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Virasoro Action on Imaginary Verma Modules and the Operator Form of the KZ-Equation

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Abstract. We define the Virasoro algebra action on imaginary Verma modules for affine $\mathfrak{sl}(2)$ and construct an analogue of the Knizhnik–Zamolodchikov equation in the operator form. Both these results are based on a realization of imaginary Verma modules in terms of sums of partial differential operators.

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1. Introduction

The root system of an affine Kac–Moody algebra has a standard partition into positive and negative roots. Corresponding to this partition is a standard Borel subalgebra, from which one may induce the standard Verma modules. However, it is possible to take other closed partitions of the root system, and form the corresponding non-standard Borel subalgebras. For finite-dimensional simple Lie algebras, we discover nothing new, but for affine Kac–Moody algebras, the induced Verma-type modules typically contain both finite and infinite-dimensional weight spaces. The classification of closed subsets of the root system for affine Kac–Moody algebras was obtained by Jakobsen and Kac [12], and by Futorny [10,11]. In this paper, we are concerned with the affine Lie algebra of type $A_1^{(1)}$. In this case, the only non-standard modules of Verma-type are the *imaginary Verma modules*. Their structure was fully understood in [2,7,10].

In [3], a realization for imaginary Verma modules was constructed for type A in terms of sums of partial differential operators. It was then generalized to other partitions of the root system of type A [4,5]. These realizations (which we called *imaginary Wakimoto modules*) generically are isomorphic to imaginary Verma modules and this resembles the connection between the standard Verma and Wakimoto modules.

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In this paper, we proceed with the study of the properties of imaginary Wakimoto modules in the $\mathfrak{sl}(2)$ case. In particular, we discover a rather unexpected property: these modules admit a Virasoro algebra action resembling very much a vertex operator algebra-like structure. This is unexpected as we do not know how to obtain a Sugawara type construction for the action of the Virasoro algebra on an imaginary Verma module. As our realizations are not in terms in fields but rather formal distributions (and not fields), they are not modules for the affine VOA of $A_1^{(1)}$. In our previous work, we showed that imaginary Verma modules and imaginary Wakimoto realizations are generically isomorphic (i.e. when the central charge is not zero), and so the action of the Virasoro algebra on the imaginary Wakimoto module can be transported over to an action on the imaginary Verma module. Furthermore, we construct the intertwining operators from an imaginary Verma module to the tensor product of an imaginary Verma module with a finite dimensional module and obtain an analogue of the operator form of the Knizhnik–Zamolodchikov equations. Our proof of this later theorem in the setting of the imaginary Wakimoto module differs in essential ways from what appears in the literature for the standard Wakimoto realization [6].

2. Preliminaries

Let $e:=E_{12}$, $f:=E_{21}$ and $h:=E_{11}-E_{22}$ the usual basis for $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$, where E_{ij} is the standard basis for $\mathfrak{gl}(2,\mathbb{C})$. We have

$$\mathfrak{g}=\mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^- \tag{2.1}$$

where $\mathfrak{n}^+=\mathbb{C}e$, $\mathfrak{h}=\mathbb{C}h$ and $\mathfrak{n}^-=\mathbb{C}f$.

Now define

$$\hat{\mathfrak{g}}=\hat{\mathfrak{sl}}(2,\mathbb{C}):=(\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}])\oplus\mathbb{C}c=\hat{\mathfrak{n}}^+\oplus\hat{\mathfrak{h}}\oplus\hat{\mathfrak{n}}^- \tag{2.2}$$

where $\hat{\mathfrak{n}}^+=\mathfrak{n}^+\otimes\mathbb{C}[t,t^{-1}]\oplus\mathfrak{h}\otimes t\mathbb{C}[t]$, $\hat{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{C}c$ and $\hat{\mathfrak{n}}^-=\mathfrak{n}^-\otimes\mathbb{C}[t,t^{-1}]\oplus\mathfrak{h}\otimes t^{-1}\mathbb{C}[t^{-1}]$. The algebra $\hat{\mathfrak{g}}$ has generators e_m, f_m, h_m, $m\in\mathbb{Z}$, and central element c with the product

$$[X_m,Y_n]=[X,Y]_{m+n}+\delta_{m+n,0}m(X|Y)c,$$

where $X_m:=X\otimes t^m$ for $X,Y\in\mathfrak{g}$ and $m\in\mathbb{Z}$ and

$$(X|Y)=\mathrm{tr}(XY)$$

is the Killing form. Now define

$$\tilde{\mathfrak{g}}:=\hat{\mathfrak{g}}\oplus\mathbb{C}d=\hat{\mathfrak{n}}^+\oplus\tilde{\mathfrak{h}}\oplus\hat{\mathfrak{n}}^-, \tag{2.3}$$

where $\tilde{\mathfrak{h}}=\hat{\mathfrak{h}}\oplus\mathbb{C}d$, $[d,X_m]=mX_m$ and $[d,c]=0$.

A subalgebra $\mathfrak{b}_{\text{nat}} = \tilde{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+$ is the *natural Borel subalgebra* introduced by Jakobsen and Kac [12].

In $\tilde{\mathfrak{h}}$, we have the dual space $\tilde{\mathfrak{h}}^*$ such that

$$\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta \tag{2.4}$$

where for all $h \in \mathfrak{h}$ we have

$$\Lambda_0(c) = 1, \quad \Lambda_0(d) = \Lambda_0(h) = 0,$$
$$\delta(d) = 1, \quad \delta(c) = \delta(h) = 0.$$

Let Δ denote the root system of $\tilde{\mathfrak{g}}$, and let $\{\alpha_0, \alpha_1\}$ be a basis for Δ. Then $\delta = \alpha_0 + \alpha_1$ is the indivisable positive imaginary root and

$$\Delta = \{\pm\alpha_1 + n\delta \mid n \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}\backslash\{0\}\}.$$

2.1. IMAGINARY VERMA MODULES

Let $\Lambda \in \tilde{\mathfrak{h}}^*$ be a weight and let $\mathbb{C}_\Lambda$ be a one-dimensional representation of $\tilde{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+$ such that $\mathbb{C}_\Lambda$ is generated by a vector v_Λ, $\hat{\mathfrak{n}}^+$ acts by zero and $\tilde{h}v_\Lambda = \Lambda(\tilde{h})v_\Lambda$, for all $\tilde{h} \in \tilde{\mathfrak{h}}$.

The *imaginary Verma module* is defined as follows [2,10]:

$$V_\Lambda = \text{Ind}_{\tilde{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+}^{\tilde{\mathfrak{g}}} \mathbb{C}_\Lambda. \tag{2.5}$$

When a representation V of $\tilde{\mathfrak{g}}$ is generated by a vector v_Λ of weight Λ that is annihilated by $\hat{\mathfrak{n}}^+$, we say that V has the (imaginary) highest weight Λ. For any weight Λ we can write

$$\Lambda = \lambda + \kappa\Lambda_0 - \Delta\delta, \quad \lambda \in \mathfrak{h}^* \text{ and } \kappa, \Delta \in \mathbb{C}, \tag{2.6}$$

where by (2.4), we have $\lambda(h) = \Lambda(h)$, $\kappa = \Lambda(c)$ and $\Delta = -\Lambda(d)$. In the following we will only consider weights Λ such that

$$\Delta = \Delta(\lambda) = \frac{\langle \lambda, \lambda + 2\rho \rangle}{2(k + h^v)} \tag{2.7}$$

and for this Λ we will denote $V_{\lambda,\kappa}$ instead of V_Λ. When κ and Δ are fixed, we just call $V_{\lambda,\kappa}$ the imaginary highest weight module with imaginary highest weight λ.

Let $H = (\mathfrak{h} \otimes \mathbb{C}[t]t) \oplus (\mathfrak{h} \otimes \mathbb{C}[t^{-1}]t^{-1}) \oplus \mathbb{C}c$ be a Heisenberg subalgebra of $\tilde{\mathfrak{g}}$. Let V_λ be the Verma H-module generated by v_Λ. Then

$$V_{\lambda,\kappa} = \text{Ind}_{H + \mathfrak{b}_{\text{nat}}}^{\tilde{\mathfrak{g}}} V_\lambda, \tag{2.8}$$

where d acts on V_λ by multiplication by $-\Delta(\lambda)$. In $V_{\lambda,\kappa}$ we have a $\mathbb{Z}$-grading:

$$V_{\lambda,\kappa} = \bigoplus_{n \geq 0} V_{\lambda,\kappa}[-n].$$

Observe that $V_{\lambda,\kappa}[-n]$ is the eigenspace of d with the eigenvalue $-n-\Delta(\lambda)$, each $V_{\lambda,\kappa}[-n]$ is an H-module and $V_{\lambda,\kappa}[0]=V_\lambda$.

Let V be a $\mathfrak{g}$-module and fix $z\in\mathbb{C}^*$. For any polynomial $P(t)\in\mathbb{C}[t]$ and $x\in\hat{\mathfrak{g}}, u\in V$ set

$$x\otimes P(t)\cdot u=P(z)xu, \quad cu=0. \tag{2.9}$$

This is called the evaluation representation of $\hat{\mathfrak{g}}$ and we denote it by $V(z)$. Unfortunately, we cannot extend it to an action of $\tilde{\mathfrak{g}}$. Let Δ be a convenient complex number (that we will define in Proposition 5.8) and z a formal variable. Consider the space

$$z^{-\Delta}V[z,z^{-1}]=V\otimes z^{-\Delta}\mathbb{C}[z,z^{-1}].$$

This space has a module structure for $\tilde{\mathfrak{g}}$, where d acts by $z\frac{\partial}{\partial_z}$. For any $z_0\neq 0$, we have the evaluation map

$$\epsilon_{z_0}: z^{-\Delta}V[z,z^{-1}]\longrightarrow V(z_0) \tag{2.10}$$

that is a $\hat{\mathfrak{g}}$-epimorphism. Denote by

$$V_{\lambda,\kappa}\hat{\otimes}V(z_0) \tag{2.11}$$

the completed tensor product generated by all infinite expressions of the form $\sum_{i=1}^{\infty} w_i\otimes v_i$, where $w_i\in V_{\lambda,\kappa}$, is a homogeneous vector, $\{\text{degree}(w_i)\}\longrightarrow -\infty$ and $v_i\in V(z_0)$, $\forall i\in\mathbb{N}^*$.

