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REALIZATIONS OF THE FOUR POINT AFFINE LIE ALGEBRA $\mathfrak{sl}(2, R) \oplus (\Omega_R/dR)$

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This paper is dedicated to Thomas J. Enright.

We construct free field realizations of the four point algebra $\mathfrak{sl}(2, R) \oplus (\Omega_R/dR)$, where $R = \mathbb{C}[t, t^{-1}, u \mid u^2 = t^2 - 2bt + 1]$.

1. Introduction

It is well known from Kassel and Loday [1982] and [Kassel 1984] that if R is a commutative algebra and $\mathfrak{g}$ is a simple Lie algebra, both defined over the complex numbers, then the universal central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g} \otimes R$ is the vector space $(\mathfrak{g} \otimes R) \oplus \Omega^1_R/dR$, where Ω^1_R/dR is the space of Kähler differentials modulo exact forms; see [Kassel 1984]. The vector space $\hat{\mathfrak{g}}$ is made into a Lie algebra by defining

$$[x \otimes f, y \otimes g] := [xy] \otimes fg + (x, y)\overline{f\,dg} \quad \text{with} \quad [x \otimes f, \omega] = 0$$

for $x, y \in \mathfrak{g}$, $f, g \in R$, and $\omega \in \Omega^1_R/dR$, where $(\cdot\,,\cdot)$ denotes the Killing form on $\mathfrak{g}$. Here $\bar{a}$ denotes the image of $a \in \Omega^1_R$ in the quotient Ω^1_R/dR. A somewhat vague (due to the underspecified R and the resulting imprecisely described basis of Ω_R/dR) but natural question arises as to whether there exists free field or Wakimoto type realizations of these algebras. The answer is well known from the work of M. Wakimoto [1986] when R is the ring of Laurent polynomials in one variable. We answer this question below when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $R = \mathbb{C}[t, t^{-1}, u \mid u^2 = t^2 - 2bt + 1]$ is the four point algebra. H. P. Jakobsen and V. Kac [1985] have related work on $\mathfrak{sl}(2, R)$, and we review the relevant material in Section 7.

Before we begin, we mention a little of the genesis of four point algebras. In Kazhdan and Lustzig's explicit study [1993; 1991] of the tensor structure of modules for affine Lie algebras, the ring of functions regular everywhere but a finite number of points appears naturally. M. Bremner named this algebra the *n-point algebra*. In particular, consider now the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with coordinate function s, and fix four distinct points a_1, a_2, a_3, a_4 on it. Let R denote the ring of rational functions with poles only in the set $\{a_1, a_2, a_3, a_4\}$. The

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automorphism group $\mathrm{PGL}_2(\mathbb{C})$ of $\mathbb{C}(s)$ is simply 3-transitive and R is a subring of $\mathbb{C}(s)$, so that R is isomorphic to the ring of rational functions with poles at $\{\infty, 0, 1, a_4\}$. Motivated by this isomorphism, one sets $a = a_4$, and here the 4-*point ring* is $R = R_a = \mathbb{C}[s, s^{-1}, (s-1)^{-1}, (s-a)^{-1}]$, where $a \in \mathbb{C}\backslash\{0, 1\}$. Let $S := S_b = \mathbb{C}[t, t^{-1}, u]$, where $u^2 = t^2 - 2bt + 1$ with b a complex number not equal to ± 1. Then M. Bremner has shown us that $R_a \cong S_b$. The latter, being $\mathbb{Z}_2$-graded, is a cousin to super Lie algebras and is thus more immediately amenable to the theatrics of conformal field theory; we therefore choose to work with S_b. Moreover Bremner has explicitly described the universal central extension of $\mathfrak{g} \otimes R$ in terms of ultraspherical (Gegenbauer) polynomials, where R is the four point algebra. His description is recapitulated in what follows. Our main result Theorem 6.1 provides a natural free-field realization in terms of a β-γ-system and the oscillator algebra of the four point affine Lie algebra when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Just as in the case of intermediate Wakimoto modules defined in [Cox and Futorny 2006], there are two realizations depending on two normal orderings.

2. The 4-point ring $\mathbb{C}[t, t^{-1}, u \mid u^2 = t^2 - 2bt + 1]$.

Recall $R = R_a = \mathbb{C}[s, s^{-1}, (s-1)^{-1}, (s-a)^{-1}]$, where $a \in \mathbb{C}\backslash\{0, 1\}$, and $S := S_b = \mathbb{C}[t, t^{-1}, u]$, where $u^2 = t^2 - 2bt + 1$ with b a complex number not equal to ± 1.

Proposition 2.1 [Bremner 1995, Proposition 1.1]. *If $b = (a+1)/(a-1)$ with $a \in \mathbb{C}\backslash\{0, 1\}$, then $R_a \cong S_b$ and $b \neq \pm 1$.*

Proposition 2.2 [Bremner 1995, Proposition 1.2]. *For $a \in \mathbb{C}\backslash\{0, 1\}$, the functions*

$$i(s) := s, \qquad p(s) := \frac{s-a}{s-1}, \qquad q(s) := \frac{a(s-1)}{s-a}. \tag{2-1}$$

uniquely determine automorphisms of R_a, and $\{i, p, q, p \circ q\}$ is a subgroup of $\mathrm{Aut}(R_a)$.

Via the isomorphism $R_a \cong S_b$, the automorphisms above are transformed into automorphisms on S_b where they take the form

$$p(t) := t^{-1}, \quad p(u) = t^{-1}u, \quad q(t) := t^{-1}, \quad q(u) = -t^{-1}u. \tag{2-2}$$

3. A triangular decomposition of the 4-point loop algebras $\mathfrak{g} \otimes R$

From now on, we identify R_a with S_b which has a basis $\{t^i, t^i u, i\} \in \mathbb{Z}$. One can decompose $R = R^0 \oplus R^1$, where

$$R^0 = \mathbb{C}[t^{\pm 1}] = \{r \in R \mid pq(r) = r\},$$
$$R^1 = \mathbb{C}[t^{\pm 1}]u = \{r \in R \mid pq(r) = -r\}$$

are the eigenspaces of $p \circ q$. From now on, $\mathfrak{g}$ will denote a simple Lie algebra over $\mathbb{C}$ with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$; then the 4-*point loop algebra* $L(\mathfrak{g}) := \mathfrak{g} \otimes R$ has a corresponding $\mathbb{Z}/2\mathbb{Z}$-grading $L(\mathfrak{g})^i := \mathfrak{g} \otimes R^i$ for $i = 0, 1$. However the degree of t does not render $L(\mathfrak{g})$ a $\mathbb{Z}$-graded Lie algebra. This leads us to the following notion.

Suppose I is an additive subgroup of the rational numbers $\mathbb{P}$ and $\mathscr{A}$ is a $\mathbb{C}$-algebra such that $\mathscr{A} = \oplus_{i \in I} \mathscr{A}_i$. Suppose also there exists a fixed $l \in \mathbb{N}$ such that

$$\mathscr{A}_i \mathscr{A}_j \subset \oplus_{|k-(i+j)| \leq l} \mathscr{A}_k$$

for all $i, j \in \mathbb{Z}$. Then $\mathscr{A}$ is said to be an l-*quasigraded algebra*. For $0 \neq x \in \mathscr{A}_i$, one says that x is *homogeneous of degree* i and writes $\deg x = i$.

For example, R has the structure of a 1-quasigraded algebra in which $I = \frac{1}{2}\mathbb{Z}$, $\deg t^i = i$, and $\deg t^i u = i + 1/2$.

A *weak triangular decomposition* of a Lie algebra $\mathfrak{l}$ is a triple $(\mathfrak{H}, \mathfrak{l}_+, \sigma)$ satisfying

(1) $\mathfrak{H}$ and $\mathfrak{l}_+$ are subalgebras of $\mathfrak{l}$,

(2) $\mathfrak{H}$ is abelian and $[\mathfrak{H}, \mathfrak{l}_+] \subset \mathfrak{l}_+$,

(3) σ is an antiautomorphism of $\mathfrak{l}$ of order 2 which is the identity on $\mathfrak{h}$, and

(4) $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{H} \oplus \sigma(\mathfrak{l}_+)$.

We denote $\sigma(\mathfrak{l}_+)$ by $\mathfrak{l}_-$.

Theorem 3.1 [Bremner 1995, Theorem 2.1]. *The* 4-*point loop algebra* $L(\mathfrak{g})$ *is* 1-*quasigraded Lie algebra such that* $\deg(x \otimes f) = \deg f$ *when* f *is homogeneous. Set* $R_+ = \mathbb{C}(1+u) \oplus \mathbb{C}[t, u]t$ *and* $R_- = p(R_+)$. *Then* $L(\mathfrak{g})$ *has a weak triangular decomposition given by*

$$L(\mathfrak{g})_\pm = \mathfrak{g} \otimes R_\pm, \quad \mathscr{H} := \mathfrak{h} \otimes \mathbb{C}.$$

4. The universal central extension of $\mathfrak{g} \otimes R$

Recall how the universal central extension of $L(\mathfrak{g})$ was realized in the introduction.

Theorem 4.1 [Bremner 1995, Theorem 3.3]. *The space* $\omega \in \Omega^1_R/dR$ *has basis*

$$\omega_0 := \overline{t^{-1}dt}, \qquad \omega_- := \overline{t^{-2}u\,dt}, \qquad \omega_+ := \overline{t^{-1}u\,dt}.$$

The automorphisms p, q descend to the space Ω^1_R/dR where they take the form

$$\begin{aligned} p(\omega_0) &= -\omega_0, & p(\omega_-) &= -\omega_+, & p(\omega_+) &= -\omega_-, \\ q(\omega_0) &= -\omega_0, & q(\omega_-) &= \omega_+, & q(\omega_+) &= \omega_-. \end{aligned}$$

M. Bremner gave a Fourier mode description of the relations satisfied by the basis elements $x \otimes t^n$, $x \otimes t^n u$, ω_0, and $\omega_\pm$ of $\hat{\mathfrak{g}}$. First we recall the *ultraspherical*

(Gegenbauer) polynomials $P_k^\lambda(b)$, which are defined to be the coefficient of t^k in the Taylor series of $P^{-\lambda}(b,z) = (1-2bt+t^2)^{-\lambda}$. We will only need the case where $\lambda = -1/2$, and we will simplify the notation by setting $P_k(b) = P_k^{-1/2}(b)$ and $P = P^{-1/2}(b,z)$. Define for $b \neq \pm 1$

$$Q_k(b) := -\frac{P_{k+2}(b)}{b^2-1},$$

which is a polynomial in b. The final result of [Bremner 1995] is

Theorem 4.2 [Bremner 1995, Theorem 3.6]. *The* 4*-point affine Lie algebra* $\hat{\mathfrak{g}}$ *has a* $\mathbb{Z}/2\mathbb{Z}$*-grading where*

$$\hat{\mathfrak{g}}^0 = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\omega_0, \quad \hat{\mathfrak{g}}^1 = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]u \oplus \mathbb{C}\omega_- \oplus \mathbb{C}\omega_+.$$

For $x, y \in \mathfrak{g}$, *the commutation relations defining* $\hat{\mathfrak{g}}$ *are*

$$\begin{aligned}
[x\otimes t^i, y\otimes t^j] &= [xy]\otimes t^{i+j} + \delta_{i+j,0}\, j\omega_0,\\
[x\otimes t^{i-\frac12}u, y\otimes t^{j-\frac12}u] &= [xy]\otimes \left(t^{i+j-1} - 2bt^{i+j} + t^{i+j+1}\right)\\
&\quad + (x,y)\omega_0\left(-2jb\delta_{i+j,0} + \tfrac12(j-i)(\delta_{i+j,-1}+\delta_{i+j,1})\right),\\
[x\otimes t^{i-\frac12}u, y\otimes t^j] &= [xy]\otimes t^{i+j-\frac12}u\\
&\quad + (x,y)j\Big(Q_{i+j-\frac32}(b)(b\omega_+ + \omega_-)\delta_{i+j\geq\frac32})\\
&\qquad + \omega_\pm\delta_{i+j,\pm\frac12} + Q_{-i-j-\frac32}(b)(\omega_+ + b\omega_0)\delta_{i+j\leq-\frac32}\Big).
\end{aligned}$$

They hold whenever the indices i *and* j *raise* t *in the commutators to an integer power. Here* $\delta_{l\geq a}$ *is the Heaviside function that is zero for* $l<a$ *and* 1 *otherwise. In addition, the elements* $\omega_0, \omega_\pm$ *are central.*

4.1. ***Formal distributions.*** More notation will simplify some later arguments. It roughly follows [Kac 1998] and [Matsuo and Nagatomo 1999]: The *formal delta function* $\delta(z/w)$ is the formal distribution

$$\delta(z/w) = z^{-1}\sum_{n\in\mathbb{Z}}\left(\frac{z}{w}\right)^n.$$

For any sequence of elements $\{a_m\}_{m\in\mathbb{Z}}$ in the ring $\mathrm{End}(V)$, where V is a vector space, the formal distribution

$$a(z) := \sum_{m\in\mathbb{Z}} a_m z^{-m-1}$$

is called a *field* if, for any $v \in V$, we have $a_m v = 0$ for $m \gg 0$. If $a(z)$ is a field, then we set

$$a(z)_- := \sum_{m\geq 0} a_m z^{-m-1} \quad \text{and} \quad a(z)_+ := \sum_{m<0} a_m z^{-m-1}. \tag{4-3}$$

The *normal ordered product* of two distributions $a(z)$ and $b(w)$ (and their coefficients) is defined by

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} :a_m b_n: z^{-m-1}w^{-n-1} = :a(z)b(w): = a(z)_+ b(w) + b(w)a(z)_-. \tag{4-4}$$

While $:a^1(z_1)\cdots a^m(z_m):$ is always defined as a formal series, we will only define $:a(z)b(z):$ as $\lim_{w\to z} :a(z)b(w):$ for certain pairs $(a(z), b(w))$.

