,dt},\\
(6+2k)\overline{t^ku\,dt} &= -2(k-3)\overline{t^{k-4}u\,dt} + 4kc\overline{t^{k-2}u\,dt}.
\end{aligned}$$

Hence, when $i+j-2=k=0,1,2,3,4,5,$

$$\begin{aligned}
\overline{u\,dt} &= \omega_{-4},\\
\overline{tu\,dt} &= \frac{1}{2}\left(\omega_{-3}+c\omega_{-1}\right),\\
\overline{t^2u\,dt} &= \frac{1}{5}\omega_{-2} + \frac{4c}{5}\omega_{-4},\\
\overline{t^3u\,dt} &= \frac{c}{2}\left(\omega_{-3}+c\omega_{-1}\right),\\
\overline{t^4u\,dt} &= -\frac{1}{7}\overline{u\,dt} + \frac{8}{7}c\overline{t^2u\,dt} = -\frac{1}{7}\omega_{-4} + \frac{8}{7}c\left(\frac{1}{5}\omega_{-2} + \frac{4c}{5}\omega_{-4}\right)\\
&= \left(\frac{32c^2-5}{35}\right)\omega_{-4} + \frac{8}{35}c\omega_{-2},\\
\overline{t^5u\,dt} &= -\frac{1}{8}\left(\omega_{-3}+c\omega_{-1}\right) + \frac{5c^2}{8}\left(\omega_{-3}+c\omega_{-1}\right)\\
&= \frac{5c^2-1}{8}\left(\omega_{-3}+c\omega_{-1}\right),\\
\overline{t^ku\,dt} &= \frac{-2(k-3)\overline{t^{k-4}u\,dt} + 4kc\overline{t^{k-2}u\,dt}}{6+2k}.
\end{aligned}$$

Thus by induction using the last equation above for $i+j-2=k=2n-3\geq 1$, $n\in\mathbb{Z}$, we have

$$\omega_{ij}(c) = P_{-3,i+j-2}(c)\left(\omega_{-3}+c\omega_{-1}\right), \tag{4.4}$$

and for $i+j-2=k=2n-2\geq 0$, $n\in\mathbb{Z}$, we have

$$\omega_{ij}(c) = P_{-4,i+j-2}(c)\omega_{-4} + P_{-2,i+j-2}(c)\omega_{-2}. \tag{4.5}$$

Applying σ to (4.4) for $i+j-2=k=2n-3\geq 1$ we obtain

$$\begin{aligned}
j\sigma(\omega_{ij}(c)) = \overline{t^{-i+1}u\,d(t^{-j})} &= -j\overline{t^{-i-j-2}u\,dt}\\
&= j\sigma\left(P_{-3,i+j-2}(c)\left(\omega_{-3}+c\omega_{-1}\right)\right)\\
&= -jP_{-3,i+j-2}(c)\left(\omega_{-1}+c\omega_{-3}\right).
\end{aligned}$$

Hence for $i+j-2=2n-3\geq 1$,

$$\omega_{-i,-j}(c) = \overline{t^{-i-j-2}u\,dt} = P_{-3,i+j-2}(c)\left(\omega_{-1}+c\omega_{-3}\right).$$

Setting $i'=-i$ and $j'=-j$ we get for $i'+j'-2=-k-4=-2n+3\leq -5$,

$$\omega_{i'j'}(c) = \overline{t^{i'+j'-2}u\,dt} = P_{-3,|i'+j'|-2}(c)\left(\omega_{-1}+c\omega_{-3}\right).$$

Similarly, if we apply σ to (4.5) for $i+j=2n\geq 2$, $n\in\mathbb{Z}$, we obtain

$$\begin{aligned}j\sigma(\omega_{ij}(c)) = \overline{t^{-i+1}u\,d(t^{-j})} &= -j\overline{t^{-i-j-2}u\,dt}\\ &= j\sigma\left(P_{-4,i+j-2}(c)\omega_{-4}+P_{-2,i+j-2}(c)\omega_{-2}\right)\\ &= -j\left(P_{-4,i+j-2}(c)\omega_{-4}+P_{-2,i+j-2}(c)\omega_{-2}\right).\end{aligned}$$

Hence for $i+j=2n\geq 2$,

$$\omega_{-i,-j}(c)=\overline{t^{-i-j-2}u\,dt}=P_{-4,i+j-2}(c)\omega_{-4}+P_{-2,i+j-2}(c)\omega_{-2}.$$

Setting $i'=-i$ and $j'=-j$, we get for $i'+j'=-2n\leq -2$:

$$\omega_{i'j'}(c)=\overline{t^{i'+j'-2}u\,dt}=P_{-4,|i'+j'|-2}(c)\omega_{-4}+P_{-2,|i'+j'|-2}(c)\omega_{-2}. \qquad \square$$

One might want to compare the above theorem with the results that M. Bremner obtained for the elliptic and four-point affine Lie algebra cases ([Bre94, Theorem 4.6] and [Bre95, Theorem 3.6] respectively).

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