2.2. FORMAL DISTRIBUTIONS

DEFINITION 2.1. A formal distribution is an expression of the form

$$a(z,w,\ldots)=\sum_{m_1,m_2,\ldots\in\mathbb{Z}} a_{m_1,m_2,\ldots}z^{m_1}w^{m_2}\cdots$$

where the $a_{m_1,m_2,\ldots}$ lie in some fixed vector space V.

If $A(z)$ is a field, then we set

$$A(z)_-:=\sum_{m\geq 0}A_m z^{-m-1}, \quad \text{and} \quad A(z)_+:=\sum_{m<0}A_m z^{-m-1}. \tag{2.12}$$

The normal ordered product of two formal distributions $A(z)$ and $B(w)$ is defined by

$$:A(z)B(w):=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}:A_mB_n:z^{-m-1}w^{-n-1}=A(z)_+B(w)+B(w)A(z)_-.$$

3. Realizations

3.1. REALIZATIONS

We now give a description of how our work is related to other people's research. To get a realization of the imaginary Verma module for $\hat{sl}(2)$, Bernard and Felder used heuristically a Borel–Weil type of construction [1]. The realization given in this later paper is in terms of formal distributions and not all are free fields (see Section 3.3 below for a description of the required operators that are not fields). Note that these operators do not take an element in the ring of polynomials in countably many variables to another polynomial, but to an infinite sum lying in some type of formal completion of the ring of polynomials in infinitely many variables. We do not try to describe what this completion should be, but get around this by conjugating the Borel–Weil type of construction by two anti-automorphisms. Then both the domain and codomain of the new operators are the same ring of countably many variables. We describe this below.

Let $\hat{B}_-$ be the Borel subgroup of the loop group $\hat{\mathrm{SL}}(2)$ corresponding to a Borel subalgebra $\mathfrak{b}_{\mathrm{nat}}$. Then $\hat{B}_-$ consists of the elements

$$\exp\left(\sum_{n\in\mathbb{Z}} x_n e_n\right)\exp\left(\sum_{m>0} y_m h_m\right),$$

where x_n, y_m are coordinate functions. One does not claim that the above expressions form a group, but use these expressions to obtain formulae on how the Lie algebra should act. In the end, we will need to check that defining relations for the algebra are satisfied. Now, consider a one-dimensional representation $\chi : \hat{B}_- \to \mathbb{C}$, where c acts by scalar K, h acts by scalar J ($J/2$ is called the *spin*) and all other elements act trivially. Then one can construct a line bundle over $\hat{\mathrm{SL}}(2)/\hat{B}_-$ by taking a fiber product

$$\mathcal{L}_\chi = \hat{\mathrm{SL}}(2)\times_{\hat{B}_-}\mathbb{C}$$

and a map $g : \hat{\mathrm{SL}}(2)\times_{\hat{B}_-}\mathbb{C}\to\hat{\mathrm{SL}}(2)/\hat{B}_-$ such that $(x,z)\mapsto x\hat{B}_-$. Differentiating the action of the group $\hat{\mathrm{SL}}(2)$ acts on the sections of the line bundle to an action of the Lie algebra $\mathfrak{g}$ and applying two anti-involutions

$$e_n \leftrightarrow -f_{-n}, \quad h_n \leftrightarrow h_{-n}, \quad c\leftrightarrow c$$

and

$$x_{-n}\leftrightarrow \partial x_n, \quad y_k\leftrightarrow -\partial y_k$$

we obtain the following realization of $\mathfrak{g}$ in the Fock space $\mathbb{C}[x_m, m\in\mathbb{Z}]\otimes\mathbb{C}[y_n, n>0]$:

$$f_n\mapsto x_n, \quad h_n\mapsto -2\sum_{m\in\mathbb{Z}} x_{m+n}\partial x_m + \delta_{n<0}y_{-n} + \delta_{n>0}2nK\partial y_n + \delta_{n,0}J,$$

$$e_n \mapsto -\sum_{m,k\in\mathbb{Z}} x_{k+m+n}\partial x_k \partial x_m + \sum_{k>0} y_k \partial x_{-k-n} + 2K\sum_{m>0} m\partial y_m \partial x_{m-n} + (Kn+J)\partial x_{-n}.$$

This module is irreducible if and only if $K\neq 0$. If we let $K=0$ and quotient out the submodule generated by $y_m, m>0$ then we obtain what is usually called *the first realization* of $\hat{sl}(2)$.

This quotient is irreducible if and only if $J\neq 0$ (cf. [10]). This has been generalized for all affine Lie algebras in [3].

3.2. WAKIMOTO MODULES

We recall the *second realization* of $\hat{sl}(2)$, that is Wakimoto module construction [14]. Set

$$a^*(z) := \sum_{m\in\mathbb{Z}} a^*_m z^{-m}, \quad a(z) := \sum_{m\in\mathbb{Z}} a_m z^{-m-1}, \quad b(z) := \sum_{m\in\mathbb{Z}} b_m z^{-m-1}.$$

Let now

$$a_n = \begin{cases} x_n, & n<0 \\ \partial x_n & n\geq 0, \end{cases} \quad a^*_n = \begin{cases} x_{-n}, & n\leq 0 \\ -\partial x_{-n}, & n>0, \end{cases} \quad b_m = \begin{cases} m\partial y_m, & m\geq 0 \\ y_{-m}, & m<0. \end{cases}$$

Here $[a_n, a^*_m] = \delta_{n+m,0}$ and $[b_n, b_m] = n\delta_{n+m,0}$.

THEOREM 3.1 ([14]). *The formulas*

$$c\mapsto K, \quad e(z)\mapsto a(z), \quad h(z)\mapsto -2:a^*(z)a(z):+b(z),$$
$$f(z)\mapsto -:a^*(z)^2 a(z):+K\partial_z a^*(z) + a^*(z)b(z)$$

define the second (*free field*) realization *of the affine* $sl(2)$ *acting on the space* $\mathbb{C}[x_n, n\in\mathbb{Z}]\otimes\mathbb{C}[y_m, m>0]$.

Thus, one can see that this realization is truly in terms of fields, i.e. given any $v\in\mathbb{C}[x_n, n\in\mathbb{Z}]\otimes\mathbb{C}[y_m, m>0]$, there is an k such that $f_l v=0, e_l v=0$ and $h_l v=0$ for $l\geq k$.

These modules are celebrated *Wakimoto modules*. They were defined for an arbitrary affine Lie algebra by Feigin and Frenkel [8,9]. Generically, Wakimoto modules are isomorphic to Verma modules.

3.3. IMAGINARY VERMA MODULES REVISITED

We will re-write the first realization in terms of formal distributions.

Let $\hat{\mathfrak{a}}$ be the infinite dimensional Heisenberg algebra with generators a_m, a^*_m, and $\mathbf{1}, m\in\mathbb{Z}$, subject to the relations

$$[a_m, a_n] = [a^*_m, a^*_n] = 0,$$
$$[a_m, a^*_n] = \delta_{m+n,0}\mathbf{1},$$
$$[a_m, \mathbf{1}] = [a^*_m, \mathbf{1}] = 0.$$

This algebra acts on $\mathbb{C}[x_m | m\in\mathbb{Z}]$ by

$$a_m \mapsto x_m, \quad a_m^* \mapsto -\partial/\partial x_{-m}$$

and $\mathbf{1}$ acts as an identity. Hence we have an $\hat{\mathfrak{a}}$-module generated by v such that

$$a_m^* v = 0, \quad m\in\mathbb{Z}.$$

Observe that $a(z)$ is not a field whereas $a^*(z)$ is a field. We will call $a(z)$ (resp. $a^*(z)$) a *pure creation* (resp. *annihilation*) *operator*. Set

$$\begin{aligned} a(z)_+ &= a(z), \quad a(z)_- = 0, \\ a^*(z)_+ &= 0, \quad a^*(z)_- = a^*(z). \end{aligned}$$

Define

$$e(z) = \sum_{n\in\mathbb{Z}} e_n z^{-n-1}, \quad f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n-1}, \quad h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1}.$$

THEOREM 3.2 ([5]). *Let $\lambda\in\mathfrak{H}^*$, $\gamma\in\mathbb{C}$. The generating functions*

$$\begin{aligned} &f(z)\mapsto a(z), \\ &h(z)\mapsto 2 : a(z)a^*(z) : + b(z), \\ &e(z)\mapsto : a^*(z)^2 a(z) : - a^*(z)b(z) - (1-\gamma^2)\partial_z a^*(z), \\ &c\mapsto \gamma^2 - 1, \end{aligned} \tag{3.1}$$

define a representation $\rho : \hat{\mathfrak{sl}}(2)\to\mathfrak{gl}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$, where $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_m, m\in\mathbb{Z}]$, $\mathbb{C}[\mathbf{y}] = \mathbb{C}[y_n, n>0]$.

In other words, the first and the second realizations can be obtained from the same (Wakimoto) formulas by taking different representations of the Heisenberg algebra $\hat{\mathfrak{a}}$.

We will denote this $\hat{\mathfrak{sl}}(2)$-module by $W_{\lambda,\kappa}$, where $\kappa = \gamma^2 - 1$ and call it *imaginary Wakimoto module*. We can make $W_{\lambda,\kappa}$ into a $\tilde{\mathfrak{sl}}(2,\mathbb{C})$-module by defining

$$d\cdot a_{n_1}\cdots a_{n_k} b_{-m_1}\cdots b_{-m_l}\cdot 1 = \left(\sum_{j=1}^{k} n_k - \sum_{i=1}^{l} m_i - \Delta(\lambda)\right) a_{n_1}\cdots a_{n_k} b_{-m_1}\cdots b_{-m_l}\cdot 1$$

so that in particular $d\cdot 1 = -\Delta(\lambda)\cdot 1$. From now on, we will let $w_{\lambda,\kappa}$ denote the generator 1 in $W_{\lambda,\kappa}$.