Then one defines recursively

$$:a^1(z_1)\cdots a^k(z_k): = :a^1(z_1)\left(:a^2(z_2)\left(:\cdots :a^{k-1}(z_{k-1})a^k(z_k):\right)\cdots :\right):,$$

while the normal ordered product

$$:a^1(z)\cdots a^k(z): = \lim_{z_1,z_2,\cdots,z_k\to z} :a^1(z_1)\left(:a^2(z_2)\left(:\cdots :a^{k-1}(z_{k-1})a^k(z_k):\right)\cdots\right):$$

will only be defined for certain k-tuples $(a^1,\dots,a^k)$. Let

$$\lfloor ab\rfloor = a(z)b(w) - :a(z)b(w): = [a(z)_-, b(w)], \tag{4-5}$$

(half of $[a(z), b(w)]$) denote the *contraction* of any two formal distributions $a(z)$ and $b(w)$.

Theorem 4.3 (Wick's theorem; see [Bogoliubov and Shirkov 1983], [Huang 1998] or [Kac 1998]). *Let $a^i(z)$ and $b^j(z)$ be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$, *satisfying*:

(1) $[\lfloor a^i(z)b^j(w)\rfloor, c^k(x)_\pm] = [\lfloor a^i b^j\rfloor, c^k(x)_\pm] = 0$ *for all* i, j, k *and* $c^k(x) = a^k(z)$ *or* $c^k(x) = b^k(w)$.

(2) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ *for all* i *and* j.

(3) *The products*

$$\lfloor a^{i_1}b^{j_1}\rfloor\cdots\lfloor a^{i_s}b^{i_s}\rfloor :a^1(z)\cdots a^M(z)b^1(w)\cdots b^N(w):_{(i_1,\dots,i_s;j_1,\dots,j_s)}$$

have coefficients in $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$ *for all subsets* $\{i_1,\dots,i_s\}\subset\{1,\dots,M\}$ *and* $\{j_1,\dots,j_s\}\subset\{1,\cdots N\}$. *Here the subscript* $(i_1,\dots,i_s; j_1,\dots,j_s)$ *means those factors* $a^i(z)$, $b^j(w)$ *with indices* $i\in\{i_1,\dots,i_s\}$, $j\in\{j_1,\dots,j_s\}$ *are to be omitted from the product* $:a^1\cdots a^M b^1\cdots b^N:$ *and when* $s=0$ *we do not omit any factors.*

Then

$$:a^1(z)\cdots a^M(z)\colon :b^1(w)\cdots b^N(w)\colon = \sum_{\substack{s=0,\dots,\min(M,N)\\ i_1<\cdots<i_s\\ j_1\neq\cdots\neq j_s}} \lfloor a^{i_1}b^{j_1}\rfloor\cdots\lfloor a^{i_s}b^{i_s}\rfloor :a^1(z)\cdots a^M(z)b^1(w)\cdots b^N(w)\colon_{(i_1,\dots,i_s;j_1,\dots,j_s)}.$$

For $m = i-\frac{1}{2}$, $i\in\mathbb{Z}+\frac{1}{2}$ and $x\in\mathfrak{g}$, define

$$x_{m+\frac{1}{2}} = x\otimes t^{i-\frac{1}{2}}u \quad \text{and} \quad x_m := x\otimes t^m.$$

Motivated by conformal field theory we set

$$x^1(z) := \sum_{m\in\mathbb{Z}} x_{m+\frac{1}{2}} z^{-m-1} \qquad \text{and} \qquad x(z) := \sum_{m\in\mathbb{Z}} x_m z^{-m-1}.$$

Then the relations in Theorem 4.2 can be rewritten as

$$[x(z), y(w)] = [xy](w)\delta\big(\tfrac{z}{w}\big) - (x,y)\omega_0\partial_w\delta\big(\tfrac{z}{w}\big),$$

$$[x^1(z), y^1(w)] \\ = (w^2-2bw+1)\big([x,y](w)\delta\big(\tfrac{z}{w}\big) - (x,y)\omega_0\partial_w\delta\big(\tfrac{z}{w}\big)\big) + (x,y)\omega_0(b-w)\delta\big(\tfrac{z}{w}\big),$$

$$\begin{aligned}[x^1(z), y(w)] &= [x,y]^1(w)\delta\big(\tfrac{z}{w}\big) \\ &\quad + (x,y)\frac{b-w}{1-b^2}\left(\frac{\omega_+b+\omega_-}{w(1-2bw^{-1}+w^{-2})^{1/2}} + \frac{\omega_++\omega_-b}{(1-2bw+w^2)^{1/2}}\right)\delta\big(\tfrac{z}{w}\big) \\ &\quad - (x,y)\left(\left(\frac{\omega_+b+\omega_-}{1-b^2}\right)w(1-2bw^{-1}+w^{-2})^{1/2} \right. \\ &\qquad \left. + \left(\frac{\omega_++\omega_-b}{1-b^2}\right)(1-2bw+w^2)^{1/2}\right)\partial_w\delta\big(\tfrac{z}{w}\big),\end{aligned}$$

$$\begin{aligned}[x(z), y^1(w)] &= [x,y]^1(w)\delta\big(\tfrac{w}{z}\big) - (x,y)\left(\left(\frac{\omega_+b+\omega_-}{1-b^2}\right)w(1-2bw^{-1}+w^{-2})^{1/2} \right. \\ &\qquad \left. + \left(\frac{\omega_++\omega_-b}{1-b^2}\right)(1-2bw+w^2)^{1/2}\right)\partial_w\delta\big(\tfrac{w}{z}\big).\end{aligned}$$

One interprets in the last summand the expansion of $(1-2bw+w^2)^{1/2}$ as a Taylor series in w about 0. Similarly $(1-2bw^{-1}+w^{-2})^{1/2}$ is interpreted as a Taylor series in w^{-1}. It is somewhat amusing to see that the variants of the ultraspherical polynomials $Q_k(b)$ disappear in the formal power series formulation above.

5. Oscillator algebras

5.1. *$\beta - \gamma$ system.* Let $\hat{\mathfrak{a}}$ be the infinite dimensional oscillator algebra with generators $a_n, a_n^*, a_n^1, a_n^{1*}$ for $n \in \mathbb{Z}$ that, together with $\mathbf{1}$, satisfies the relations

$$
\begin{aligned}
&[a_n, a_m] = [a_m, a_n^1] = [a_m, a_n^{1*}] = [a_n^*, a_m^*] = [a_n^*, a_m^1] = [a_n^*, a_m^{1*}] = 0,\\
&[a_n^1, a_m^1] = [a_n^{1*}, a_m^{1*}] = 0 = [\mathfrak{a}, \mathbf{1}],\\
&[a_n, a_m^*] = \delta_{m+n,0}\mathbf{1} = [a_n^1, a_m^{1*}].
\end{aligned}
$$

For $c = a, a^1$ and respectively $X = x, x^1$ with $r = 0$ or $r = 1$, we define the representation $\rho : \hat{\mathfrak{a}} \to \mathfrak{gl}(\mathbb{C}[x_n, x_n^1 \mid n \in \mathbb{Z}])$ by

$$
\rho_r(c_m) := \begin{cases} \partial/\partial X_m & \text{if} \quad m \geq 0 \ \text{ and } \ r = 0,\\ X_m & \text{otherwise,} \end{cases}
$$

$$
\rho_r(c_m^*) := \begin{cases} X_{-m} & \text{if} \quad m \leq 0 \ \text{ and } \ r = 0,\\ -\partial/\partial X_{-m} & \text{otherwise,} \end{cases}
$$

and $\rho_r(\mathbf{1}) = 1$. These two representations can be constructed using induction: For $r = 0$ the representation ρ_0 is the $\hat{\mathfrak{a}}$-module generated by $1 =: |0\rangle$, where

$$
a_m|0\rangle = a_m^1|0\rangle = 0, \quad m \geq 0 \quad \text{and} \quad a_m^*|0\rangle = a_m^{1*}|0\rangle = 0, \quad m > 0.
$$

For $r = 1$ the representation ρ_1 is the $\hat{\mathfrak{a}}$-module generated by $1 =: |0\rangle$, where

$$
a_m^*|0\rangle = a_m^{1*}|0\rangle = 0, \quad m \in \mathbb{Z}.
$$

If we write

$$
\alpha(z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad \alpha^*(z) := \sum_{n\in\mathbb{Z}} a_n^* z^{-n},
$$

$$
\alpha^1(z) := \sum_{n\in\mathbb{Z}} a_n^1 z^{-n-1}, \qquad \alpha^{1*}(z) := \sum_{n\in\mathbb{Z}} a_n^{1*} z^{-n},
$$

then

$$
\begin{aligned}
&[\alpha(z), \alpha(w)] = [\alpha^*(z), \alpha^*(w)] = [\alpha^1(z), \alpha^1(w)] = [\alpha^{1*}(z), \alpha^{1*}(w)] = 0,\\
&[\alpha(z), \alpha^*(w)] = [\alpha^1(z), \alpha^{1*}(w)] = \mathbf{1}\delta(z/w).
\end{aligned}
$$

We observe that $\rho_1(\alpha(z))$ and $\rho_1(\alpha^1(z))$ are not fields, whereas $\rho_r(\alpha^*(z))$ and $\rho_r(\alpha^{1*}(z))$ are always fields. Corresponding to these two representations there are two possible normal orderings. For $r = 0$ we use the usual normal ordering given by Equation (4-3) and for $r = 1$ we define the *natural normal ordering* to be

$$
\begin{aligned}
&\alpha(z)_+ = \alpha(z), && \alpha(z)_- = 0, && \alpha^*(z)_+ = 0, && \alpha^*(z)_- = \alpha^*(z),\\
&\alpha^1(z)_+ = \alpha^1(z), && \alpha^1(z)_- = 0, && \alpha^{1*}(z)_+ = 0, && \alpha^{1*}(z)_- = \alpha^{1*}(z).
\end{aligned}
$$

This means in particular that for $r=0$ we get

$$\lfloor \alpha\alpha^* \rfloor = \delta_-(z/w) = \iota_{z,w}\Big(\frac{1}{z-w}\Big),$$
$$\lfloor \alpha^*\alpha \rfloor = -\delta_+(w/z) = \iota_{z,w}\Big(\frac{1}{w-z}\Big),$$

and, for $r=1$,

$$\lfloor \alpha\alpha^* \rfloor = [\alpha(z)_-, \alpha^*(w)] = 0,$$
$$\lfloor \alpha^*\alpha \rfloor = [\alpha^*(z)_-, \alpha(w)] = -\delta(w/z).$$

Similar results hold for α^1. Notice that in both cases we have

$$\lfloor \alpha(z)\alpha^*(w) \rfloor - \lfloor \alpha^*(w)\alpha(z) \rfloor = \delta(z/w).$$

We will also need the following two results.

Theorem 5.1 (Taylor's theorem, [Kac 1998, 2.4.3]). *Let $a(z)$ be a formal distribution. Then in the region $|z-w|<|w|$,*

$$a(z) = \sum_{j=0}^{\infty} \partial_w^{(j)} a(w)(z-w)^j. \tag{5-6}$$

Theorem 5.2 [Kac 1998, Theorem 2.3.2]. *Define* $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_n, x_n^1 \mid n \in \mathbb{Z}]$ *and* $\mathbb{C}[\mathbf{y}] = C[y_m, y_m^1 \mid m \in \mathbb{N}^*]$. *Let $a(z)$ and $b(z)$ be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$, *where we are using the usual normal ordering. Then, for* $c^j(w) \in \mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])[\![w, w^{-1}]\!]$, *these two are equivalent*:

(i) $[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z-w) c^j(w).$

(ii) $\lfloor ab \rfloor = \sum_{j=0}^{N-1} \iota_{z,w}\Big(\frac{1}{(z-w)^{j+1}}\Big) c^j(w).$

In other words, the singular part of the *operator product expansion*

$$\lfloor ab \rfloor = \sum_{j=0}^{N-1} \iota_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right) c^j(w)$$

completely determines the bracket of mutually local formal distributions $a(z)$ and $b(w)$. One writes

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}.$$

5.2. ***The 4-point Heisenberg algebra.*** The Cartan subalgebra $\mathfrak{h}$ tensored with R generates a subalgebra of $\hat{\mathfrak{g}}$ which is an extension of an oscillator algebra. This extension motivates defining a Lie algebra with generators b_m, b^1_m, $m\in\mathbb{Z}$, $\mathbf{1}_0$, $\mathbf{1}_\pm$ and with relations

$$[b_m, b_n] = 2n\,\delta_{m+n,0}\mathbf{1}_0, \tag{5-7}$$

$$[b^1_m, b^1_n] = (n-m)(\delta_{m+n,0} - 2b\,\delta_{m+n,-1} + \delta_{m+n,-2})\mathbf{1}_0, \tag{5-8}$$

$$[b^1_m, b_n] = 2n\big(Q_{m+n-1}(b)(b\mathbf{1}_+ + \mathbf{1}_-)\delta_{m+n\geq 1} + \mathbf{1}_+\delta_{m+n,0} + \mathbf{1}_-\delta_{m+n,-1} + Q_{-m-n-2}(b)(b\mathbf{1}_- + \mathbf{1}_+)\delta_{m+n\leq -2}\big), \tag{5-9}$$

$$[b_m, \mathbf{1}_0] = [b^1_m, \mathbf{1}_0] = [b_m, \mathbf{1}_\pm] = [b^1_m, \mathbf{1}_\pm] = 0. \tag{5-10}$$

W call this the 4-*point (affine) Heisenberg algebra* and denote it by $\hat{\mathfrak{h}}_4$. This algebra has an involutive antiautomorphism induced by $-p$:

$$\sigma(b_n) = -b_{-n},\quad \sigma(b^1_n) = -b^1_{-n-1},\quad \sigma(\mathbf{1}_0) = \mathbf{1}_0,\quad \sigma(\mathbf{1}_\pm) = \mathbf{1}_\pm.$$

The antiautomorphisms $-q$ and $-p\circ q$ also induce antiautomorphisms on $\hat{\mathfrak{h}}_4$, which we leave to the reader to formulate.