THEOREM 3.3 ([5]). *Fix Λ as in (2.6). For $\kappa = \gamma^2 - 1 \neq 0$, the map $\Psi_{\lambda,\kappa} : V_{\lambda,\kappa} \longrightarrow W_{\lambda,\kappa}$ given by*

$$\Psi_{\lambda,\kappa}(f_{n_1}\cdots f_{n_k} h_{-m_1}\cdots h_{-m_l}\cdot v_{\lambda,\kappa}) = a_{n_1}\cdots a_{n_k} b_{-m_1}\cdots b_{-m_l}\cdot w_{\lambda,\kappa}$$

is an isomorphism of $\tilde{\mathfrak{sl}}(2,\mathbb{C})$-modules.

4. The Virasoro Action

Our first goal is to define an action of the Virasoro algebra on imaginary Verma module. We follow the construction done in [13]. The Virasoro algebra is defined to have basis $L_n, n\in\mathbb{Z}$ and center spanned by c with relations

$$[L_m, L_n]:=(m-n)L_{m+n}+\frac{1}{12}(m^3-m)\delta_{m+n,0}c.$$

Define

$$\bar{L}_k:=\sum_{j\in\mathbb{Z}}(j-k)a_j a^*_{k-j}$$

which is to be viewed as acting on the imaginary Wakimoto module. Now we have

LEMMA 4.1.

$$[a_k, \bar{L}_n]=ka_{k+n}, \quad [a^*_k, \bar{L}_n]=(k+n)a^*_{k+n}. \tag{4.1}$$

Define $\psi:\mathbb{R}\to\{0, 1\}$ by

$$\psi(x):=\begin{cases}1 & \text{if } |x|\le 1;\\ 0 & \text{!if } |x|>1.\end{cases}$$

Then set

$$\bar{L}_k(\epsilon):=\sum_{j\in\mathbb{Z}}(j-k)a_j a^*_{k-j}\psi(\epsilon j).$$

Now $\bar{L}_n(\epsilon)$ has only a finite number of summands and as $\epsilon\to 0$, $\bar{L}_n(\epsilon)\to\bar{L}_n$.

Proof.

$$[a_k, \bar{L}_n(\epsilon)]=\sum_{j\in\mathbb{Z}}(j-n)[a_k, a_j a^*_{n-j}]\psi(\epsilon j)=ka_{k+n}\psi(\epsilon(n+k))$$
$$[a^*_k, \bar{L}_n(\epsilon)]=\sum_{j\in\mathbb{Z}}(j-n)[a^*_k, a_j a^*_{n-j}]\psi(\epsilon j)=(k+n)a^*_{k+n}\psi(\epsilon(-k))$$

□

LEMMA 4.2. $[\bar{L}_m, \bar{L}_n]=(m-n)\bar{L}_{m+n}$.

Proof.

$$[\bar{L}_m(\epsilon), \bar{L}_n]=\sum_{j\in\mathbb{Z}}(j-m)[a_j a^*_{m-j}, \bar{L}_n]\psi(\epsilon j)$$
$$=\sum_{j\in\mathbb{Z}}(j-m)[a_j, \bar{L}_n]a^*_{m-j}\psi(\epsilon j)+\sum_{j\in\mathbb{Z}}(j-m)a_j[a^*_{m-j}, \bar{L}_n]\psi(\epsilon j)$$

$$
\begin{aligned}
&= \sum_{j\in\mathbb{Z}}(j-m)ja_{j+n}a^*_{m-j}\psi(\epsilon j) + \sum_{j\in\mathbb{Z}}(j-m)(m+n-j)a_ja^*_{m+n-j}\psi(\epsilon j) \\
&= \sum_{j\in\mathbb{Z}}(j-n-m)(j-n)a_ja^*_{m+n-j}\psi(\epsilon(j-n)) \\
&\quad + \sum_{j\in\mathbb{Z}}(j-m)(m+n-j)a_ja^*_{m+n-j}\psi(\epsilon j) \\
&= \sum_{j\in\mathbb{Z}}((j-n-m)(j-n)\psi(\epsilon(j-n)) + (j-m)(m+n-j)\psi(\epsilon j))a_ja^*_{m+n-j}.
\end{aligned}
$$

Now for ϵ small enough and for j fixed, we have

$$
\begin{aligned}
&(j-n-m)(j-n)\psi(\epsilon(j-n)) + (j-m)(m+n-j)\psi(\epsilon j) \\
&\quad = (j-n-m)(j-n) + (j-m)(m+n-j) = (m-n)(j-m-n).
\end{aligned}
$$

Hence,

$$
[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}.
$$

□

Let $\bar{L}(z) = \sum_{n\in\mathbb{Z}}\bar{L}_n z^{-n-2}$. We have

$$
\begin{aligned}
[\bar{L}(z), \bar{L}(w)] &= \sum_{k\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}[\bar{L}_k, \bar{L}_q]z^{-k-2}w^{-q-2} \\
&= \sum_{k\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}(k-q)\bar{L}_{k+q}z^{-k-2}w^{-q-2} \\
&= \sum_{l\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}(l-2q)\bar{L}_l z^{-l+q-2}w^{-q-2} \\
&= \left(\sum_{l\in\mathbb{Z}}(l+2)\bar{L}_l z^{-l-3}\right)\left(\sum_{q\in\mathbb{Z}} z^{q+1}w^{-q-2}\right) \\
&\quad + \left(\sum_{l\in\mathbb{Z}}\bar{L}_l z^{-l-2}\right)\left(\sum_{q\in\mathbb{Z}}(-2q-2)z^q w^{-q-2}\right) \\
&= -\partial_z\bar{L}(z)\delta(z/w) + 2\bar{L}(z)\partial_w\delta(z/w) \\
&= \partial_w\bar{L}(w)\delta(z/w) + 2\bar{L}(w)\partial_w\delta(z/w).
\end{aligned}
$$

Let $\mu\in\mathbb{C}$ be fixed. Now define L_k by

$$
L(z) := \sum_{k\in\mathbb{Z}} L_k z^{-k-2} = \bar{L}(z) + \frac{1}{4}:b(z)^2: + \frac{\mu}{2}\partial_z b(z) \tag{4.2}
$$

$$
L(z) := \sum_{k\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}}(j-k)a_ja^*_{k-j} + \frac{1}{4}\sum_{j\in\mathbb{Z}}:b_jb_{k-j}: - \frac{\mu}{2}(k+1)b_k\right)z^{-k-2} \tag{4.3}
$$

where $\sum_{n\in\mathbb{Z}}:b_nb_{m-n}:=\sum_{n>m}b_{m-n}b_n+\sum_{n\leq m}b_nb_{m-n}$ because

$$\begin{aligned}:b(z)b(z):&=b(z)_+b(z)+b(z)b(z)_-\\&=\sum_{m<0,n\in\mathbb{Z}}b_mb_nz^{-m-n-2}+\sum_{m\geq 0,n\in\mathbb{Z}}b_nb_mz^{-m-n-2}\\&=\sum_{m'<n,n\in\mathbb{Z}}b_{m'-n}b_nz^{-m'-2}+\sum_{m'\geq n,n\in\mathbb{Z}}b_nb_{m'-n}z^{-m'-2}\end{aligned}$$

and then as we shall see below, the center of the Virasoro acts by the scalar $\mathcal{c}=6-6\mu^2\in\mathbb{C}$. We have b_i satisfying the relation

$$[b_m,b_p]=2m\delta_{m+p,0},$$

so that

$$\begin{aligned}&[b(z),b(w)]=2\partial_w\delta(z/w),\\&b(z)b(w)=:b(z)b(w):+\frac{2}{(z-w)^2}.\end{aligned}$$

PROPOSITION 4.3. *Take* $b_0v=\lambda v$, $cv=\mathcal{c}v$ *where* $v=w_{\lambda,\kappa}$ *is the vacuum vector. Then*

$$[L(z),L(w)]=\frac{c}{12}\partial_w^3\delta(z/w)+2L(w)\partial_w\delta(z/w)+\partial_wL(w)\delta(z/w).\tag{4.4}$$

Proof. We first calculate using Kac's theorem, Wick's theorem and Taylor's theorem

$$\begin{aligned}:b(z)^2::b(w)^2&=:b(z)^2b(w)^2:+\frac{8}{(z-w)^2}:b(z)b(w):+\frac{8}{(z-w)^4}\\&=:b(z)^2b(w)^2:+\frac{8}{(z-w)^2}:b(w)b(w):+\frac{8}{(z-w)}:b(w)\partial_wb(w):+\frac{8}{(z-w)^4}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left[\frac{1}{4}:b(z)^2:,\frac{1}{4}:b(w)^2:\right]\\&=2\left(\frac{1}{4}\right):b(w)b(w):\partial_w\delta(z/w)+\frac{1}{2}:b(w)\partial_wb(w):+\frac{1}{2}\partial_w^3\delta(z/w).\end{aligned}$$

Next, we have

$$:b(z)^2:b(w)=:b(z)^2b(w):+\frac{4}{(z-w)^2}b(w)+\frac{4}{z-w}\partial_wb(w),$$

so that

$$\begin{aligned}&\left[\frac{1}{4}:b(z)^2:,\frac{\mu}{2}\partial_wb(w)\right]+\left[\frac{\mu}{2}\partial_zb(z),\frac{1}{4}:b(w)^2:\right]\\&=\partial_w\left(\frac{\mu}{2}b(w)\partial_w\delta(z/w)+\frac{\mu}{2}\partial_wb(w)\delta(z/w)\right)+\partial_z\left(\frac{\mu}{2}b(z)\partial_z\delta(z/w)+\frac{\mu}{2}\partial_zb(z)\delta(z/w)\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\mu}{2}\partial_w b(w)\partial_w\delta(z/w) + \frac{\mu}{2}\partial_w^2 b(w)\delta(z/w) + \frac{\mu}{2}b(w)\partial_w^2\delta(z/w) - \frac{\mu}{2}\partial_z^2(b(z)\delta(z/w))\\
&= \mu\partial_w b(w)\partial_w\delta(z/w) + \frac{\mu}{2}\partial_w^2 b(w)\delta(z/w) + \frac{\mu}{2}b(w)\partial_w^2\delta(z/w) - \frac{\mu}{2}\partial_z^2(b(w)\delta(z/w))\\
&= \mu\partial_w b(w)\partial_w\delta(z/w) + \frac{\mu}{2}\partial_w^2 b(w)\delta(z/w) + \frac{\mu}{2}b(w)\partial_w^2\delta(z/w) - \frac{\mu}{2}b(w)\partial_z^2\delta(z/w)\\
&= \mu\partial_w b(w)\partial_w\delta(z/w) + \frac{\mu}{2}\partial_w^2 b(w)\delta(z/w) + \frac{\mu}{2}b(w)\partial_w^2\delta(z/w) - \frac{\mu}{2}b(w)\partial_w^2\delta(z/w)
\end{aligned}
$$

Lastly, we compute $[\frac{\mu}{2}\partial_z b(z), \frac{\mu}{2}\partial_w b(w)] = -\frac{\mu^2}{2}\partial_w^3\delta(z/w)$.

Putting all these calculations together, we get

$$
\begin{aligned}
[L(z), L(w)] &= \partial_w \bar{L}(w)\delta(z/w) + 2\bar{L}(w)\partial_w\delta(z/w) + \frac{1}{2}:b(w)b(w):\partial_w\delta(z/w)\\
&\quad + \frac{1}{2}:b(w)\partial_w b(w): + \frac{1}{2}\partial_w^3\delta(z/w) + \mu\partial_w b(w)\partial_w\delta(z/w) + \frac{\mu}{2}\partial_w^2 b(w)\delta(z/w)\\
&\quad + \frac{\mu}{2}b(w)\partial_w^2\delta(z/w) - \frac{\mu}{2}b(w)\partial_w^2\delta(z/w) - \frac{\mu^2}{2}\partial_w^3\delta(z/w)\\
&= \left(\partial_w\bar{L}(w) + \frac{1}{2}:b(w)\partial_w b(w): + \frac{\mu}{2}\partial_w^2 b(w)\right)\delta(z/w)\\
&\quad + \left(2\bar{L}(w) + \frac{1}{2}:b(w)^2: + \frac{\mu}{2}\partial_w b(w)\right)\partial_w\delta(z/w) + \left(\frac{1}{2} - \frac{\mu^2}{2}\right)\partial_w^3\delta(z/w)\\
&= \partial_w L(w)\delta(z/w) + 2L(w)\partial_w\delta(z/w) + \frac{c}{12}\partial_w^3\delta(z/w).
\end{aligned}
$$

□

Next, we have

PROPOSITION 4.4.

$$
\begin{aligned}
&[a(z), L(w)] = a(w)\partial_w\delta(z/w),\\
&[a^*(z), L(w)] = -\partial_w a^*(w)\delta(z/w),\\
&[b(z), L(w)] = b(w)\partial_w\delta(z/w) + \mu\partial_w^2\delta(z/w).
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&[a(z), L(w)] = [a(z), \bar{L}(w)] = \sum_{k,n\in\mathbb{Z}}[a_k, \bar{L}_n]z^{-k-1}w^{-n-2} = \sum_{k,n\in\mathbb{Z}} k a_{k+n} z^{-k-1}w^{-n-2}\\
&\quad = \sum_{k,n\in\mathbb{Z}} k a_{k+n} z^{-k-1}w^{k-1}w^{-n-k-1} = \sum_{k,n\in\mathbb{Z}} k a_{n-k} z^{k-1}w^{-k-1}w^{-n+k-1}\\
&\quad = a(w)\partial_w\delta(z/w),\\
&[a^*(z), L(w)] = [a^*(z), \bar{L}(w)] = \sum_{k,n\in\mathbb{Z}}[a_k^*, \bar{L}_n]z^{-k}w^{-n-2} = \sum_{k,n\in\mathbb{Z}}(k+n)a_{k+n}^* z^{-k}w^{-n-2}\\
&\quad = \sum_{k,n\in\mathbb{Z}}(k+n)a_{k+n}^* z^{-k}w^{k}w^{-1}w^{-n-k-1}
\end{aligned}
$$

$$= -\partial_w a^*(w)\delta(z/w)$$

and using calculations from Proposition 4.3 we have

$$\begin{aligned}[b(z), L(w)] &= \left[b(z), \frac{1}{4}:b(w)^2: + \frac{\mu}{2}\partial_w b(w)\right] \\ &= \frac{1}{4}(-4\partial_z b(z)\delta(z/w) - 4b(z)\partial_z\delta(z/w)) + \mu\partial_w^2\delta(z/w) \\ &= -\partial_w b(w)\delta(z/w) + b(z)\partial_w\delta(z/w) + \mu\partial_w^2\delta(z/w) \\ &= b(w)\partial_w\delta(z/w) + \mu\partial_w^2\delta(z/w).\end{aligned}$$

□

Note that the above implies that

$$[L_{-1}, a(z)] = \partial_z a(z), \quad [L_{-1}, a^*(z)] = \partial_z a^*(z), \quad [L_{-1}, b(z)] = \partial_z b(z), \tag{4.5}$$

$$[L_0, a(z)] = a(z) + z\partial_z a(z), \quad [L_0, a^*(z)] = z\partial_z a^*(z), \quad [L_0, b(z)] = b(z) + z\partial_z b(z). \tag{4.6}$$

Indeed, for example from the proof we have

$$[L_{-1}, a^*(z)] = -\sum_{k\in\mathbb{Z}}(k-1)a^*_{k-1}z^{-k} = -\sum_{s\in\mathbb{Z}} s a^*_s z^{-s-1} = \partial_z a^*(z).$$

Note also that

$$L(z) = a(z)\partial_z a^*(z) + \frac{1}{2}:b(z)^2: + \mu\partial_z b(z). \tag{4.7}$$

PROPOSITION 4.5.

$$\begin{aligned}[\rho(f(z)), L(w)] &= \rho(f(w))\partial_w\delta(z/w) \\ [\rho(h(z)), L(w)] &= \rho(h(w))\partial_w\delta(z/w) + \mu\partial_w^2\delta(z/w) \\ [\rho(e(z)), L(w)] &= \rho(e(w))\partial_w\delta(z/w) - \mu\partial_w^2(a^*(w)\delta(z/w)).\end{aligned}$$

Proof. Here are the calculations:

$$[\rho(f(z)), L(w)] = [a(z), L(w)] = a(w)\partial_w\delta(z/w) = \rho(f(w))\partial_w\delta(z/w)$$

$$\begin{aligned}[\rho(h(z)), L(w)] &= [2a(z)a^*(z) + b(z), L(w)] \\ &= -2a(z)(\partial_w a^*(w))\delta(z/w) + 2a(w)a^*(w)\partial_z\delta(z/w) + [b(z), L(w)] \\ &= 2a(w)(-\partial_w a^*(w)\delta(z/w) + \partial_w(a^*(w)\delta(z/w)) + [b(z), L(w)] \\ &= 2a(w)a^*(w)\partial_w\delta(z/w) + b(w)\partial_w\delta(z/w) + \mu\partial_w^2\delta(z/w) \\ &= \rho(h(w))\partial_w\delta(z/w) + \mu\partial_w^2\delta(z/w)\end{aligned}$$

$$
\begin{aligned}
[\rho(e(z)), L(w)] &= -[a(z)a^*(z)^2 + a^*(z)b(z) + (1-\gamma^2)\partial_z a^*(z), L(w)] \\
&= -a(w)a^*(z)^2\partial_w\delta(z/w) + 2a(z)a^*(z)\partial_w a^*(w)\delta(z/w) \\
&\quad -(b(w)\partial_w\delta(z/w) + \mu\partial_w^2\delta(z/w))a^*(z) + b(z)\partial_w a^*(w)\delta(z/w) \\
&\quad +(1-\gamma^2)\partial_z(\partial_w a^*(w)\delta(z/w)) \\
&= -a(w)a^*(w)^2\partial_w\delta(z/w) - b(w)a^*(w)\partial_w\delta(z/w) \\
&\quad -\mu\partial_w^2\delta(z/w)a^*(z) + (1-\gamma^2)\partial_w a^*(w)\partial_z\delta(z/w) \\
&= -a(w)a^*(w)^2\partial_w\delta(z/w) - b(w)a^*(z)\partial_w\delta(z/w) \\
&\quad -\mu(a^*(w)\partial_w^2\delta(z/w) + 2\partial_w a^*(w)\partial_w\delta(z/w) + \partial_w^2 a^*(z)\delta(z/w)) \\
&\quad -(1-\gamma^2)\partial_w a^*(w)\partial_w\delta(z/w) \\
&= \rho(e(w))\partial_w\delta(z/w) \\
&\quad -\mu\left(a^*(w)\partial_w^2\delta(z/w) + 2\partial_w a^*(w)\partial_w\delta(z/w) + \partial_w^2 a^*(z)\delta(z/w)\right) \\
&= \rho(e(w))\partial_w\delta(z/w) - \mu\partial_w^2(a^*(w)\delta(z/w))
\end{aligned}
$$

□

COROLLARY 4.6.

$$
\begin{aligned}
&[L_{-1}, \rho(f(z))] = \partial_z\rho(f(z)), \quad [L_{-1}, \rho(e(z))] = \partial_z\rho(e(z)), \\
&[L_{-1}, \rho(h(z))] = \partial_z\rho(h(z)), \quad [L_{-1}, b(z)] = \partial_z b(z), \\
&[L_0, \rho(f(z))] = z\partial_z\rho(f(z)) + \rho(f(z)), \quad [L_0, \rho(e(z))] = z\partial_z\rho(e(z)) + \rho(e(z)), \\
&[L_0, \rho(h(z))] = z\partial_z\rho(h(z)) + \rho(h(z)), \quad [L_0, b(z)] = z\partial_z b(z) + b(z),
\end{aligned}
$$

as operators on any imaginary Wakimoto module.