If we introduce the formal distributions

$$\beta(z) := \sum_{n\in\mathbb{Z}} b_n z^{-n-1} \quad\text{and}\quad \beta^1(z) := \sum_{n\in\mathbb{Z}} b^1_n z^{-n-1}. \tag{5-11}$$

then using calculations done earlier for the 4-point Lie algebra we can see that the relations above can be rewritten in the form

$$\begin{aligned}
[\beta(z), \beta(w)] &= 2\mathbf{1}_0\partial_z\delta(z/w) = -2\mathbf{1}_0\partial_w\delta(z/w),\\
[\beta^1(z), \beta^1(w)] &= \big(\mathbf{1}_0\partial_z((z^2-2bz+1)\delta(z/w)) + 2\mathbf{1}_0(b-z)\delta(z/w)\big)\\
&= 2\big(\mathbf{1}_0(b-w)\delta(z/w) - (w^2-2bw+1)\mathbf{1}_0\partial_w(\delta(z/w)\big),\\
[\beta^1(z), \beta(w)] &= 2\big((\mathbf{1}_+b+\mathbf{1}_-)\,z(1-2bz^{-1}+z^{-2})^{1/2} + (\mathbf{1}_+ + \mathbf{1}_-b)\,(1-2bz+z^2)^{1/2}\big)\\
&\qquad\times(1-b^2)^{-1}\partial_z\delta(z/w)\\
&= 2(b-w)\left(\frac{\mathbf{1}_+b+\mathbf{1}_-}{w(1-2bw^{-1}+w^{-2})^{1/2}} + \frac{\mathbf{1}_+ + \mathbf{1}_-b}{(1-2bw+w^2)^{1/2}}\right)(1-b^2)^{-1}\delta(z/w)\\
&\quad - 2\big((\mathbf{1}_+b+\mathbf{1}_-)\,w(1-2bw^{-1}+w^{-2})^{1/2} + (\mathbf{1}_+ + \mathbf{1}_-b)(1-2bw+w^2)^{1/2}\big)\\
&\qquad\times(1-b^2)^{-1}\partial_w\delta(z/w),\\
[\beta(z), \beta^1(w)] &= -2\big((\mathbf{1}_+b+\mathbf{1}_-)\,w(1-2bw^{-1}+w^{-2})^{1/2} +\\
&\qquad (\mathbf{1}_+ + \mathbf{1}_-b)\,(1-2bw+w^2)^{1/2}\big)(1-b^2)^{-1}\partial_w\delta(z/w).
\end{aligned}$$

Setting

$$\hat{\mathfrak{h}}_4^{\pm} := \sum_{n \gtrless 0} \left(\mathbb{C}b_n + \mathbb{C}b_n^1\right) \quad \text{and} \quad \hat{\mathfrak{h}}_4^0 := \mathbb{C}\mathbf{1}_- \oplus \mathbb{C}\mathbf{1}_0 \oplus \mathbb{C}\mathbf{1}_+ \oplus \mathbb{C}b_0 \oplus \mathbb{C}b_0^1,$$

we introduce a Borel type subalgebra $\hat{\mathfrak{b}}_4 = \hat{\mathfrak{h}}_4^+ \oplus \hat{\mathfrak{h}}_4^0$. The defining relations above show that $\hat{\mathfrak{b}}_4$ is a subalgebra.

Lemma 5.3. *Let $\mathcal{V} = \mathbb{C}\mathbf{v}_0 \oplus \mathbb{C}\mathbf{v}_1$ be a two-dimensional representation of $\hat{\mathfrak{b}}_4$, where $\mathfrak{h}_4^+ \mathbf{v}_i = 0$ for $i = 0, 1$. Suppose $\lambda, \mu, \nu, \varkappa, \chi_\pm, \kappa_0 \in \mathbb{C}$ are such that*

$$\begin{aligned} b_0\mathbf{v}_0 &= \lambda\mathbf{v}_0, & b_0^1\mathbf{v}_0 &= \mu\mathbf{v}_0 + \nu\mathbf{v}_1, & \mathbf{1}_\pm\mathbf{v}_i &= \chi_\pm\mathbf{v}_i, \\ b_0\mathbf{v}_1 &= \lambda\mathbf{v}_1, & b_0^1\mathbf{v}_1 &= \varkappa\mathbf{v}_0 + \mu\mathbf{v}_1, & \mathbf{1}_0\mathbf{v}_i &= \kappa_0\mathbf{v}_i, \quad i = 0, 1. \end{aligned}$$

Then $b\chi_+ + \chi_- = 0$.

Proof. Since b_m acts by scalar multiplication for $m, n \geq 0$, the first defining relation (5-7) is satisfied for $m, n \geq 0$. The second relation (5-8) is also satisfied as the right hand side is zero if $m \geq 0$, $n \geq 0$. If $n = 0$, then since b_0 acts by a scalar, the relation (5-9) leads to no condition on $\lambda, \mu, \nu, \varkappa, \chi_\pm, \kappa_0 \in \mathfrak{h}_4^0$. However, If $m \geq 0$ and $n > 0$, the third relation becomes

$$0 = b_m^1 b_n\mathbf{v} - b_n b_m^1\mathbf{v} = [b_m^1, b_n]\mathbf{v} = 2n\left(Q_{m+n-1}(b)(b\mathbf{1}_+ + \mathbf{1}_-)\right)\mathbf{v}. \qquad \square$$

Let B_0^1 denote the linear transformation on $\mathcal{V}$ that agrees with the action of b_0^1. If we define the notion of a $\mathfrak{b}_4$-submodule as in [Sheĭnman 1995, Definition 1.2], then $\mathcal{V}$ above is an irreducible $\mathfrak{b}_4$-module if $\varkappa\nu \neq 0$ that is, if $\det B_0^1 \neq \mu^2$. Later we will induce up from $\mathcal{V}$ and the resulting representation (whose irreducibility we will study in future work) that the four point affine algebra would not have a chance of being irreducible if $\mathcal{V}$ were not irreducible (in the sense of Sheĭnman) itself.

Proposition 5.4. *Let $\mathcal{M} = \mathbb{C}[y_{-n}, y_{-m}^1 \mid m, n \in \mathbb{N}^*] \otimes \mathcal{V}$. Then for $k > 0$ the relations*

$$\rho(b_{-k}) = -y_{-k},$$

$$\rho(b_{-k}^1) = -y_{-k}^1 - 2(b\chi_- + \chi_+)\sum_{q<-k}(q+k)Q_{-q-2}(b)\partial_{y_{q+k}},$$

$$\rho(b_k) = 2k\Big(\kappa_0\,\partial_{y_{-k}} + \chi_+\,\partial_{y_{-k}^1} + \chi_+\,\partial_{y_{-1-k}^1} + (b\chi_- + \chi_+)\sum_{q\leq -2}Q_{-q-2}(b)\partial_{y_{q-k}^1}\Big),$$

$$\begin{aligned}\rho(b_k^1) = 2\Big(k\,\chi_+\partial_{y_{-k}} + (k+1)\chi_-\,\partial_{y_{-1-k}} + (b\chi_- + \chi_+)\sum_{q\leq -2}(k-q)Q_{-q-2}(b)\partial_{y_{q-k}}\Big)\\ + \big((2k\partial_{y_{-k}^1} - (2k+1)b\,\partial_{y_{-1-k}^1} + 2(k+1)\partial_{y_{-k-2}^1})\kappa_0\big),\end{aligned}$$

$$\rho(b_0^1) = 2\Big(\chi_-\,\partial_{y_{-1}} - (b\chi_- + \chi_+)\sum_{q\leq -2}q\,Q_{-q-2}(b)\partial_{y_q}\Big) + \big(-b\,\partial_{y_{-1}^1} + 2\partial_{y_{-2}^1}\big)\kappa_0 + B_0^1,$$

$$\rho(b_0) = \lambda, \qquad \rho(\mathbf{1}_0) = \kappa_0, \qquad \rho(\mathbf{1}_\pm) = \chi_\pm$$

define a representation of $\mathfrak{b}_4$.

Proof. We begin with the first relation. For this we may assume $m \geq 0$ and $p > 0$:

$$[\rho(b_m), \rho(b_{-p})] = \Big[2m\Big(\kappa_0\,\partial_{y_{-m}} + \chi_+\,\partial_{y^1_{-m}} + \chi_-\,\partial_{y^1_{-1-m}} + (b\chi_- + \chi_+)\sum_{q\leq -2} Q_{-q-2}(b)\partial_{y^1_{q-m}}\Big), -y_{-p}\Big] = -2m\,\delta_{m-p,0}\rho(\mathbf{1}_0).$$

Now consider the third relation. For $m > 0$, $n > 0$ we have

$$\begin{aligned}
[\rho(b^1_{-m}), \rho(b_{-n})] &= \Big[-y^1_{-m} - 2(b\chi_- + \chi_+)\sum_{q<-m}(q+m)Q_{-q-2}(b)\partial_{y_{q+m}}, -y_{-n}\Big] \\
&= -2n(b\chi_- + \chi_+)Q_{m+n-2}(b) \\
&= -2n\left(\delta_{-m-n,0}\chi_+ + \chi_-\,\delta_{-m-n-1,0} + (b\chi_- + \chi_+)Q_{m+n-2}(b)\delta_{-m-n\leq -2}\right) \\
&= \rho([b^1_{-m}, b_{-n}]),
\end{aligned}$$

which agrees with (5-9). Next we have (with $m \geq 0$ and $n > 0$)

$$\begin{aligned}
[\rho(b^1_m), \rho(b_n)] &= 2\Big[\Big(m\,\partial_{y_{-m}}\chi_+ + (m+1)\chi_-\,\partial_{y_{-1-m}} + (b\chi_- + \chi_+)\sum_{q\leq -2}(m-q)Q_{-q-2}(b)\partial_{y_{q-m}}\Big) \\
&\qquad + \big((2m\partial_{y^1_{-m}} - (2m+1)b\,\partial_{y^1_{-1-m}} + 2(m+1)\partial_{y^1_{-m-2}})\kappa_0\big), \\
&\qquad 2n\Big(\kappa_0\,\partial_{y_{-n}} + \chi_+\,\partial_{y^1_{-n}} + \chi_-\,\partial_{y^1_{-1-n}} + (b\chi_- + \chi_+)\sum_{q\leq -2} Q_{-q-2}(b)\partial_{y^1_{q-n}}\Big)\Big] \\
&= 0 \\
&= 2n\big(Q_{m+n-1}(b)(b\chi_+ + \chi_-)\delta_{m+n\geq 1} + \chi_+\delta_{m+n,0} + \chi_-\delta_{m+n,-1} \\
&\qquad + Q_{-m-n-2}(b)(b\chi_- + \chi_+)\delta_{m+n\leq -2}\big),
\end{aligned}$$

because $b\chi_+ + \chi_- = 0$.

The third case for (5-9) is (still $m \geq 0$ and $n > 0$)

$$\begin{aligned}
[\rho(b^1_m), \rho(b_{-n})] &= 2\Big[\Big(m\,\partial_{y_{-m}}\chi_+ + (m+1)\chi_-\,\partial_{y_{-1-m}} + (b\chi_- + \chi_+)\sum_{q\leq -2}(m-q)Q_{-q-2}(b)\partial_{y_{q-m}}\Big) \\
&\qquad + \big(2m\partial_{y^1_{-m}} - (2m+1)b\,\partial_{y^1_{-1-m}} + 2(m+1)\partial_{y^1_{-m-2}})\kappa_0\big), -y_{-n}\Big] \\
&= 2\Big[\Big(m\,\partial_{y_{-m}}\chi_+ + (m+1)\chi_-\,\partial_{y_{-1-m}} \\
&\qquad + (b\chi_- + \chi_+)\sum_{q\leq -2}(m-q)Q_{-q-2}(b)\partial_{y_{q-m}}\Big), -y_{-n}\Big]
\end{aligned}$$

$$= -2n\big(\delta_{m-n,0}\chi_+ + \chi_-\,\delta_{m-n,-1} + (b\chi_- + \chi_+)Q_{-m+n-2}(b)\delta_{m-n\le -2}\big)$$
$$= -2n\big(Q_{m-n-1}(b)(b\chi_+ + \chi_-)\delta_{m-n\ge 1} + \chi_+\delta_{m-n,0} + \chi_-\delta_{m-n,-1} + Q_{-m+n-2}(b)(b\chi_- + \chi_+)\delta_{m-n\le -2}\big).$$

The last case that we consider for (5-8) is

$$\begin{aligned}
&[\rho(b^1_{-m}), \rho(b_n)]\\
&= \Big[-y^1_{-m} - 2(b\chi_- + \chi_+)\sum_{q<-m}(q+m)Q_{-q-2}(b)\partial_{y_{q+m}},\\
&\qquad 2n\Big(\kappa_0\,\partial_{y_{-n}} + \chi_+\,\partial_{y^1_{-n}} + \chi_-\,\partial_{y^1_{-1-n}} + (b\chi_- + \chi_+)\sum_{q\le -2}Q_{-q-2}(b)\partial_{y^1_{q-n}}\Big)\Big]\\
&= 2n\Big[-y^1_{-m}, \Big(\kappa_0\,\partial_{y_{-n}} + \chi_+\,\partial_{y^1_{-n}} + \chi_-\,\partial_{y^1_{-1-n}} + (b\chi_- + \chi_+)\sum_{q\le -2}Q_{-q-2}(b)\partial_{y^1_{q-n}}\Big)\Big]\\
&= 2n\big(\delta_{-m+n,0}\chi_+ + \chi_-\,\delta_{-m+n,-1} + (b\chi_- + \chi_+)Q_{m-n-2}(b)\delta_{-m+n\le -2}\big)\\
&= \rho([b^1_{-m}, b_n]).
\end{aligned}$$