5. Intertwining Operators

5.1. TOPOLOGICAL MODULES

Following definitions given in [15] page 22, we have:

DEFINITION 5.1. A set $\mathcal{N}$ of subsets of a set S is called a filter on S if $S \in \mathcal{N}, \emptyset \notin \mathcal{N}$, the intersection of two elements of $\mathcal{N}$ is again in $\mathcal{N}$, and any subset of S containing a set in $\mathcal{N}$ is again in $\mathcal{N}$. Suppose $\mathcal{B}$ is a collection of subsets of a given set S and consider the set $\mathcal{N} = \{F \subseteq S \mid \exists B \in \mathcal{B}, B \subseteq F\}$. If $\mathcal{N}$ is a filter on S, then $\mathcal{B}$ is called a filter base on S. A fundamental system of neighborhoods of zero in a vector space is any filter base generating the neighborhoods of zero.

THEOREM 5.2 ([15, Theorem 3.5]). *Suppose A is a ring. If $\mathcal{N}$ is a filter base of neighborhoods of zero for A, then $\mathcal{N}$ satisfies the following conditions:*

(TRN1) *For each $N \in \mathcal{N}$ there exists $U \in \mathcal{N}$ such that $U + U \subseteq N$.*

(TRN2) *$N \in \mathcal{N}$ implies that there exists $U \in \mathcal{N}$ such that $U \subseteq -N$.*
(TRN3) *For each $N \in \mathcal{N}$ there exists $U \in \mathcal{N}$ such that $UU \subseteq N$.*
(TRN4) *For each $N \in \mathcal{N}, b \in A$ there exists $U \in \mathcal{N}$ such that $bU \subseteq N$ and $Ub \subseteq N$.*

Conversely if $\mathcal{N}$ is a filter base on A satisfying (TRN1)–(TRN4)*, then there is a unique ring topology on A for which $\mathcal{N}$ is a fundamental system of neighborhoods of zero.*

Now to define topological modules M for a topological ring A, we need a collection of neighborhoods of zero $\mathfrak{n}$ of subsets N of M such that

(M1) For each $N \in \mathfrak{n}$, there exists $U \in \mathfrak{n}$ such that $U + U \subseteq N$.
(M2) For each $N \in \mathfrak{n}$, there exists $U \in \mathfrak{n}$ such that $U \subseteq -N$.
(M3) For each $N \in \mathfrak{n}$, there exists a neighborhood T of zero in A and $U \in \mathfrak{n}$ such that $TU \subseteq N$.
(M4) For each $N \in \mathfrak{n}$ and each $b \in N$ there exists a neighborhood T of zero in A such that $Tb \subseteq N$.
(M5) For each $N \in \mathfrak{n}$ and each $\beta \in A$ there exists $U \in \mathfrak{n}$ such that $\beta U \subseteq \mathfrak{n}$.

The set $\{N\}$ satisfying the above (M1)–(M5) forms a base for a topological module M. We want to find a completion of $M = U(\hat{\mathfrak{g}}_-) \cdot 1 \subseteq V_{\lambda-m}$ and $W_{\lambda-m}$ such that

$$f_{m_1} \cdots f_{m_k} \left(\sum_{n \in \mathbb{N}} \sum_{\pi \in P_n} \alpha_\pi h_\pi 1 \right) \in \hat{V}_{\lambda-m}.$$

Let

$$U_n = \operatorname{span}\left\langle h_{n_1} \cdots h_{n_k} | n \leq \sum_{i=1}^{k} n_i \right\rangle \subseteq A, \quad \mathfrak{n} = \{U_n | n \in \mathbb{N}\}$$

be a fundamental system of neighborhoods of zero for A,

$$\begin{aligned} &\mathfrak{n}' = \{U_n \cdot 1 | n \in \mathbb{N}\} \text{ topology for } M, \\ &U_n \subseteq U(\hat{\mathfrak{g}}_-) = A, \quad M_n = U_n \cdot 1, \quad M = U(\mathfrak{g}) \cdot 1, \\ &\hat{M} := \varprojlim_n M/M_n \end{aligned} \tag{5.1}$$

completion of M. So we have:

(M1) If $N = M_n \in \mathfrak{n}$, then $U = M_n$ satisfies $M_n + M_n \subseteq M_n$.
(M2) If $N = M_n \in \mathfrak{n}$, then $U = M_n$ satisfies $M = M_n \subseteq -N$.
(M3) If $N = M_n \in \mathfrak{n}$, then for $T = U_n$ and $U = M_n \in \mathfrak{n}$ we have $TM = U_n U_n \cdot 1 \subseteq U_{2n} \cdot 1 \subseteq U_n \cdot 1 = M_n$.
(M4) For $N = M_k \in \mathfrak{n}$, $b = \sum_{n \in \mathbb{N}} \sum_{\pi \in P_n} \alpha_\pi h_\pi \cdot 1$, take $T = U_k$ and then we have $U_k b \subseteq M_k$.
(M5) For $N = M_k \in \mathfrak{n}, \beta = \sum_{n \in \mathbb{N}} \sum_{\pi \in P_n} \alpha_\pi h_\pi \in A$, take $U = M_k$ and we have $bM_k \subseteq M_k$.

We need to make sure $\hat{M}$ can be made into an $\hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}}$-module. Now we have to check if left multiplication by $h_n, n \in \mathbb{N}$ is continuous: $h_n M_m \subseteq M_{n+m}$.

So h_n extends to a map $h_n : \hat{M} \longrightarrow \hat{M}$.

DEFINITION 5.3. Define $\hat{V}_{\lambda,\kappa} = \varprojlim_n V_{\lambda,\kappa}/(V_{\lambda,\kappa})_n$ and similar to $\hat{W}_{\lambda,\kappa}$.

DEFINITION 5.4. Let $\mathfrak{F}_m$ be the finite dimensional $\mathfrak{sl}(2,\mathbb{C})$-module with highest weight $m \in \mathbb{N}$ and highest weight vector u^m.

Observe that the definitions of $\mathfrak{F}_m(z)$ and $V_{\lambda-m,\kappa} \hat{\otimes} \mathfrak{F}_m(z)$ are similar to the definitions given in (2.9) and (2.11). Set $u_k^m = u^m \otimes z^k \in \mathfrak{F}_m(z)$ for $k \in \mathbb{Z}$.

DEFINITION 5.5. Define $P_n := \{(-n_1, -n_2, \ldots, -n_p) | n_1 + \cdots + n_p = n\}$ and for a fixed $\pi = (-n_1, \ldots, -n_p) \in P_n$, define $h_\pi := h_{n_1} \cdots h_{n_p}$.

PROPOSITION 5.6. *Suppose* $\kappa \neq 0$ *and fix* $\beta_{(1)} \in \mathbb{C}$. *Let* $m \in \mathbb{N}$, $V_{\lambda-m,\kappa}$ *the imaginary Verma module for* $\hat{\mathfrak{g}}$ *and* $v_{\lambda-m,\kappa} \in V_{\lambda-m,\kappa}$ *be an imaginary highest weight vector of highest weight* $\lambda - m$. *For* $(n_1, n_2, \ldots, n_r) = (n_1, n_2, \ldots, n_r, 0, 0, \ldots)$ *with* $n_i \in \mathbb{N}$, *define*

$$\beta_{(n_1,\ldots,n_r)} = \frac{m^{n_r+\cdots+n_1-1}}{(-\kappa)^{n_r+\cdots+n_1-1}(r^{n_r} \cdot \cdots \cdot 2^{n_2}) \cdot n_r! \cdots n_1!} \cdot \beta_{(1)}.$$

For a partition

$$\pi = (\underbrace{-1, -1, \ldots, -1}_{n_1}, \underbrace{-2, -2, \ldots, -2}_{n_2}, \ldots, \underbrace{-r, -r, \ldots, -r}_{n_r})$$

of $-n$ *(so* $\sum_{l=1}^r l n_l = n$*), define*

$$\alpha_{(\underbrace{-1,-1,\ldots,-1}_{n_1}, \underbrace{-2,-2,\ldots,-2}_{n_2}, \ldots, \underbrace{-r,-r,\ldots,-r}_{n_r})} = \beta_{(n_1,n_2,\ldots,n_r)}$$

and

$$h_\pi := h_{-1}^{n_1} h_{-2}^{n_2} \cdots h_{-r}^{n_r}.$$

The vector

$$v_{\lambda,\kappa}^\sharp := \sum_{n \in \mathbb{N}} \sum_{\pi \in P_n} \alpha_\pi h_\pi v_{\lambda-m,\kappa} \otimes u_n^m \in V_{\lambda-m,\kappa} \hat{\otimes} \mathfrak{F}_m(z)$$

is an imaginary highest weight vector of weight λ. *Hence there is a nonzero* $\hat{\mathfrak{g}}$*-module homomorphism* $\Phi^V(z) : V_{\lambda,\kappa} \to V_{\lambda-m,\kappa} \hat{\otimes} \mathfrak{F}_m(z)$ *such that* $\Phi^V(z)(v_{\lambda,\kappa}) = v_{\lambda,\kappa}^\sharp$.