Now we move on to (5-10). It's straightforward to check that $[b^1_m, b^1_n] = 0$ for $m \ge 0$ and $n \ge 0$, and $[b^1_{-m}, b^1_{-n}] = 0$ for $m > 0$ and $n > 0$. Thus we have to check for $m > 0$ and $n \ge 0$ that

$$\begin{aligned}
&[\rho(b^1_{-m}), \rho(b^1_n)]\\
&= \Big[-y^1_{-m} - 2(b\chi_- + \chi_+)\sum_{q<-m}(q+m)Q_{-q-2}(b)\partial_{y_{q+m}},\\
&\qquad 2\Big(n\,\partial_{y_{-n}}\chi_+ + (n+1)\chi_-\,\partial_{y_{-1-n}} + (b\chi_- + \chi_+)\sum_{q\le -2}(n-q)Q_{-q-2}(b)\partial_{y_{q-n}}\Big)\\
&\qquad\qquad + \big(2n\partial_{y^1_{-n}} - (2n+1)b\,\partial_{y^1_{-1-n}} + 2(n+1)\partial_{y^1_{-n-2}})\kappa_0\big)\Big]\\
&= -\big[y^1_{-m}, \big(2n\partial_{y^1_{-n}} - (2n+1)b\,\partial_{y^1_{-1-n}} + 2(n+1)\partial_{y^1_{-n-2}}\big)\kappa_0\big]\\
&= \big(2n\delta_{-m+n,0} - (2n+1)b\,\delta_{-m+n,-1} + 2(n+1)\delta_{-m+n,-2}\big)\,\kappa_0\\
&= (n+m)\,\big(\delta_{-m+n,0} - 2b\,\delta_{-m+n,-1} + \delta_{-m+n,-2}\big)\,\kappa_0.
\end{aligned}$$

□

6. Two realizations of the affine 4-point algebra $\hat{\mathfrak{g}}$

This is our main result:

Theorem 6.1. *Fix $r \in \{0, 1\}$, which then fixes the normal ordering convention defined in the previous section. Set $\hat{\mathfrak{g}} = (\mathfrak{sl}(2, \mathbb{C}) \otimes R) \oplus \mathbb{C}\omega_+ \oplus \mathbb{C}\omega_0 \oplus \mathbb{C}\omega_-$, and assume $b\chi_+ + \chi_- = 0$ with $\chi_\pm, \chi_0 \in \mathbb{C}$ and that $\mathcal{V}$ is as in Proposition 5.4. Then we have a representation of the four point algebra $\mathfrak{g}$ on $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}] \otimes \mathcal{V}$:*

$$\begin{aligned}
\tau(\omega_\pm) &= \chi_\pm, \\
\tau(\omega_0) &= \chi_0 = \kappa_0 - 4\delta_{r,0}, \\
\tau(f(z)) &= -\alpha, \\
\tau(f^1(z)) &= -\alpha^1, \\
\tau(h^1(z)) &= 2\left(:\alpha^1\alpha^*: + P^2 :\alpha\alpha^{1*}:\right) + \beta^1, \\
\tau(h(z)) &= 2\left(:\alpha\alpha^*: + :\alpha^1\alpha^{1*}:\right) + \beta, \\
\tau(e^1(z)) &= :\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*}\right), \\
\tau(e(z)) &= :\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*} \\
&\qquad + P^2 :\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*.
\end{aligned}$$

It might appear that terms above such as $\chi_+(1-2bz+z^2)^{1/2}\partial_z\alpha^{1*}(z) = \chi_+ P\partial\alpha^{1*}$ are not well defined. But if we expand these summands out, we get infinite sums that are meaningful when applied to a polynomial in x_n, x_n^1, y_m, y_m^1. For example,

$$\begin{aligned}
(1-2bz+z^2)^{1/2}\partial_z\alpha^{1*}(z) &= -\sum_{k\geq 0}\sum_{n\in\mathbb{Z}} nP_k(b)\alpha_n^{1*}z^{k-n-1} \\
&= -\sum_{k\in\mathbb{N}^*}\sum_{m\in\mathbb{Z}}(m+k)P_k(b)\alpha_{k+m}^{1*}z^{-m-1}.
\end{aligned}$$

Now for a fixed $m\in\mathbb{Z}$, all but a finite number of summands of

$$\sum_{k\in\mathbb{N}^*}(m+k)P_k(b)\alpha_{k+m}^{1*}$$

applied to a polynomial in the x_n, x_n^1 are zero. This is because the α_l^{1*} all act as partial derivatives when $l > 0$.

Before we go through the proof it will be fruitful to introduce V. Kac's λ-notation (see [Kac 1998, Section 2.2] and [Wakimoto 2001] for some of its properties) used in operator product expansions. If $a(z)$ and $b(w)$ are formal distributions, then

$$[a(z), b(w)] = \sum_{j=0}^{\infty}\frac{(a_{(j)}b)(w)}{(z-w)^{j+1}}$$

is transformed under the *formal Fourier transform*

$$F_{z,w}^{\lambda}a(z,w) = \mathrm{Res}_z e^{\lambda(z-w)}a(z,w),$$

into the sum

$$[a_\lambda b] = \sum_{j=0}^{\infty}\frac{\lambda^j}{j!}a_{(j)}b.$$

Proof. We need to check the invariance under τ of the table

$[\cdot_\lambda\cdot]$	$f(w)$	$f^1(w)$	$h(w)$	$h^1(w)$	$e(w)$	$e^1(w)$
$f(z)$	0	0	$*$	$*$	$*$	$*$
$f^1(z)$		0	$*$	$*$	$*$	$*$
$h(z)$			$*$	$*$	$*$	$*$
$h^1(z)$				$*$	$*$	$*$
$e(z)$					0	0
$e^1(z)$						0

Here $*$ represent nonzero formal distributions obtained from the defining relations. The proof uses Wick's theorem together with Taylor's theorem, as follows:

$$
\begin{aligned}
&[\tau(f)_\lambda\tau(f)]=[\tau(f)_\lambda\tau(f^1)]=0,\\
&[\tau(f)_\lambda\tau(h)]=-\big[\alpha_\lambda(2(\alpha\alpha^*+\alpha^1\alpha^{1*})+\beta)\big]=-2\alpha=2\tau(f),\\
&[\tau(f)_\lambda\tau(h^1)]=-\big[\alpha_\lambda(2(\alpha^1\alpha^*+P^2\alpha\alpha^{1*})+\beta^1)\big]=-2\alpha^1=2\tau(f^1),\\
&[\tau(f)_\lambda\tau(e)]=-\big[\alpha_\lambda\big(:\alpha(\alpha^*)^2:+\beta\alpha^*+\chi_+\partial(P\alpha^{1*})+P^2:\alpha(\alpha^{1*})^2:\\
&\qquad\qquad\qquad\qquad\qquad +2:\alpha^1\alpha^*\alpha^{1*}:+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*\big)\big]\\
&\qquad =-2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)-\beta-\chi_0\partial=-\tau(h)-\chi_0\lambda,\\
&[\tau(f)_\lambda\tau(e^1)]=-\big[\alpha_\lambda\big(:\alpha^1(\alpha^*)^2:+\beta^1\alpha^*+\chi_0P(\partial P)\alpha^{1*}+\chi_+P\partial\alpha^*\\
&\qquad\qquad\qquad\qquad\qquad +P^2(:\alpha^1(\alpha^{1*})^2:+2:\alpha\alpha^*\alpha^{1*}:+\beta\alpha^{1*}+\chi_0\partial\alpha^{1*})\big)\big]\\
&\qquad =-2(:\alpha^1\alpha^*:+P^2:\alpha\alpha^{1*}:)-\beta^1-\chi_+P\partial=-\tau(h^1)-\chi_+P\lambda,\\
&[\tau(f^1)_\lambda\tau(f^1)]=0,\\
&[\tau(f^1)_\lambda\tau(h)]=-\big[\alpha^1_\lambda(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta)\big]=-2\alpha^1=2\tau(f^1),\\
&[\tau(f^1)_\lambda\tau(h^1)]=-\big[\alpha^1_\lambda(2(:\alpha^1\alpha^*:+P^2:\alpha\alpha^{1*}:)+\beta^1)\big]=-2P^2\alpha^1=P^2\tau(f^1),\\
&[\tau(f^1)_\lambda\tau(e)]=-\big[\alpha^1_\lambda\big(:\alpha(\alpha^*)^2:+\beta\alpha^*+\partial(P\alpha^{1*})+P^2:\alpha(\alpha^{1*})^2:\\
&\qquad\qquad\qquad\qquad\qquad +2:\alpha^1\alpha^*\alpha^{1*}:+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*\big)\big]\\
&\qquad =-\big(\chi_+(\partial P+P\lambda)+2P^2:\alpha\alpha^{1*}:+2:\alpha^1\alpha^*:+\beta^1\big)\\
&\qquad =-\tau(h^1)-\chi_+(\partial P+P\partial),\\
&[\tau(f^1)_\lambda\tau(e^1)]=-\big[\alpha^1_\lambda\big(:\alpha^1(\alpha^*)^2:+\beta^1\alpha^*+\chi_0P(\partial P)\alpha^{1*}+\chi_+P\partial\alpha^*\\
&\qquad\qquad\qquad\qquad\qquad +P^2(:\alpha^1(\alpha^{1*})^2:+2:\alpha\alpha^*\alpha^{1*}:+\beta\alpha^{1*}+\chi_0\partial\alpha^{1*})\big)\big]\\
&\qquad =-\big(\chi_0P(\partial P)\alpha^{1*}+P^2(2(:\alpha^1\alpha^{1*}:+:\alpha\alpha^*:)+\beta+\chi_0\lambda)\big)\\
&\qquad =-(P^2\tau(h)+\chi_0P\partial P+P^2\chi_0\lambda),\\
&[\tau(h)_\lambda\tau(h)]=\big[(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta)_\lambda(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta(w))\big]\\
&\qquad =4\big(-:\alpha\alpha^*:+:\alpha^*\alpha:-:\alpha^1\alpha^{1*}:+:\alpha^{1*}\alpha^1:\big)-8\delta_{r,0}\lambda+[\beta_\lambda\beta]\\
&\qquad =-2(4\delta_{r,0}+\kappa_0)\lambda.
\end{aligned}
$$

Thus $[\tau(h(z)),\tau(h(w))]=-2(4\delta_{r,0}+\kappa_0)\partial_w\delta(z/w)$. Next we calculate

$$\begin{aligned}[\tau(h)_\lambda\tau(h^1)]&=\big[(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta)_\lambda(2(:\alpha^1\alpha^*:+P^2:\alpha\alpha^{1*}:)+\beta^1)\big]\\&=4\big((:\alpha^*\alpha^1:-:\alpha^1\alpha^*:)+P^2(-:\alpha\alpha^{1*}:+:\alpha^{1*}\alpha:)\big)+[\beta_\lambda\beta^1].\end{aligned}$$

Since $[a_n,a_m^{1*}]=[a_n^1,a_m^*]=0$, we have

$$[\tau(h(z)),\tau(h^1(w))]=[\beta(z),\beta^1(w)]=-2\chi_+(1-2bw+w^2)^{1/2}\partial_w\delta(z/w).$$

We continue with

$$\begin{aligned}[\tau(h^1)_\lambda\tau(h^1)]&=\big[\big(2(:\alpha^1\alpha^*:+P^2:\alpha\alpha^{1*}:)+\beta^1\big)_\lambda\\&\qquad\times\big(2(:\alpha^1\alpha^*:+P^2:\alpha\alpha^{1*}:)+\beta^1\big)\big]\\&=4P^2\left(-:\alpha\alpha^*:+:\alpha^{1*}\alpha^1:\right)+4P^2\left(-:\alpha^1\alpha^{1*}:+:\alpha^*\alpha:\right)\\&\qquad+8\delta_{r,0}P^2\lambda+8\delta_{r,0}P(\partial P)+[\beta^1_\lambda\beta^1]\\&=8\delta_{r,0}P^2\lambda+8\delta_{r,0}P(\partial P)-2\kappa_0(P^2\lambda+P(\partial P)).\end{aligned}$$

Note that $:a(z)b(z):$ and $:b(z)a(z):$ are usually not equal, but

$$:\alpha^1(w)\alpha^{*1}(w):=:\alpha^{1*}(w)\alpha^1(w):\quad\text{and}\quad:\alpha(w)\alpha^*(w):=:\alpha^*(w)\alpha(w):.$$

Thus we have

$$\big[\tau(h^1(z)),\tau(h^1(w))\big]=(4\delta_{r,0}+\kappa_0)(-2(w^2-2bw+1)\partial_w\delta(z/w)+2(b-w)\delta(z/w)).$$