Proof. We write out an indication of the calculation in the case of the action of h_1:

$$\begin{aligned}
&h_1(\alpha_0 v_{\lambda-m,\kappa}\otimes u_0^m+\alpha_{(-1)}h_{-1}v_{\lambda-m,\kappa}\otimes u_1^m+(\alpha_{(-1,-1)}h_{-1}^2+\alpha_{(-2)}h_{-2})v_{\lambda-m,\kappa}\otimes u_2^m\\
&\quad+(\alpha_{(-1,-1,-1)}h_{-1}^3+\alpha_{(-2,-1)}h_{-1}h_{-2}+\alpha_{(-3)}h_{-3})v_{\lambda-m,\kappa}\otimes u_3^m\\
&\quad+(\alpha_{(-1,-1,-1,-1)}h_{-1}^4+\alpha_{(-2,-1,-1)}h_{-1}^2h_{-2}+\alpha_{(-2,-2)}h_{-2}^2\\
&\quad+\alpha_{(-3,-1)}h_{-1}h_{-3}+\alpha_{(-4)}h_{-4})v_{\lambda-m,\kappa}\otimes u_4^m+\cdots)\\
&=0+m\alpha_0 v_{\lambda-m,\kappa}\otimes u_1^m+\alpha_{(-1)}\kappa v_{\lambda-m,\kappa}\otimes u_1^m+m\alpha_{(-1)}h_{-1}v_{\lambda-m,\kappa}\otimes u_2^m\\
&\quad+2\alpha_{(-1,-1)}\kappa h_{-1}v_{\lambda-m,\kappa}\otimes u_2^m+m\alpha_{(-1,-1)}h_{-1}^2v_{\lambda-m,\kappa}\otimes u_3^m\\
&\quad+m\alpha_{(-2)}h_{-2}v_{\lambda-m,\kappa}\otimes u_3^m\\
&\quad+3\alpha_{(-1,-1,-1)}\kappa h_{-1}^2v_{\lambda-m,\kappa}\otimes u_3^m+m\alpha_{(-1,-1,-1)}h_{-1}^3v_{\lambda-m,\kappa}\otimes u_4^m\\
&\quad+\alpha_{(-2,-1)}\kappa h_{-2}v_{\lambda-m,\kappa}\otimes u_3^m+m\alpha_{(-2,-1)}h_{-1}h_{-2}v_{\lambda-m,\kappa}\otimes u_4^m\\
&\quad+m\alpha_{(-3)}h_{-3}v_{\lambda-m,\kappa}\otimes u_4^m\\
&\quad+\kappa(4\alpha_{(-1,-1,-1,-1)}h_{-1}^3+2\alpha_{(-2,-1,-1)}h_{-1}h_{-2}+\alpha_{(-3,-1)}h_{-3})v_{\lambda-m,\kappa}\otimes u_4^m+\cdots
\end{aligned}$$

For $1\le k\le r$, we have

$$\begin{aligned}
&h_k(\beta_{(n_1,n_2,\dots,n_r)}h_{-1}^{n_1}\cdots h_{-k}^{n_k}\cdots h_{-r}^{n_r}v_{\lambda-m,\kappa}\otimes u_n^m)\\
&=k.n_k.\kappa.\beta_{(n_1,\dots,n_k,\dots,n_r)}h_{-1}^{n_1}\cdots h_{-k}^{n_k-1}\cdots h_{-r}^{n_r}v_{\lambda-m,\kappa}\otimes u_n^m\\
&\quad+m\beta_{(n_1,\dots,n_k,\dots,n_r)}h_{-1}^{n_1}\cdots h_{-k}^{n_k}\cdots h_{-r}^{n_r}v_{\lambda-m,\kappa}\otimes u_{n+k}^m.
\end{aligned}$$

For this reason, we want in general to have

$$m\beta_{(n_1,\dots,n_k-1,\dots,n_r)}+k.n_k.\kappa\beta_{(n_1,\dots,n_k,\dots,n_r)}=0. \tag{5.2}$$

In this case, when we apply h_k, the coefficient of $h_{-1}^{n_1}\cdots h_{-k}^{n_k-1}\cdots h_{-r}^{n_r}v_{\lambda-m,\kappa}\otimes u_n^m$, where $n=\sum_{i=1}^r in_i$, is $m\beta_{(n_1,\dots,n_k-1,\dots,n_r)}+k.n_k.\kappa\beta_{(n_1,\dots,n_k,\dots,n_r)}$ and we want it to be zero. Then we can fix $\beta_{(1)}$ and find all the others using this condition:

$$\begin{aligned}
&m\beta_{(1)}+l.\kappa\beta_{(1,0,\dots,0,1)}=0 \quad\Longrightarrow\quad \beta_{(1,0,\dots,0,1)}=\frac{-m}{l\cdot\kappa}\beta_{(1)}\\
&m\beta_{(0,\dots,0,1)}+1.\kappa\beta_{(1,0,\dots,0,1)}=0 \quad\Longrightarrow\quad \beta_{(0,0,\dots,0,1)}=\frac{1}{l}\beta_{(1)},
\end{aligned}$$

where 1 occurs in the (last) l-entry of $(1,0,\dots,0,1)$ and $(0,\dots,0,1)$. Similarly by applying h_k repeatedly, we need to require

$$m^{n_k}\beta_{(n_1,\dots,n_{k-1},0,n_{k+1},\dots,n_r)}=(-1)^{n_k}\cdot k^{n_k}\cdot\kappa^{n_k}\cdot n_k!\beta_{(n_1,\dots,n_k,\dots,n_r)}.$$

Without lost of generality, we can assume $n_1\neq 0$ and require

$$\beta_{(1)}=\frac{(-\kappa)^{n_r+\cdots+n_1-1}\cdot(r^{n_r}\cdots 2^{n_2})\cdot n_r!\cdots n_1!}{m^{n_r+\cdots+n_1-1}}\cdot\beta_{(n_1,\dots,n_r)}.$$

So, under the hypothesis $\kappa \neq 0$,

$$\beta_{(n_1,\ldots,n_r)} = \frac{m^{n_r+\cdots+n_1-1}}{(-\kappa)^{n_r+\cdots+n_1-1}(r^{n_r}\cdots 2^{n_2})\cdot n_r!\cdots n_1!}\cdot \beta_{(1)}$$

is well defined and forces $v^{\sharp}_{\lambda,\kappa}$ to be annihilated by the h_k for all $k \geq 1$.

Now, we have

$$\begin{aligned} h_m(v^{\sharp}_{\lambda,\kappa}) &= 0,\\ e_m(v^{\sharp}_{\lambda,\kappa}) &= \sum_{n\in\mathbb{N}^*}\sum_{\pi\in P_n}\alpha_\pi(e_m h_\pi v_{\lambda-m,\kappa})\otimes u^m_n + \sum_{n\in\mathbb{N}^*}\sum_{\pi\in P_n}\alpha_\pi h_\pi v_{\lambda-m,\kappa}\otimes e_m u^m_n\\ &= 0 \end{aligned}$$

for all $m>0$ (because $e_m h_k = -2e_{m+k} + h_k e_m,\ \ \forall k$). Thus $v^{\sharp}_{\lambda,\kappa}$ is an imaginary highest weight vector of weight $\lambda + \kappa\Lambda_0$. By the universal mapping property of imaginary Verma modules, there exists a $\hat{\mathfrak{g}}$-module homomorphism $\varphi : V_{\lambda,\kappa} \longrightarrow V_{\lambda-m,\kappa}\hat{\otimes}\mathfrak{F}_m(z)$, for $\kappa \neq 0$, sending $v_{\lambda,\kappa}$ to $v^{\sharp}_{\lambda,\kappa}$. □

COROLLARY 5.7.

$$((zh(z))_-\otimes 1)\hat{\Phi}^V(z)(v_{\lambda,\kappa}) = -m\hat{\Phi}^V(z)(v_{\lambda,\kappa}). \tag{5.3}$$

Proof. By (5.2) we have

$$\begin{aligned} &(h_k\otimes 1)(\beta_{(n_1,n_2,\ldots,n_r)}h^{n_1}_{-1}\cdots h^{n_k}_{-k}\cdots h^{n_r}_{-r}v_{\lambda-m,\kappa}\otimes u^m_n)\\ &\quad = k.n_k.\kappa.\beta_{(n_1,\ldots,n_k,\ldots,n_r)}h^{n_1}_{-1}\cdots h^{n_k-1}_{-k}\cdots h^{n_r}_{-r}v_{\lambda-m,\kappa}\otimes u^m_n\\ &\quad = -m\beta_{(n_1,\ldots,n_k-1,\ldots,n_r)}h^{n_1}_{-1}\cdots h^{n_k-1}_{-k}\cdots h^{n_r}_{-r}v_{\lambda-m,\kappa}\otimes u^m_n \end{aligned}$$

and hence

$$\begin{aligned} ((zh(z))_-\otimes 1)\hat{\Phi}^V(z)(v_{\lambda,\kappa}) &= \sum_{k>0}\sum_{n\in\mathbb{N}}\sum_{\pi\in P_n}\alpha_\pi[h_k,h_\pi]v_{\lambda-m,\kappa}\otimes z^{-\Delta}u^m_n z^{-k}\\ &= -m\sum_{k>0}\sum_{n\in\mathbb{N}}\sum_{\pi\in P_n}\alpha_\pi h_\pi v_{\lambda-m,\kappa}\otimes z^{-\Delta-k}u^m_{n+k}\\ &= -mz^{-\Delta}\sum_{n\in\mathbb{N}}\sum_{\pi\in P_n}\alpha_\pi h_\pi v_{\lambda-m,\kappa}\otimes u^m_n\\ &= -m\hat{\Phi}^V(z)(v_{\lambda,\kappa}). \end{aligned}$$

□

In particular, note the intertwining property of $\Phi^V(z)$: If we let $*$ denote the tensor product action, then

$$\Phi^V(z)x_n = x_n * \Phi(z) = (x_n\otimes 1 + z^n(1\otimes x))\Phi^V(z) \tag{5.4}$$

for $x\in\mathfrak{sl}(2,\mathbb{C})$.

5.2. OPERATOR FORM OF THE KZ EQUATION

We define $\Phi^W(z): W_{\lambda,\kappa} \to W_{\lambda-m,\kappa}\hat{\otimes}\mathfrak{F}_m(z)$ through the diagram

$$\begin{array}{ccc} W_{\lambda,\kappa} & \xrightarrow{\Phi^W(z)} & W_{\lambda-m,\kappa}\hat{\otimes}\mathfrak{F}_m(z) \\ \Psi^{-1}_{\lambda,\kappa}\big\downarrow & & \Psi_{\lambda-m,\kappa}\otimes 1\big\uparrow \\ V_{\lambda,\kappa} & \xrightarrow{\Phi^V(z)} & V_{\lambda-m,\kappa}\hat{\otimes}\mathfrak{F}_m(z) \end{array}$$

where $\Psi_{\lambda,\kappa}: V_{\lambda,\kappa} \to W_{\lambda,\kappa}$ is the canonical isomorphism given in Theorem 3.3 and

$$(\Psi_{\lambda-m,\kappa}\otimes 1)(v_1\otimes v_2) := \Psi_{\lambda-m,\kappa}(v_1)\otimes v_2.$$

Then

$$\Phi^W(z) := (\Psi_{\lambda-m,\kappa}\otimes 1)\circ\Phi^V(z)\circ\Psi^{-1}_{\lambda,\kappa}. \tag{5.5}$$

Now consider z as a formal variable and set

$$\mathfrak{F}_m\otimes z^{-\Delta}\mathbb{C}[z,z^{-1}] = z^{-\Delta}\mathfrak{F}_m[z_1,z_1^{-1}].$$

This space is an infinite dimensional representation of $\hat{\mathfrak{g}}$ with a basis $v\otimes z^{n-\Delta}$, $n\in\mathbb{Z}$, $v\in\mathfrak{F}_m$.

PROPOSITION 5.8. *The* $\hat{\mathfrak{g}}$*-module map* $z^{-\Delta}\Phi^W(z): W_{\lambda,\kappa} \longrightarrow W_{\lambda-m,\kappa}\hat{\otimes}z^{-\Delta}\mathfrak{F}_m[z,z^{-1}]$ *such that*

$$z^{-\Delta}\Phi^W(z)(w_{\lambda,\kappa}) = \sum_{n\in\mathbb{N}^*}\sum_{\pi\in P_n}\alpha_\pi b_\pi w_{\lambda-m,\kappa}\otimes z^{-\Delta}u_n^m$$

is a $\tilde{\mathfrak{g}}$*-module homomorphism (here* d *acts by* $z\frac{\partial}{\partial z}$ *on the second factor) if and only if*

$$\Delta = \Delta(\lambda) - \Delta(\lambda - m). \tag{5.6}$$

Proof. By the definition, we have $d\cdot w_{\lambda,\kappa} = -\Delta(\lambda)\cdot w_{\lambda,\kappa}$, for $w_{\lambda,\kappa}\in W_{\lambda,\kappa}$ (see (2.8)). Now $z^{-\Delta}\Phi^W(z)$ is a $\tilde{\mathfrak{g}}$-intertwiner if and only if it satisfies

$$z^{-\Delta}\Phi^W(z)d - (d\otimes 1)z^{-\Delta}\Phi^W(z) = \left(1\otimes z\frac{\partial}{\partial z}\right)z^{-\Delta}\Phi^W(z). \tag{5.7}$$

We will apply both sides of this equality to the imaginary highest weight vector $w_{\lambda,\kappa}$ and see that they agree if and only if the condition (5.6) is satisfied. Then since $W_{\lambda,\kappa}$ is generated by this vector, the two sides agree on all of $W_{\lambda,\kappa}$ if and only if (5.6) is satisfied.

The left hand side of (5.7) applied to $w_{\lambda,\kappa}$ gives us

$$\begin{aligned}
&(z^{-\Delta}\Phi^W(z)d-(d\otimes 1)z^{-\Delta}\Phi^W(z))(w_{\lambda,\kappa})\\
&\quad=-\Delta(\lambda)\sum_{n\in\mathbb{N}^*}\sum_{\pi\in P_n}\alpha_\pi b_\pi w_{\lambda-m,\kappa}\otimes u_n^m z^{-\Delta}\\
&\qquad-\sum_{n\in\mathbb{N}^*}\sum_{\pi\in P_n}\alpha_\pi(-n-\Delta(\lambda-m))b_\pi w_{\lambda-m,\kappa}\otimes u_n^m z^{-\Delta}\\
&\quad=-(\Delta(\lambda)-\Delta(\lambda-m))\sum_{n\in\mathbb{N}^*}\sum_{\pi\in P_n}\alpha_\pi b_\pi w_{\lambda-m,\kappa}\otimes u_n^m z^{-\Delta}\\
&\qquad+\sum_{n\in\mathbb{N}^*}\sum_{\pi\in P_n}n\alpha_\pi b_\pi w_{\lambda-m,\kappa}\otimes u_n^m z^{-\Delta},
\end{aligned}$$

while the right hand side is

$$\begin{aligned}
\left(1\otimes z\frac{\partial}{\partial z}\right)z^{-\Delta}\Phi^W(z)(w_{\lambda,\kappa})&=\sum_{\pi\in P_n}\alpha_\pi b_\pi w_{\lambda-m,\kappa}\otimes z\frac{\partial}{\partial z}u_n^m z^{-\Delta}\\
&=-\Delta\sum_{\pi\in P_n}\alpha_\pi b_\pi w_{\lambda-m,\kappa}\otimes u_n^m z^{-\Delta}\\
&\quad+\sum_{\pi\in P_n}n\alpha_\pi b_\pi w_{\lambda-m,\kappa}\otimes u_n^m z^{-\Delta}.
\end{aligned}$$

To finish the proof, we recall (5.5) and note that $\Psi_{\lambda,\kappa}$ is a $\tilde{\mathfrak{g}}$-module homomorphism. □

For $x\in\mathfrak{F}_m^*$, define $\Phi_x^V(z):V_{\lambda,\kappa}\longrightarrow\hat{V}_{\lambda-m,\kappa}$ by

$$\Phi_x^V(z)(w)=x(\Phi^V(z)(w)). \tag{5.8}$$

For example,

$$\Phi_x^V(z)(v_{\lambda,\kappa})=\sum_{n\in\mathbb{N}^*}\sum_{\pi\in P_n}\alpha_\pi h_\pi v_{\lambda-m,\kappa}x(u_n^m), \tag{5.9}$$

where $xu_n^m=(xu^m)z^n$. We similarly define $\Phi_x^W(z):W_\lambda\to\hat{W}_{\lambda-m,\kappa}$. The intertwining property (5.4) for $x=f$ becomes

$$\begin{aligned}
\Phi_x^W(z)a_m=\Phi_x^W(z)f_m=f_m*\Phi_x^W(z)&=(f_m\otimes 1)\Phi_x^W(z)+z^m(1\otimes f)\Phi_x^V(z)\\
&=(a_m\otimes 1)\Phi_x^W(z)-z^m\Phi_{fx}^W(z).
\end{aligned}$$

where fx means the action of f on an element in the dual space $\mathfrak{F}_m^*$. Thus

$$[a_m,\Phi_x^W(z)]=z^m\Phi_{fx}^W(z). \tag{5.10}$$

A similar computation shows that

$$[b_m,\Phi_x^W(z)]=z^m\Phi_{hx}^W(z). \tag{5.11}$$

The Equations (5.10) and (5.11) show that

$$[b(\zeta)_+, \Phi^W_x(z)] = \frac{1}{z-\zeta}\Phi^W_{hx}(z), \quad [b(\zeta)_-, \Phi^W_x(z)] = \frac{1}{\zeta - z}\Phi^W_{hx}(z),$$
$$[a(\zeta), \Phi^W_x(z)] = \Phi^W_{fx}(z)\delta(\zeta/z). \tag{5.12}$$

For $x \in \mathfrak{F}^*_m$ and $\Delta = \Delta(\lambda) - \Delta(\lambda - m)$ set $\tilde{\Phi}^W_x(z) := z^{-\Delta}\Phi^W_x(z)$.

THEOREM 5.9 (Operator form of the KZ-Equations). *For $x \in \mathfrak{F}^*_m$ of weight $\alpha \in \mathbb{C}$, set*

$$\Delta(\alpha, \mu) := \frac{\alpha(\alpha - 2\mu)}{4} \quad \textit{and} \quad \hat{\Phi}^W_x(z) := z^{-\Delta(\mu,\alpha)}\tilde{\Phi}^W_x(z),$$

where μ is fixed (see Section 4.2). The operator $\hat{\Phi}^W_x(z) : W_{\lambda,\kappa} \longrightarrow W_{\lambda-m,\kappa}\hat{\otimes}\mathfrak{F}_m[z, z^{-1}]\cdot z^{-\Delta}$ satisfies the differential equation

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}z}\hat{\Phi}^W_x(z) = \hat{\Phi}^W_{fx}(z)\partial_z a^*(z) + \frac{\alpha}{2} : b(z)\hat{\Phi}^W_x(z) :} \tag{5.13}$$