Next we calculate the h's paired with the e's:

$$\begin{aligned}&[\tau(h)_\lambda\tau(e^1)]\\&=\big[(2(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:)+\beta)_\lambda\\&\qquad\times\big(:\alpha^1(\alpha^*)^2:+\beta^1\alpha^*+\chi_0P(\partial P)\alpha^{1*}+\chi_+P\partial\alpha^*\\&\qquad\qquad+P^2(:\alpha^1(\alpha^{1*})^2:+2:\alpha\alpha^*\alpha^{1*}:+\beta\alpha^{1*}+\chi_0\partial\alpha^{1*})\big)\big]\\&=4:\alpha^1(\alpha^*)^2:-2:\alpha^1(\alpha^*)^2:+2\beta^1\alpha^*+2\chi_0P(\partial P)\alpha^{1*}\\&\quad+2\chi_+P\alpha^*\lambda+2\chi_+\partial P\\&\quad+4P^2(:\alpha^*\alpha\alpha^{1*}:-:\alpha\alpha^*\alpha^{1*}:+:\alpha\alpha^*\alpha^{1*}:-\delta_{r,0}\alpha^{1*}\lambda-\delta_{r,0}\alpha^{1*}\lambda)\\&\quad-2P^2(-:\alpha^1(\alpha^{1*})^2:+2:\alpha^1\alpha^{1*}\alpha^{1*}:+\beta\alpha^{1*}+\chi_0\lambda)\\&\quad+\alpha^*[\beta_\lambda\beta^1]+P^2\alpha^{1*}[\beta_\lambda\beta]\\&=2:\alpha^1(\alpha^*)^2:+2\beta^1\alpha^*+2\chi_0P(\partial P)\alpha^{1*}+2\chi_+P\alpha^*\lambda+2\chi_+\partial P\\&\quad+4P^2(:\alpha^*\alpha\alpha^{1*}:-2\delta_{r,0}\alpha^{1*}\lambda)-2P^2(:\alpha^1(\alpha^{1*})^2:+\beta\alpha^{1*}+\chi_0\lambda)\\&\quad+\alpha^*[\beta_\lambda\beta^1]+P^2\alpha^{1*}[\beta_\lambda\beta].\end{aligned}$$

From this we can conclude

$$\begin{aligned}\big[\tau(h(z)),\,\tau(e^1(w))\big]&\\ =2\,&\big(:\alpha^1(w)\alpha^*(w)^2:+\beta^1(w)\alpha^*(w)+\chi_0(w-b)\alpha^{1*}(w)\big)\,\delta(z/w)\\ &+2\big(\chi_+(1-2bw+w^2)^{1/2}\partial_w\alpha^*(w)\big)\delta(z/w)\\ &+2(w^2-2bw+1)\big(2:\alpha(w)\alpha^*(w)\alpha^{1*}(w):+\chi_0\partial_w\alpha^{1*}(w)\big)\delta(z/w)\\ &+2(w^2-2bw+1)\big(:\alpha^1(w)\alpha^{1*}(w)\alpha^{1*}(w):+\beta(w)\alpha^{1*}(w)\big)\delta(z/w)\\ &-2\left(w^2-2bw+1\right)(4\delta_{r,0}+\kappa_0-\chi_0)\alpha^{1*}(w)\partial_w\delta(z/w)\\ =2\tau&(e^1(w))\delta(z/w)-2\left(w^2-2bw+1\right)(4\delta_{r,0}+\kappa_0-\chi_0)\alpha^{1*}(w)\partial_w\delta(z/w).\end{aligned}$$

Next we must calculate

$$\begin{aligned}\big[\tau(h)_\lambda\tau(e)\big]&\\ =2\big[&:\alpha\alpha^*:{}_\lambda\big(:\alpha(\alpha^*)^2:+\beta\alpha^*+\chi_+\partial P\alpha^{1*}+\chi_+P\partial\alpha^{1*}\\ &\qquad+P^2:\alpha(\alpha^{1*})^2:+2:\alpha^1\alpha^*\alpha^{1*}:+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*\big)\big]\\ &+2\big[:\alpha^1\alpha^{1*}:{}_\lambda\big(:\alpha(\alpha^*)^2:+\beta\alpha^*+\chi_+(\partial P)\alpha^{1*}+\chi_+P\partial\alpha^{1*}\\ &\qquad+P^2:\alpha(\alpha^{1*})^2:+2:\alpha^1\alpha^*\alpha^{1*}:+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*\big)\big]\\ &+\big[\beta_\lambda\big(:\alpha(\alpha^*)^2:+\beta\alpha^*+\chi_+(\partial P)\alpha^{1*}+\chi_+P\partial\alpha^{1*}\\ &\qquad+P^2:\alpha(\alpha^{1*})^2:+2:\alpha^1\alpha^*\alpha^{1*}:+\beta^1\alpha^{1*}+\chi_0\partial_w\alpha^*\big)\big]\\ =2\big[&:\alpha\alpha^*:{}_\lambda\big(:\alpha(\alpha^*)^2:+\beta\alpha^*+P:\alpha(\alpha^{1*})^2:+2:\alpha^1\alpha^*\alpha^{1*}:+\chi_0\partial\alpha^*\big)\big]\\ &+2\big[:\alpha^1\alpha^{1*}:{}_\lambda\big(\chi_+(\partial P)\alpha^{1*}+\chi_+P\partial\alpha^{1*}\\ &\qquad+P^2:\alpha(\alpha^{1*})^2:+2:\alpha^1\alpha^*\alpha^{1*}:+\beta^1\alpha^{1*}\big)\big]\\ &+\big[\beta_\lambda\big(\beta\alpha^*+\beta^1\alpha^{1*}\big)\big]\\ =2\big(&:\alpha(\alpha^*)^2:-2\delta_{r,0}\alpha^*\lambda+\beta\alpha^*-P^2:\alpha(\alpha^{1*})^2:\\ &\qquad+2:\alpha^1\alpha^*\alpha^{1*}:+\chi_0\partial\alpha^*+\chi_0\alpha^*\lambda\big)\\ &+2\chi_+(\partial P)\alpha^{1*}+2\chi_+P\partial\alpha^{1*}+2\chi_+P\alpha^{1*}\lambda+4P^2:\alpha(\alpha^{1*})^2:\\ &-4\delta_{r,0}\alpha^*\lambda+2\beta^1\alpha^{1*}-2\kappa_0\alpha^*\lambda-2\chi_+P\alpha^{1*}\lambda\\ =2\big(&:\alpha(\alpha^*)^2:+\beta\alpha^*+\chi_+\partial(P\alpha^{1*})+P^2:\alpha(\alpha^{1*})^2:+2:\alpha^1\alpha^*\alpha^{1*}:\\ &\qquad+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*\big)\\ &+2\chi_0\alpha^*\lambda-8\delta_{r,0}\alpha^*\lambda-2\kappa_0\alpha^*\lambda.\end{aligned}$$

As a consequence we get

$$[\tau(h(z)),\,\tau(e(w))]=2\tau(e(w))\delta(z/w)+2(\chi_0-\kappa_0-4\delta_{r,0})\alpha^*(w)\partial_w\delta(z/w).$$

Next we calculate

$$
\begin{aligned}
&[\tau(h^1)_\lambda \tau(e^1)]\\
&=\big[\big(2(:\alpha^1\alpha^*: + P^2 :\alpha\alpha^{1*}:) + \beta^1\big)_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^*\\
&\qquad\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*}\right)\big)\big]\\
&=2\big[:\alpha^1\alpha^*:_\lambda\big(\chi_0 P(\partial P)\alpha^{1*} + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial_w\alpha^{1*}\right)\big)\big]\\
&\quad+2\big[P^2 :\alpha\alpha^{1*}:_\lambda\big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_+ P\partial\alpha^* + P^2(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}:)\big)\big]\\
&\quad+\big[\beta^1_\lambda\big(\beta^1\alpha^* + P^2\beta\alpha^{1*}\big)\big]\\
&=2\big(\chi_0 P(\partial P)\alpha^* + P^2\big(2:\alpha^1\alpha^*\alpha^{1*}: + 2:\alpha(\alpha^*)^2:\\
&\qquad\qquad - 2:\alpha^1\alpha^*\alpha^{1*}: - 2\delta_{r,0}\alpha^*\lambda + \beta\alpha^* + \chi_0\partial\alpha^* + \chi_0\alpha^*\lambda\big)\big)\\
&\quad+4P^2 :\alpha^1\alpha^{1*}\alpha^*: - 2P^2 :\alpha(\alpha^*)^2: - 4P^2\delta_{r,0}\alpha^*\lambda - 8P(\partial P)\delta_{r,0}\alpha^* + 2P^2\beta^1\alpha^{1*}\\
&\quad+2P^4 :\alpha(\alpha^{1*})^2: + 2\chi_+ P^3\partial\alpha^{1*} + 2\chi_+ P^3\partial\alpha^{1*} + 2\chi_+ P^3\alpha^{1*}\lambda + 4\chi_+ P^2(\partial P)\alpha^{1*}\\
&\quad- 2\kappa_0\alpha^*(P(\partial P) + P^2\lambda) - 2\chi_+ P^2\alpha^{1*}(\partial P + P\lambda)\\
&=2P^2 :\alpha(\alpha^*)^2: + 2P^2\beta\alpha^* + 2\chi_0 P^2\partial\alpha^* + 2P^4 :\alpha(\alpha^{1*})^2:\\
&\quad+4P^2 :\alpha^1\alpha^{1*}\alpha^*: + 2P^2\beta^1\alpha^{1*} + 2\chi_+ P^3\partial\alpha^{1*} + 2\chi_+ P^2(\partial P)\alpha^{1*}\\
&\quad+2\chi_0\alpha^*(P(\partial P) + P^2\lambda) - 8\delta_{r,0}\alpha^*(P(\partial P) + P^2\lambda) - 2\kappa_0\alpha^*(P(\partial P) + P^2\lambda)\\
&=2P^2\tau(e).
\end{aligned}
$$

The final calculation for the Cartan generators is

$$
\begin{aligned}
&[\tau(h^1)_\lambda \tau(e)]\\
&\quad=\big[\big(2(:\alpha^1\alpha^*: + P^2 :\alpha\alpha^{1*}:) + \beta^1\big)_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*}\\
&\qquad\qquad + P^2 :\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big]\\
&\quad=2\big[(:\alpha^1\alpha^*:)_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*}\\
&\qquad\qquad + P^2 :\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big]\\
&\qquad+2\big[(P^2 :\alpha\alpha^{1*}:)_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*}\\
&\qquad\qquad + P^2 :\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big]\\
&\qquad+\big[\beta^1_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*}\\
&\qquad\qquad + P^2 :\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big]\\
&\quad=2\big[(:\alpha^1\alpha^*:)_\lambda\big(:\alpha(\alpha^*)^2: + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*}\\
&\qquad\qquad + P^2 :\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*}\big)\big]\\
&\qquad+2\big[(P^2 :\alpha\alpha^{1*}:)_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + 2:\alpha^1\alpha^*\alpha^{1*}: + \chi_0\partial\alpha^*\big)\big]\\
&\qquad+\big[\beta^1_\lambda(\beta\alpha^* + \beta^1\alpha^{1*})\big]
\end{aligned}
$$

$$
\begin{aligned}
&= -2:\alpha^1(\alpha^*)^2: +2\chi_+(\partial P)\alpha^* + 2\chi_+ P\partial\alpha^* + 2\chi_+ P\alpha^*\lambda \\
&\quad -2P^2:\alpha^1(\alpha^{1*})^2: +4P^2:\alpha\alpha^*\alpha^{1*}: -4\delta_{r,0}P^2\alpha^{1*}\lambda + 4:\alpha^1(\alpha^*)^2: +\beta^1\alpha^* \\
&\quad +4P^2:\alpha\alpha^*\alpha^{*1}: +2P^2\beta\alpha^{*1} + 4P^2:\alpha^1(\alpha^{1*})^2: -4P^2:\alpha\alpha^*\alpha^{1*}: \\
&\quad -4\delta_{r,0}P^2\alpha^{1*}\lambda - 8\delta_{r,0}P(\partial P)\alpha^{1*} + 2\chi_0 P^2\alpha^{*1}\lambda + 2\chi_0 P^2\partial\alpha^{1*} + 4\chi_0 P(\partial P)\alpha^{1*} \\
&\quad -2\chi_+(P\lambda + \partial P)\alpha^* - 2\kappa_0\alpha^{1*}(P(\partial P) + P^2\lambda) \\
&= 2:\alpha^1(\alpha^*)^2: + 2\chi_+ P\partial\alpha^* + 2P^2:\alpha^1(\alpha^{1*})^2: + \beta^1\alpha^* + 4P^2:\alpha\alpha^*\alpha^{*1}: + 2P^2\beta\alpha^{*1} \\
&\quad + 2\chi_0 P^2\partial\alpha^{1*} + 2\chi_0 P(\partial P)\alpha^{1*} + 2(\chi_0 - 4\delta_{r,0} - \kappa_0)\alpha^{*1}(P(\partial P) + P^2\lambda) \\
&= 2\tau(e^1).
\end{aligned}
$$

Next we start on the Chevalley generators e:

$$
\begin{aligned}
[\tau(e)_\lambda \tau(e)] = &\big[:\alpha(\alpha^*)^2:{}_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*} \\
&\qquad + P^2:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big] \\
&+ \big[\beta\alpha^*_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*} \\
&\qquad + P^2:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big] \\
&+ \chi_+\big[(\partial P)\alpha^{1*}_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*} \\
&\qquad + P^2:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big] \\
&+ \chi_+\big[P\partial_z\alpha^{1*}(z)_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*} \\
&\qquad + P^2:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big] \\
&+ \big[P^2:\alpha(\alpha^{1*})^2:{}_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*} \\
&\qquad + P^2:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big] \\
&+ 2\big[:\alpha^1\alpha^*\alpha^{1*}:{}_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*} \\
&\qquad + P^2:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big] \\
&+ \big[\beta^1\alpha^{1*}_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*} \\
&\qquad + P^2:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big] \\
&+ \chi_0\big[\partial\alpha^*_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^{1*} \\
&\qquad + P^2:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\big)\big]
\end{aligned}
$$

which in turn is equal to

$$
\begin{aligned}
&\big[:\alpha(\alpha^*)^2:{}_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + P^2:\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \chi_0\partial\alpha^*\big)\big] \\
&\quad + \big[\beta\alpha^*_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + P^2:\alpha(\alpha^{1*})^2: + \beta^1\alpha^{1*}\big)\big] \\
&\quad + \chi_+\big[\partial(P\alpha^{1*})_\lambda(2P^2:\alpha^1\alpha^*\alpha^{1*}:)\big] \\
&\quad + \big[P^2:\alpha(\alpha^{1*})^2:{}_\lambda\big(:\alpha(\alpha^*)^2: + \beta\alpha^* + 2:\alpha^1\alpha^*\alpha^{1*}: + \chi_0\partial\alpha^*\big)\big]
\end{aligned}
$$

$$
\begin{aligned}
&+2\big[\,{:}\alpha^1\alpha^*\alpha^{1*}{:}\,_\lambda\big({:}\alpha(\alpha^*)^2{:}+\chi_+(\partial P)\alpha^{1*}+\chi_+P\partial\alpha^{1*}\\
&\qquad\qquad\qquad\qquad+P^2\,{:}\alpha(\alpha^{1*})^2{:}+2\,{:}\alpha^1\alpha^*\alpha^{1*}{:}+\beta^1\alpha^{1*}\big)\big]\\
&+\big[\beta^1\alpha^{1*}_\lambda\big({:}\beta\alpha^*{:}+2\,{:}\alpha^1\alpha^*\alpha^{1*}{:}+\beta^1\alpha^{1*}\big)\big]\\
&+\chi_0\big[\partial\alpha^*_\lambda\big({:}\alpha(\alpha^*)^2{:}+P^2\,{:}\alpha(\alpha^{1*})^2{:}\big)\big]
\end{aligned}
$$

$$
\begin{aligned}
=\,&2\,{:}\alpha(\alpha^*)^3{:}-2\,{:}\alpha(\alpha^*)^3{:}-4\delta_{r,0}\,{:}\alpha^*(\partial\alpha^*){:}-4\delta_{r,0}\,{:}(\alpha^*)^2{:}\,\lambda\\
&+\,{:}\beta(\alpha^*)^2{:}-2P^2\,{:}\alpha\alpha^*(\alpha^{1*})^2{:}+2\,{:}\alpha^1(\alpha^*)^2\alpha^{1*}{:}+2\chi_0\,{:}\alpha^*(\partial\alpha^*){:}\\
&+\chi_0\,{:}(\alpha^*)^2{:}\,\lambda-{:}\beta(\alpha^*)^2{:}-2\kappa_0\,{:}(\alpha^*)^2{:}\,\lambda-2\kappa_0\,{:}\alpha^*(\partial\alpha^*){:}-P^2\,{:}\beta(\alpha^{1*})^2{:}\\
&-2\chi_+P\,{:}\alpha^*\alpha^{1*}{:}\,\lambda-2\chi_+P\,{:}(\partial\alpha^*)\alpha^{1*}{:}+2\chi_+P\,{:}\alpha^*\alpha^{1*}{:}\,\lambda\\
&+2P^2\,{:}\alpha\alpha^*(\alpha^{1*})^2{:}+P^2\,{:}\beta(\alpha^{1*})^2{:}+2P^2\,{:}\alpha^1(\alpha^{1*})^3{:}\\
&-4P^2\,{:}\alpha\alpha^*(\alpha^{1*})^2{:}-4P^2\delta_{r,0}\,{:}(\alpha^{1*})^2{:}\,\lambda-4P^2\delta_{r,0}\,{:}\alpha^{1*}\partial\alpha^{1*}{:}\\
&-8\delta_{r,0}P(\partial P)\,{:}(\alpha^{1*})^2{:}+\chi_0P^2\,{:}(\alpha^{1*})^2{:}\,\lambda+2\chi_0P^2\,{:}\alpha^{*1}\partial\alpha^{*1}{:}\\
&+2\chi_0P(\partial P)\,{:}(\alpha^{*1})^2{:}-2\,{:}\alpha^1(\alpha^*)^2\alpha^{1*}{:}+2\chi_+(\partial P)\,{:}\alpha\alpha^{1*}{:}\\
&+2\chi_+P\,{:}\alpha^*\partial\alpha^{1*}{:}+2\chi_+P\,{:}\alpha^{1*}\partial\alpha^*{:}+2\chi_+P\,{:}\alpha^*\alpha^{1*}{:}\,\lambda\\
&+4P^2\,{:}\alpha\alpha^*(\alpha^{1*})^2{:}-2P^2\,{:}\alpha^1(\alpha^{1*})^3{:}-4\delta_{r,0}P^2\,{:}(\alpha^{1*})^2{:}\,\lambda\\
&-4\delta_{r,0}P^2\,{:}\alpha^{1*}(\partial\alpha^{1*}){:}+4\,{:}\alpha^1\alpha^*\alpha^{1*}{:}-4\,{:}\alpha^1\alpha^*\alpha^{1*}{:}\\
&-4\delta_{r,0}\,{:}(\alpha^*)^2{:}\,\lambda-4\delta_{r,0}\,{:}\alpha^*(\partial\alpha^*){:}+2\,{:}\beta^1\alpha^*\alpha^{1*}{:}\\
&-2\chi_+(\partial P+P\lambda)\,{:}\alpha^*\alpha^{1*}{:}-2\chi_+P\,{:}\alpha^*(\partial\alpha^{1*}){:}-2\kappa_0(P\partial P+P^2\lambda)\,{:}(\alpha^{1*})^2{:}\\
&-2\kappa_0P^2\,{:}\alpha^{1*}(\partial\alpha^{1*}){:}-2\,{:}\beta^1\alpha^*\alpha^{1*}{:}+\lambda\chi_0\big({:}(\alpha^*)^2{:}+P^2\,{:}(\alpha^{1*})^2{:}\big)
\end{aligned}
$$

$$
\begin{aligned}
=&-4\delta_{r,0}\,{:}\alpha^*(\partial\alpha^*){:}-4\delta_{r,0}\,{:}(\alpha^*)^2{:}\,\lambda-4P^2\delta_{r,0}\,{:}(\alpha^{1*})^2{:}\,\lambda\\
&-4P^2\delta_{r,0}\,{:}\alpha^{1*}\partial\alpha^{1*}{:}-8\delta_{r,0}P(\partial P)\,{:}(\alpha^{1*})^2{:}-4\delta_{r,0}P^2\,{:}(\alpha^{1*})^2{:}\,\lambda\\
&-4\delta_{r,0}P^2\,{:}\alpha^{1*}(\partial\alpha^{1*}){:}-4\delta_{r,0}\,{:}(\alpha^*)^2{:}\,\lambda-4\delta_{r,0}\,{:}\alpha(\partial\alpha^*){:}\\
&-2\kappa_0\,{:}(\alpha^*)^2{:}\,\lambda-2\kappa_0\,{:}\alpha^*(\partial\alpha^*){:}-2\chi_+P\,{:}\alpha^*\alpha^{1*}{:}\,\lambda-2\chi_+P\,{:}(\partial\alpha^*)\alpha^{1*}{:}\\
&-2\chi_+(\partial P+P\lambda)\,{:}\alpha^*\alpha^{1*}{:}-2\chi_+P\,{:}\alpha^*(\partial\alpha^{1*}){:}-2\kappa_0(P\partial P+P^2\lambda)\,{:}(\alpha^{1*})^2{:}\\
&-2\kappa_0P^2\,{:}\alpha^{1*}(\partial\alpha^{1*}){:}+2\chi_+(\partial P)\,{:}\alpha\alpha^{1*}{:}+2\chi_+P\,{:}\alpha^*\partial\alpha^{1*}{:}\\
&+2\chi_+P\,{:}\alpha^{1*}\partial\alpha^*{:}+2\chi_+P\,{:}\alpha^*\alpha^{1*}{:}\,\lambda+2\chi_+P\,{:}\alpha^*\alpha^{1*}{:}\,\lambda\\
&+\chi_0P^2\,{:}(\alpha^{1*})^2{:}\,\lambda+2\chi_0P^2\,{:}\alpha^{*1}\partial\alpha^{*1}{:}+2\chi_0P(\partial P)\,{:}(\alpha^{*1})^2{:}\\
&+2\chi_0\,{:}\alpha^*(\partial\alpha^*){:}+\chi_0\,{:}(\alpha^*)^2{:}\,\lambda+\lambda\chi_0\,\big({:}(\alpha^*)^2{:}+P^2\,{:}(\alpha^{1*})^2{:}\big)
\end{aligned}
$$

$$
\begin{aligned}
=&-8\delta_{r,0}\,\big({:}\alpha^*(\partial\alpha^*){:}+\,{:}(\alpha^*)^2{:}\,\lambda\big)\\
&-8\delta_{r,0}\,\big(P^2\,{:}(\alpha^{1*})^2{:}\,\lambda+P^2\,{:}\alpha^{1*}\partial\alpha^{1*}{:}+P(\partial P)\,{:}(\alpha^{1*})^2{:}\big)\\
&-2\kappa_0\,\big({:}\alpha^*(\partial\alpha^*){:}+\,{:}(\alpha^*)^2{:}\,\lambda\big)\\
&-2\kappa_0\,\big(P^2\,{:}(\alpha^{1*})^2{:}\,\lambda+P^2\,{:}\alpha^{*1}\partial\alpha^{*1}{:}+P(\partial P)\,{:}(\alpha^{*1})^2{:}\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 2\chi_0 \left(:\alpha^*(\partial\alpha^*): + :(\alpha^*)^2:\lambda \right) \\
&\quad + 2\chi_0 \left(P^2 :(\alpha^{1*})^2: \lambda + P^2 :\alpha^{*1}\partial\alpha^{*1}: + P(\partial P) :(\alpha^{*1})^2: \right) \\
&= 2\left(\chi_0 - \kappa_0 - 4\delta_{r,0}\right)\left(:\alpha^*(\partial\alpha^*): + :(\alpha^*)^2:\lambda \right) \\
&\quad + 2\left(\chi_0 - \kappa_0 - 4\delta_{r,0}\right)\left(P^2 :(\alpha^{1*})^2: \lambda + P^2 :\alpha^{*1}\partial\alpha^{*1}: + P(\partial P) :(\alpha^{*1})^2: \right) \\
&= 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
[\tau(e)_\lambda \tau(e^1)] = &\Big[:\alpha(\alpha^*)^2:{}_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*} \right)\big)\Big] \\
&+ \Big[:\beta\alpha^*:{}_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*} \right)\big)\Big] \\
&+ \Big[\chi_+\partial(P\alpha^{1*}) :_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*} \right)\big)\Big] \\
&+ \Big[P^2 :\alpha(\alpha^{1*})^2:{}_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*} \right)\big)\Big] \\
&+ 2\Big[:\alpha^1\alpha^*\alpha^{1*}:{}_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*} \right)\big)\Big] \\
&+ \Big[\beta^1\alpha^{1*}_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*} \right)\big)\Big] \\
&+ \Big[\chi_0\partial\alpha^*_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*} \right)\big)\Big]
\end{aligned}
$$

which equals

$$
\begin{aligned}
&\Big[:\alpha(\alpha^*)^2:{}_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_+ P\partial\alpha^* + 2P^2 :\alpha\alpha^*\alpha^{1*}: \big)\Big] \\
&\quad + \Big[\beta\alpha^*_\lambda \big(\beta^1\alpha^* + P^2\left(2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} \right)\big)\Big] \\
&\quad + \chi_+\Big[\partial(P\alpha^{1*})_\lambda \big(:\alpha^1(\alpha^*)^2: + P^2 :\alpha^1(\alpha^{1*})^2: \big)\Big] \\
&\quad + \Big[P^2 :\alpha(\alpha^{1*})^2:{}_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_+ P\partial\alpha^* \\
&\qquad\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: \right)\big)\Big] \\
&\quad + 2\Big[:\alpha^1\alpha^*\alpha^{1*}:{}_\lambda \big(:\alpha^1(\alpha^*)^2: + \chi_0 P(\partial P)\alpha^{1*} \\
&\qquad\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*} \right)\big)\Big] \\
&\quad + \Big[:\beta^1\alpha^{1*}:{}_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + P^2\left(:\alpha^1(\alpha^{1*})^2: + \beta\alpha^{1*} \right)\big)\Big] \\
&\quad + 2\chi_0\Big[:\partial\alpha^*:{}_\lambda P^2 :\alpha\alpha^*\alpha^{1*}: \Big]
\end{aligned}
$$

$$
\begin{aligned}
&= 2 :\alpha^1(\alpha^*)^3: + \beta^1(\alpha^*)^2 + 2\chi_+ P :\alpha^*(\partial\alpha^*): + \chi_+ P :(\alpha^*)^2: \lambda + 2P^2 :\alpha(\alpha^*)^2\alpha^{1*}: \\
&\quad - 4P^2 :\alpha(\alpha^*)^2\alpha^{1*}: - 4\delta_{r,0} P^2 :\alpha^*\alpha^{1*}: \lambda - 4\delta_{r,0} P^2 :(\partial\alpha^*)\alpha^{1*}: \\
&\quad - 2\chi_+ P :\alpha^*(\partial\alpha^*): - 2\chi_+ P :(\alpha^*)^2: \lambda \\
&\quad + P^2 \big(- 2 :\alpha^*\alpha^{1*}: \beta - 2\kappa_0 :\alpha^*\alpha^{1*}: \lambda - 2\kappa_0 :(\partial\alpha^*)\alpha^{1*}: \big) \\
&\quad - \chi_+ \lambda(- P :(\alpha^*)^2: - P^3 :(\alpha^{1*})^2:) \\
&\quad + 2P^2 :\alpha^1\alpha^*(\alpha^{1*})^2: - 2P^2 :\alpha(\alpha^*)^2\alpha^{1*}: - 4\delta_{r,0} P^2 :\alpha^*\alpha^{*1}: \lambda \\
&\quad - 4P^2\delta_{r,0} :\alpha^*(\partial\alpha^{*1}): - 8\delta_{r,0} P(\partial P) :\alpha^*\alpha^{*1}: + P^2\beta^1(\alpha^*)^2 \\
&\quad + 2\chi_+ P^3 :\alpha^{1*}(\partial\alpha^{1*}): + 2\chi_+ P^2(\partial P) :(\alpha^{1*})^2: + \chi_+ P^3 :(\alpha^{1*})^2: \lambda \\
&\quad + P^4 \big(- 2 :\alpha(\alpha^{1*})^3: + 2 :\alpha(\alpha^{1*})^3: \big) \\
&\quad + 2\big(- :\alpha^1(\alpha^*)^3: + \chi_0 P(\partial P) :\alpha^*\alpha^{1*}: \\
&\qquad\quad + P^2\big(- :\alpha^1\alpha^*(\alpha^{1*})^2: + 2 :\alpha^*\alpha^1(\alpha^{1*})^2: - 2\delta_{r,0} :\alpha^*\alpha^{1*}: \lambda \\
&\qquad\qquad - 2\delta_{r,0} :(\partial\alpha^*)\alpha^{1*}: + 2 :\alpha(\alpha^*)^2\alpha^{*1}: - 2 :\alpha^1\alpha^*(\alpha^{1*})^2: \\
&\qquad\qquad - 2\delta_{r,0} :\alpha^{1*}\alpha^*: \lambda - 2\delta_{r,0} :(\partial\alpha^{1*})\alpha^*: \\
&\qquad\qquad + :\beta\alpha^*\alpha^{1*}: + \chi_0 :(\partial\alpha^*)\alpha^{1*}: + \chi_0 :\alpha^*(\partial\alpha^{1*}): + \chi_0 :\alpha^*\alpha^{1*}: \lambda\big)\big) \\
&\quad - :\beta^1(\alpha^*)^2: - 2\kappa_0 P^2 :(\partial\alpha^{1*})\alpha^*: - 2\kappa_0 P^2 :\alpha^{1*}\alpha^*: \lambda - 2\kappa_0 P(\partial P) :\alpha^{1*}\alpha^*: \\
&\quad + P^2\big(- :\beta^1(\alpha^{1*})^2: - 2\chi_+(\partial P) :(\alpha^{1*})^2: \\
&\qquad\qquad\qquad\qquad - 2\chi_+ P :(\alpha^{1*})^2: \lambda - 2\chi_+ P :\alpha^{*1}(\partial\alpha^{1*}): \big) \\
&\quad + 2\chi_0\lambda P^2 :\alpha^*\alpha^{1*}: \\
&= 2\chi_+ P :\alpha^*(\partial\alpha^*): + \chi_+ P :(\alpha^*)^2: \lambda - 4\delta_{r,0} P^2 :\alpha^*\alpha^{1*}: \lambda - 4\delta_{r,0} P^2 :(\partial\alpha^*)\alpha^{1*}: \\
&\quad - 2\chi_+ P :\alpha^*(\partial\alpha^*): - 2\chi_+ P :(\alpha^*)^2: \lambda - 2\kappa_0 P^2 :\alpha^*\alpha^{1*}: \lambda - 2\kappa_0 P^2 :(\partial\alpha^*)\alpha^{1*}: \\
&\quad + \chi_+ P :(\alpha^*)^2: \lambda + \chi_+ P^3 :(\alpha^{1*})^2: \lambda \\
&\quad - 4\delta_{r,0} P^2 :\alpha^*\alpha^{*1}: \lambda - 4\delta_{r,0} P^2 :(\partial\alpha^{*1})\alpha^*: - 8\delta_{r,0} P(\partial P) :\alpha^*\alpha^{*1}: \\
&\quad + 2\chi_+ P^3 :\alpha^{1*}(\partial\alpha^{1*}): + 2\chi_+ P^2(\partial P) :(\alpha^{1*})^2: + \chi_+ P^3 :(\alpha^{1*})^2: \lambda \\
&\quad + 2\big(\chi_0 P(\partial P) :\alpha^*\alpha^{1*}: \\
&\quad + P^2\big(- 2\delta_{r,0} :\alpha^*\alpha^{1*}: \lambda - 2\delta_{r,0} :(\partial\alpha^*)\alpha^{1*}: \\
&\qquad\qquad\qquad\qquad - 2\delta_{r,0} :\alpha^{1*}\alpha^*: \lambda - 2\delta_{r,0} :(\partial\alpha^{1*})\alpha^*: \\
&\qquad\qquad\qquad\qquad + \chi_0 :(\partial\alpha^*)\alpha^{1*}: + \chi_0 :\alpha^*(\partial\alpha^{1*}): + \chi_0 :\alpha^*\alpha^{1*}: \lambda\big)\big) \\
&\quad - 2\kappa_0 P^2 :(\partial\alpha^{1*})\alpha^*: - 2\kappa_0 P^2 :\alpha^{1*}\alpha^*: \lambda - 2\kappa_0 P(\partial P) :\alpha^{1*}\alpha^*: \\
&\quad + P^2 \big(- 2\chi_+(\partial P) :(\alpha^{1*})^2: - 2\chi_+ P :(\alpha^{1*})^2: \lambda - 2\chi_+ P :\alpha^{*1}(\partial\alpha^{1*}): \big) \\
&\quad + 2\chi_0\lambda P^2 :\alpha^*\alpha^{1*}: \\
&= - 16\delta_{r,0} P^2 :\alpha^*\alpha^{1*}: \lambda - 8\delta_{r,0} P^2 :(\partial\alpha^*)\alpha^{1*}: - 8\delta_{r,0} P^2 :(\partial\alpha^{*1})\alpha^*: \\
&\quad - 4\delta_{r,0} P(\partial P) :\alpha^*\alpha^{*1}: - 4\delta_{r,0} P^2 :(\partial\alpha^{1*})\alpha^*:
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 2\chi_0 P(\partial P) :\alpha^*\alpha^{1*}: + 2\chi_0\lambda P^2 :\alpha^*\alpha^{1*}: \\
&\quad + 2\chi_0 \left(P^2 :(\partial\alpha^*)\alpha^{1*}: + P^2 :\alpha^*(\partial\alpha^{1*}): + P^2 :\alpha^*\alpha^{1*}: \lambda\right) \\
&\quad - 2\kappa_0 P^2 :(\partial\alpha^{1*})\alpha^*: - 2\kappa_0 P^2 :\alpha^{1*}\alpha^*: \lambda - 2\kappa_0 P(\partial P) :\alpha^{1*}\alpha^*: \\
&\quad - 2\kappa_0 \left(P^2 :\alpha^*\alpha^{1*}: \lambda + P^2 :(\partial\alpha^*)\alpha^{1*}:\right) \\
&= 2(\chi_0 - \kappa_0 - 4\delta_{r,0}) \left(P^2 :(\partial\alpha^{1*})\alpha^*: + P^2 :\alpha^{1*}\alpha^*: \lambda + P(\partial P) :\alpha^{1*}\alpha^*:\right) \\
&\quad + 2(\chi_0 - \kappa_0 - 4\delta_{r,0}) \left(P^2 :\alpha^*\alpha^{1*}: \lambda + P^2 :(\partial\alpha^*)\alpha^{1*}:\right) \\
&= 0.
\end{aligned}
$$

Lastly we have

$$
\begin{aligned}
[\tau(e^1)_\lambda \tau(e^1)] = \Big[&:\alpha^1(\alpha^*)^2:_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2(:\alpha^1(\alpha^{1*})^2: + 2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*})\big)\Big] \\
&+ \Big[\beta^1\alpha^*_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2(:\alpha^1\alpha^{1*})^2: + 2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*}\big)\Big] \\
&+ \Big[\chi_0 P(\partial P)\alpha^{1*}(z)_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2(:\alpha^1\alpha^{1*})^2: + 2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*}\big)\Big] \\
&+ \Big[\chi_+ P\partial\alpha^* :_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2(:\alpha^1(\alpha^{1*})^2: + 2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*})\big)\Big] \\
&+ \Big[P^2 :\alpha^1(\alpha^{1*})^2:_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2(:\alpha^1(\alpha^{1*})^2: + 2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*})\big)\Big] \\
&+ 2\Big[P^2 :\alpha\alpha^*\alpha^{1*}:_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2(:\alpha^1(\alpha^{1*})^2: + 2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*})\big)\Big] \\
&+ \Big[P^2\beta\alpha^{1*}_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2(:\alpha^1(\alpha^{1*})^2: + 2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*})\big)\Big] \\
&+ \Big[P^2\chi_0\partial\alpha^{1*}_\lambda \big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\
&\qquad + P^2(:\alpha^1(\alpha^{1*})^2: + 2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*})\big)\Big]
\end{aligned}
$$

which equals

$$
\begin{aligned}
&\Big[:\alpha^1(\alpha^*)^2:_\lambda \big(\chi_0 P(\partial P)\alpha^{1*} + P^2\big(:\alpha^1(\alpha^{1*})^2: + 2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*}\big)\big)\Big] \\
&\quad + \Big[\beta^1\alpha^*_\lambda \big(\beta^1\alpha^* + P^2 \left(2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*}\right)\big)\Big] \\
&\quad + \chi_0\Big[P(\partial P)\alpha^{1*}_\lambda \big(:\alpha^1(\alpha^*)^2: + P^2 :\alpha^1(\alpha^{1*})^2:\big)\Big] + 2\chi_+\Big[P\partial\alpha^*{}_\lambda P^2 :\alpha\alpha^*\alpha^{1*}:\Big] \\
&\quad + \Big[P^2 :\alpha^1(\alpha^{1*})^2:{}_\lambda \big(:\alpha^1(\alpha^*)^2: + \chi_0 P(\partial P)\alpha^{1*} \\
&\qquad + P^2 \left(:\alpha^1(\alpha^{1*})^2: + 2 :\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*}\right)\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+2\big[P^2 :\alpha\alpha^*\alpha^{1*}:\,{}_\lambda\big(:\alpha^1(\alpha^*)^2: +\beta^1\alpha^* + \chi_+ P\partial\alpha^* \\
&\qquad\qquad\qquad\qquad + P^2\left(:\alpha^1(\alpha^{1*})^2: +2:\alpha\alpha^*\alpha^{1*}:\right)\big)\big] \\
&+\big[P^2\beta\alpha^{1*}\,{}_\lambda\big(:\alpha^1(\alpha^*)^2: +\beta^1\alpha^* + P^2\left(:\alpha^1(\alpha^{1*})^2: +\beta\alpha^{1*}\right)\big)\big] \\
&+\big[P^2\chi_0\partial\alpha^{1*}\,{}_\lambda\big(:\alpha^1(\alpha^*)^2: + P^2 :\alpha^1(\alpha^{1*})^2:\big)\big]
\end{aligned}
$$

$$
\begin{aligned}
&= \chi_0 P(\partial P) :\alpha^{*2}: \\
&\quad + P^2\big(2:\alpha^1(\alpha^*)^2\alpha^{1*}: +2:\alpha(\alpha^*)^3: -4:\alpha^1\alpha^{*2}\alpha^{1*}: \\
&\qquad -4\delta_{r,0}:\alpha^{*2}:\lambda - 4\delta_{r,0}:\alpha^*(\partial\alpha^*): + \beta(\alpha^*)^2 + \chi_0(\alpha^*)^2\lambda + 2\chi_0:\alpha^*(\partial\alpha^*):\big) \\
&\quad - 2\kappa_0(P(\partial P):(\alpha^*)^2: + P^2:(\alpha^*)^2:\lambda + P^2:\alpha^*\partial\alpha^*:) \\
&\quad + P^2\left(-2:\beta^1\alpha^*\alpha^{1*}: -2\chi_+(\partial P + P\lambda):\alpha^*\alpha^{1*}: -2\chi_+ P:(\partial\alpha^*)\alpha^{1*}:\right) \\
&\quad - \chi_0 P(\partial P):(\alpha^*)^2: - \chi_0 P^3(\partial P):(\alpha^{1*})^2: \\
&\quad + 2\chi_+ P^2(\partial P):\alpha^*\alpha^{1*}: + 2\chi_+ P^3:\alpha^*\alpha^{1*}:\lambda \\
&\quad + P^2\big(-2:\alpha^1\alpha^{1*}(\alpha^*)^2: + \chi_0 P(\partial P):(\alpha^{1*})^2:\big) \\
&\quad - 8\delta_{r,0}P^3(\partial P):(\alpha^{1*})^2: - 4\delta_{r,0}P^4:(\alpha^{1*})^2:\lambda - 4\delta_{r,0}P^4:\alpha^{1*}(\partial\alpha^{1*}): \\
&\quad + 2P^4:\alpha\alpha^*(\alpha^{1*})^2: + P^4\beta(\alpha^{1*})^2 + \chi_0(2P^3(\partial P):(\alpha^{1*})^2: + 2P^4:\alpha^{1*}(\partial\alpha^{1*}): \\
&\quad + P^4:(\alpha^{1*})^2:\lambda) + 2P^2\big(-:\alpha\alpha^*(\alpha^*)^2: + 2:\alpha^1\alpha^{1*}(\alpha^*)^2:\big) \\
&\quad - 4\delta_{r,0}\left(2P(\partial P):(\alpha^*)^2: + P^2:\alpha^*\partial\alpha^*: + P^2:(\alpha^*)^2:\lambda\right) + 2P^2:\beta^1\alpha^*\alpha^{1*}: \\
&\quad + 2\chi_+\big(2P^2(\partial P):\alpha^*\alpha^{1*}: + P^3:(\partial\alpha^*)\alpha^{1*}: \\
&\qquad\qquad + P^3:\alpha^*(\partial\alpha^{1*}): + P^3:\alpha^*\alpha^{1*}:\lambda\big) \\
&\quad + 2P^2\big(-P^2:\alpha\alpha^*(\alpha^{1*})^2: + 2P^2:\alpha\alpha^*(\alpha^{1*})^2: - 2P^2:\alpha\alpha^*(\alpha^{1*})^2: \\
&\qquad - 2\delta_{r,0}P^2:(\alpha^{1*})^2:\lambda - 2\delta_{r,0}P^2:(\partial\alpha^{1*})\alpha^{1*}: - 4\delta_{r,0}P(\partial P):(\alpha^{1*})^2:\big) \\
&\quad - P^2:\beta(\alpha^*)^2: - 2\chi_+ P^3:\alpha^{1*}\alpha^*:\lambda - 4\chi_+ P^2(\partial P):\alpha^{*1}\alpha^*: \\
&\quad - 2\chi_+ P^3:\alpha^*(\partial\alpha^{1*}): - P^4:\beta(\alpha^{1*})^2: - 2P^4\kappa_0:(\alpha^{1*})^2:\lambda \\
&\quad - 2P^4\kappa_0:\alpha^{1*}(\partial\alpha^{1*}): - 4\kappa_0 P^3(\partial P):(\alpha^{1*})^2: \\
&\quad + \chi_0\big(P^2:(\alpha^*)^2:\lambda + 2P(\partial P):(\alpha^*)^2: + P^4:(\alpha^{1*})^2:\lambda - 2P^3(\partial P):(\alpha^{1*})^2:\big).
\end{aligned}
$$

$$
\begin{aligned}
&= \chi_0 P(\partial P) :\alpha^{*2}: \\
&\quad + P^2\big(-4\delta_{r,0}:\alpha^{*2}:\lambda - 4\delta_{r,0}:\alpha^*(\partial\alpha^*): + \chi_0(\alpha^*)^2\lambda + 2\chi_0:\alpha^*(\partial\alpha^*):\big) \\
&\quad - 2\kappa_0(P(\partial P):(\alpha^*)^2: + P^2:(\alpha^*)^2:\lambda + P^2:\alpha^*\partial\alpha^*:) \\
&\quad - \chi_0 P(\partial P):(\alpha^*)^2: - \chi_0 P^3(\partial P):(\alpha^{1*})^2: \\
&\quad + \chi_0 P^3(\partial P):(\alpha^{1*})^2: \\
&\quad - 8\delta_{r,0}P^3(\partial P):(\alpha^{1*})^2: - 4\delta_{r,0}P^4:(\alpha^{1*})^2:\lambda - 4\delta_{r,0}P^4:\alpha^{1*}(\partial\alpha^{1*}): \\
&\quad + \chi_0(2P^3(\partial P):(\alpha^{1*})^2: + 2P^4:\alpha^{1*}(\partial\alpha^{1*}): + P^4:(\alpha^{1*})^2:\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&-4\delta_{r,0}\big(2P(\partial P):(\alpha^*)^2:+P^2:\alpha^*\partial\alpha^*:+P^2:(\alpha^*)^2:\lambda\big)\\
&+2P^2\big(-2\delta_{r,0}P^2:(\alpha^{1*})^2:\lambda-2\delta_{r,0}P^2:(\partial\alpha^{1*})\alpha^{1*}:-4\delta_{r,0}P(\partial P):(\alpha^{1*})^2:\big)\\
&-2P^4\kappa_0:(\alpha^{1*})^2:\lambda-2P^4\kappa_0:\alpha^{1*}(\partial\alpha^{1*}):-4\kappa_0P^3(\partial P):(\alpha^{1*})^2:\\
&+\chi_0\big(-P^2:(\alpha^*)^2:\lambda-2P(\partial P):(\alpha^*)^2:-P^4:(\alpha^{1*})^2:\lambda-2P^3(\partial P):(\alpha^{1*})^2:\big)\\
&+2\chi_+P^2(\partial P):\alpha^*\alpha^{1*}:\\
&+4\chi_+P^2(\partial P):\alpha^*\alpha^{1*}:+2\chi_+P^3:(\partial\alpha^*)\alpha^{1*}:+2\chi_+P^3:\alpha^*(\partial\alpha^{1*}):\\
&-4\chi_+P^2(\partial P):\alpha^{*1}\alpha^*:-2\chi_+P^3:\alpha^*(\partial\alpha^{1*}):\\
&+P^2\big(-2\chi_+(\partial P):\alpha^*\alpha^{1*}:-2\chi_+P:(\partial\alpha^*)\alpha^{1*}:\big)
\end{aligned}
$$

$$
\begin{aligned}
=\;&\chi_0P(\partial P):\alpha^{*2}:+\chi_0P^2:(\alpha^*)^2:\lambda+2\chi_0P^2:\alpha^*(\partial\alpha^*):+\chi_0P^3(\partial P):(\alpha^{1*})^2:\\
&-\chi_0P(\partial P):(\alpha^*)^2:-\chi_0P^3(\partial P):(\alpha^{1*})^2:\\
&+\chi_0\big(-P^2:(\alpha^*)^2:\lambda-2P(\partial P):(\alpha^*)^2:-P^4:(\alpha^{1*})^2:\lambda-2P^3(\partial P):(\alpha^{1*})^2:\big)\\
&+\chi_0(2P^3(\partial P):(\alpha^{1*})^2:+2P^4:\alpha^{1*}(\partial\alpha^{1*}):+P^4:(\alpha^{1*})^2:\lambda)\\
&-4\delta_{r,0}P^2:\alpha^{*2}:\lambda-4\delta_{r,0}P^2:\alpha^*(\partial\alpha^*):-8\delta_{r,0}P^3(\partial P):(\alpha^{1*})^2:\\
&-4\delta_{r,0}P^4:(\alpha^{1*})^2:\lambda-4\delta_{r,0}P^4:\alpha^{1*}(\partial\alpha^{1*}):\\
&-4\delta_{r,0}\big(2P(\partial P):(\alpha^*)^2:+P^2:\alpha^*\partial\alpha^*:+P^2:(\alpha^*)^2:\lambda\big)\\
&-4P^2\delta_{r,0}\big(P^2:(\alpha^{1*})^2:\lambda+P^2:(\partial\alpha^{1*})\alpha^{1*}:+2P(\partial P):(\alpha^{1*})^2:\big)\\
&-2\kappa_0(P(\partial P):(\alpha^*)^2:+P^2:(\alpha^*)^2:\lambda+P^2:\alpha^*\partial\alpha^*:)\\
&-2P^4\kappa_0:(\alpha^{1*})^2:\lambda-2P^4\kappa_0:\alpha^{1*}(\partial\alpha^{1*}):-4\kappa_0P^3(\partial P):(\alpha^{1*})^2:
\end{aligned}
$$

$$
\begin{aligned}
=\;&2\chi_0\big(P^2:(\alpha^*)^2:\lambda+P^2:\alpha^*(\partial\alpha^*):+P(\partial P):(\alpha^*)^2:\big)\\
&+2\chi_0(P^4:(\alpha^{1*})^2:\lambda+P^4:\alpha^{1*}(\partial\alpha^{1*}):+2P^3(\partial P):(\alpha^{1*})^2:)\\
&-8\delta_{r,0}\big(P^2:(\alpha^*)^2:\lambda+P^2:\alpha^*\partial\alpha^*:+P(\partial P):(\alpha^*)^2:\big)\\
&-8\delta_{r,0}\big(P^4:(\alpha^{1*})^2:\lambda+P^4:(\partial\alpha^{1*})\alpha^{1*}:+2P^3(\partial P):(\alpha^{1*})^2:\big)\\
&-2\kappa_0(P^2:(\alpha^*)^2:\lambda+P^2:\alpha^*\partial\alpha^*:+P(\partial P):(\alpha^*)^2:)\\
&-2\kappa_0\big(P^4:(\alpha^{1*})^2:\lambda+P^4:\alpha^{1*}(\partial\alpha^{1*}):+2P^3(\partial P):(\alpha^{1*})^2:\big)
\end{aligned}
$$

$$=0. \qquad \square$$

7. Jakobsen–Kac realizations

If R is an associative (but not necessarily commutative) algebra over $\mathbb{C}$, then a linear map $\phi : R \to \mathbb{C}$ is called a *trace* on $\mathbb{R}$ if

$$\phi(ab)=\phi(ba) \quad \text{for all } a, b \in R.$$

If ϕ is a trace on R, then the space of matrices

$$\mathfrak{sl}_2(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in R \text{ and } \phi(a+d)=0 \right\}$$

is a Lie algebra under the commutator:

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right] = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} - \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix}. \tag{7-12}$$

Now from the assumption that ϕ is a trace on R, the matrix difference on the right has the property

$$\phi(ae+bg-(ea+fc)+cf+dh-(gb+hd))=0.$$

The subspace

$$\mathfrak{b} := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a,b,d \in R \text{ and } \phi(a+d)=0 \right\}$$

is also clearly a subalgebra of $\mathfrak{sl}_2(R)$. Let $\mathbb{C}_\phi = \mathbb{C}\mathbf{v}_\phi$ be the one-dimensional $\mathfrak{b}_+$ module defined by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mathbf{v}_\phi := \phi(a)\mathbf{v}_\phi.$$

One can to check that this is a representation of $\mathfrak{b}_+$ using that ϕ is a trace on R. Thus one can define an induced module

$$M(\phi) := U(\mathfrak{sl}_2(R)) \otimes_{U(\mathfrak{b}_+)} C_\phi.$$

If R is a commutative ring with basis $\{a_\beta\}_{\beta\in B}$ with structure constants $c^\gamma_{\alpha\beta}$ such that $a_\alpha a_\beta = c^\gamma_{\alpha\beta} a_\gamma$, then $M(\phi)$ has a basis in elements of the form

$$(a_{\alpha_1} f)\cdots(a_{\alpha_n} f).v_\phi, \quad \text{where } af := \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}.$$

Jakobsen and Kac [1985] realized the representation $M(\phi)$ of $\mathfrak{sl}(2,R)$ on the space of polynomials $\mathbb{C}[x_\beta \mid \beta\in B]$; it is given by

$$\begin{aligned} \rho(a_{\alpha_0} f) &= x_{\alpha_0}, \\ \rho(a_{\alpha_0} h) &= -2c^\gamma_{\alpha_0\alpha} x_\gamma \frac{\partial}{\partial x_\alpha} + \phi(a_{\alpha_0}), \\ \rho(a_{\alpha_0} e) &= -c^\gamma_{\alpha_0\alpha} c^\delta_{\gamma\beta} x_\delta \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta} + \phi(a_\gamma) c^\gamma_{\alpha_0\alpha} \frac{\partial}{\partial x_\alpha}. \end{aligned}$$

If $R=\mathbb{C}[t,t^{-1},u \mid u^2=t^2-2bt+1]$, then $\{a_n=t^n, a^1_n := t^n u \mid n\in\mathbb{Z}\}$ is a basis, where

$$a_m a_n = a_{m+n}, \qquad a_m a^1_n = a^1_{m+n}, \qquad a^1_m a^1_n = a_{m+n+2} - 2b a_{m+n+1} + a_{m+n}.$$

Define a representation of $\mathfrak{sl}(2, R)$ on $\mathbb{C}[x_n, x_m^1 \mid m, n \in \mathbb{Z}]$ by

$$\begin{aligned}
\rho(t^m f) &= x_m, \\
\rho(t^m u f) &= x_m^1, \\
\rho(t^m h) &= -2\sum_p \big(x_{m+p}\partial_{x_p} + x'_{m+p}\partial_{x'_p}\big) + \phi(t^n), \\
\rho(t^m u h) &= -2\sum_p \big(x'_{m+p}\partial_{x_p} + \big(x_{m+p+2} - 2x_{m+p+1} + x_{m+p}\big)\partial_{x'_p}\big) + \phi(t^n u), \\
\rho(t^m e) &= -\sum_{n,q}\big(x_{m+n+q}\partial_{x_n}\partial_{x_q} + x'_{m+n+q}\partial_{x_n}\partial_{x'_q} + x'_{m+n+q}\partial_{x'_n}\partial_{x_q}\big) \\
&\quad - \sum_{n,q}\big(x_{m+n+q+2}\partial_{x'_n}\partial_{x'_q} - 2b x_{m+n+q+1}\partial_{x'_n}\partial_{x'_q} + x_{m+n+q}\partial_{x'_n}\partial_{x'_q}\big) \\
&\quad + \sum_p \big(\phi(t^{m+p})\partial_{x_p} + \phi(t^{m+p}u)\partial_{x'_p}\big), \\
\rho(t^m u e) &= -\sum_{n,q} x'_{m+n+q}\partial_{x_n}\partial_{x_q} \\
&\quad - \sum_{n,q}\big(x_{m+n+q+2} - 2b x_{m+n+q+1} + x_{m+n+q}\big)\partial_{x_n}\partial_{x'_q} \\
&\quad - \sum_{n,q}\big(x_{m+n+q+2} - 2b x_{m+n+q+1} + x_{m+n+q}\big)\partial_{x'_n}\partial_{x_q} \\
&\quad - \sum_{n,q}\big(x'_{m+n+q+2} - 2b x'_{m+n+q+1} + x'_{m+n+q}\big)\partial_{x'_n}\partial_{x'_q} \\
&\quad + \sum_p \big(\phi(t^{m+p}u)\partial_{x_p} + \phi(t^{m+p}(t^2 - 2bt - 1))\partial_{x'_p}\big).
\end{aligned}$$

One can check that, up to a change in sign, Jakobsen and Kac's representation is a quotient of the representation that we have constructed in Theorem 6.1 for the universal central extension of $\mathfrak{sl}_2(R)$ when $r = 0$.

8. Further comments

Considering the elliptic affine Lie algebras,

$$\mathfrak{sl}(2, R) \oplus (\Omega_R/dR), \quad \text{where } R = \mathbb{C}[x, x^{-1}, y \mid y^2 = 4x^3 - g_2 x - g_3].$$

Bremner [1994] has also explicitly described the universal central extension of this algebra in terms of Pollaczek polynomials. Essentially the same algebras appear in recent work of A. Fialowski and M. Schlichenmaier [2005; 2006]. Together with André Bueno and Vyacheslav Futorny, the author is currently investigating whether the methods in the work above will also provide boson-type realizations of this elliptic Lie algebra or other Lie algebras whose coordinate ring is the coordinate ring of an algebraic curve. We also plan to use the above construction to help elucidate the structure of these representations of a four point algebra.

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