Proof. We have

$$\begin{aligned}
&\hat{\Phi}^W_x(z) \circ a^*_n(a_{n_1}\cdots a_{n_k} b_{-m_1}\cdots b_{-m_l}\cdot w_{\lambda,\kappa}) \\
&\quad = -\sum_{i=1}^{k} \delta_{n+n_i,0}\hat{\Phi}^W_x(z)\left(a_{n_1}\cdots a_{n_{i-1}}a_{n_{i+1}}\cdots a_{n_k}b_{-m_1}\cdots b_{-m_l}\cdot w_{\lambda,\kappa})\right),
\end{aligned}$$

while

$$\begin{aligned}
&a^*_n \circ \hat{\Phi}^W(z)(a_{n_1}\cdots a_{n_k}b_{-m_1}\cdots b_{-m_l}\cdot w_{\lambda,\kappa}) \\
&\quad = a^*_n \circ (a_{n_1} * \cdots * a_{n_k} * b_{-m_1} * \cdots * b_{-m_l} * \hat{\Phi}^W(z)(w_{\lambda,\kappa})) \\
&\quad = a^*_n((a_{n_1}\otimes 1 + z^{n_1}\otimes f)\cdots(a_{n_k}\otimes 1 + z^{n_k}\otimes f) \\
&\qquad \times (b_{-m_1}\otimes 1 + z^{-m_1}\otimes h)\cdots(b_{-m_l}\otimes 1 + z^{-m_l}\otimes h))\hat{\Phi}^W(z)(w_{\lambda,\kappa}) \\
&\quad = -\sum_{i=1}^{k}\delta_{n+n_i,0}(a_{n_1}\otimes 1 + z^{n_1}\otimes f)\cdots(a_{n_{i-1}}\otimes 1 + z^{n_{i-1}}\otimes f)(a_{n_{i+1}}\otimes 1 + z^{n_{i+1}}\otimes f) \\
&\qquad \cdots(a_{n_k}\otimes 1 + z^{n_k}\otimes f)\cdot(b_{-m_1}\otimes 1 + z^{-m_1}\otimes h)\cdots(b_{-m_l}\otimes 1 + z^{-m_l}\otimes h)\cdot\hat{\Phi}^W(z)(w_{\lambda,\kappa}) \\
&\qquad + a_{n_1} * \cdots * a_{n_k} * b_{-m_1} * \cdots * b_{-m_l} * (a^*_n \otimes 1)\hat{\Phi}^W(z)(w_{\lambda,\kappa})) \\
&\quad = -\sum_{i=1}^{k}\delta_{n+n_i,0}\hat{\Phi}^W(z)(a_{n_1}\cdots a_{n_{i-1}}a_{n_{i+1}}\cdots a_{n_k}b_{-m_1}\cdots b_{-m_l}\cdot w_{\lambda,\kappa}) \\
&\qquad + a_{n_1} * \cdots * a_{n_k} * b_{-m_1} * \cdots * b_{-m_l} * (a^*_n\otimes 1)\hat{\Phi}^W(z)(w_{\lambda,\kappa})).
\end{aligned}$$

On the other hand,

$$a^*_n\hat{\Phi}^W(z)(w_{\lambda,\kappa}) = \sum_{n\in\mathbb{N}}\sum_{\pi\in P_n}\alpha_\pi a^*_n b_\pi w_{\lambda-m,\kappa}\otimes z^{-\Delta}u^m_n = 0,$$

so that

$$
\begin{aligned}
&(a_n^*\otimes 1)\circ\hat{\Phi}_x^W(z)(a_{n_1}\cdots a_{n_k}b_{-m_1}\cdots b_{-m_l}\cdot w_{\lambda,\kappa})\\
&\quad=(1\otimes x)\circ(a_n^*\otimes 1)\circ\hat{\Phi}^W(z)(a_{n_1}\cdots a_{n_k}b_{-m_1}\cdots b_{-m_l}\cdot w_{\lambda,\kappa})\\
&\quad=-\sum_{i=1}^{k}\delta_{n+n_i,0}(1\otimes x)\hat{\Phi}^W(z)(a_{n_1}\cdots a_{n_{i-1}}a_{n_{i+1}}\cdots a_{n_k}b_{-m_1}\cdots b_{-m_l}\cdot w_{\lambda,\kappa})\\
&\quad=-\sum_{i=1}^{k}\delta_{n+n_i,0}\hat{\Phi}_x^W(z)(a_{n_1}\cdots a_{n_{i-1}}a_{n_{i+1}}\cdots a_{n_k}b_{-m_1}\cdots b_{-m_l}\cdot w_{\lambda,\kappa}).
\end{aligned}
$$

Thus $[a_n^*,\hat{\Phi}_x^W(z)]=0$. We have

$$
z\frac{\mathrm{d}}{\mathrm{d}z}\hat{\Phi}_x^W(z)=-[d,\tilde{\Phi}_x^W(z)]-\Delta(\mu,\alpha)\hat{\Phi}_x^W(z).
$$

Replace d by $-L_0$ and use (4.3):

$$
\begin{aligned}
z\frac{\mathrm{d}}{\mathrm{d}z}\hat{\Phi}_x^W(z)&=[L_0,\hat{\Phi}_x^W(z)]-\Delta(\mu,\alpha)\hat{\Phi}_x^W(z)\\
&=\left[\sum_{n\in\mathbb{Z}}na_na_{-n}^*+\frac{1}{4}\left(\sum_{n\in\mathbb{Z}}:b_nb_{-n}:\right)-\frac{\mu}{2}b_0,\hat{\Phi}_x^W(z)\right]-\Delta(\mu,\alpha)\hat{\Phi}_x^W(z)\\
&=\sum_{n\in\mathbb{Z}}n[a_n,\hat{\Phi}_x^W(z)]a_{-n}^*\\
&\quad+\left[\frac{1}{4}\left(\sum_{n>0}b_{-n}b_n\right)+\frac{1}{4}\left(\sum_{n<0}b_nb_{-n}\right)+\frac{1}{4}b_0^2-\frac{\mu}{2}b_0,\hat{\Phi}_x^W(z)\right]\\
&\quad-\Delta(\mu,\alpha)\hat{\Phi}_x^W(z)\\
&=\hat{\Phi}_{fx}^W(z)\sum_{n\in\mathbb{Z}}na_{-n}^*z^n\\
&\quad+\frac{1}{2}\left(\sum_{n>0}\left[b_{-n},\hat{\Phi}_x^W(z)\right]b_n\right)+\frac{1}{2}\left(\sum_{n>0}b_{-n}\left[b_n,\hat{\Phi}_x^W(z)\right]\right)\\
&\quad+\frac{1}{4}\left(\left[b_0,\hat{\Phi}_x^W(z)\right]b_0+b_0\left[b_0,\hat{\Phi}_x^W(z)\right]\right)\\
&\quad-\frac{\mu}{2}\left[b_0,\hat{\Phi}_x^W(z)\right]-\Delta(\mu,\alpha)\hat{\Phi}_x^W(z)\\
&=\hat{\Phi}_{fx}^W(z)z\partial_za^*(z)\\
&\quad+\frac{1}{2}\left(\sum_{n>0}\hat{\Phi}_{hx}^W(z)b_nz^{-n}\right)\\
&\quad+\frac{1}{2}\left(\sum_{n>0}b_{-n}\Phi_{hx}^W(z)z^n\right)+\frac{1}{4}\left(\hat{\Phi}_{hx}^W(z)b_0+b_0\hat{\Phi}_{hx}^W(z)\right)\\
&\quad-\frac{\mu}{2}\hat{\Phi}_{hx}^W(z)-\Delta(\mu,\alpha)\hat{\Phi}_x^W(z)
\end{aligned}
$$

$$
\begin{aligned}
&= \hat{\Phi}^W_{fx}(z) z\partial_z a^*(z) \\
&\quad + \frac{z}{2} : b(z)\hat{\Phi}^W_{hx}(z) : + \frac{1}{4}(-\hat{\Phi}^W_{hx}(z)b_0 + b_0\hat{\Phi}^W_{hx}(z)) - \frac{\mu}{2}\hat{\Phi}^W_{hx}(z) \\
&\quad - \Delta(\mu,\alpha)\hat{\Phi}^W_{x}(z) \\
&= \hat{\Phi}^W_{fx}(z) z\partial_z a^*(z) + \frac{z}{2} : b(z)\hat{\Phi}^W_{hx}(z) : + \frac{1}{4}\hat{\Phi}^W_{h^2x}(z) - \frac{\mu}{2}\hat{\Phi}^W_{hx}(z) \\
&\quad - \Delta(\mu,\alpha)\hat{\Phi}^W_{x}(z) \\
&= \hat{\Phi}^W_{fx}(z) z\partial_z a^*(z) + \frac{\alpha z}{2} : b(z)\hat{\Phi}^W_{x}(z) : .
\end{aligned}
$$

□

THEOREM 5.10.

$$
[L_m, \hat{\Phi}^W_x(z)] = \hat{\Phi}^W_{fx}(z) z^{m+1}\partial_z a^*(z) + \frac{\alpha}{2} z^{m+1} : b(z)\hat{\Phi}^W_x(z) : + (m+1)\Delta(\mu,\alpha) z^m \hat{\Phi}^W_x(z).
$$

Proof. We have

$$
\begin{aligned}
[L_m, \hat{\Phi}^W_x(z)] &= \left[\sum_{n\in\mathbb{Z}}(n-m)a_n a^*_{m-n} + \frac{1}{4}\left(\sum_{n\in\mathbb{Z}} : b_n b_{m-n} :\right) - \frac{\mu}{2}(m+1)b_m, \hat{\Phi}^W_x(z)\right] \\
&= \sum_{n\in\mathbb{Z}}(n-m)\left(\left[a_n, \hat{\Phi}^W_x(z)\right]a^*_{m-n} + a_n\left[a^*_{m-n}, \hat{\Phi}^W_x(z)\right]\right) \\
&\quad + \frac{1}{4}\left[\sum_{n\in\mathbb{Z}} : b_n b_{m-n} :, \hat{\Phi}^W_x(z)\right] - \frac{\mu}{2}(m+1)\left[b_m, \hat{\Phi}^W_x(z)\right] \\
&= \hat{\Phi}^W_{fx}(z)\sum_{n\in\mathbb{Z}}(n-m)a^*_{m-n}z^n - \frac{\mu}{2}(m+1)\left[b_m, \hat{\Phi}^W_x(z)\right] \\
&\quad + \frac{1}{4}\left[\sum_{n\in\mathbb{Z}} : b_n b_{m-n} :, \hat{\Phi}^W_x(z)\right] \\
&= \hat{\Phi}^W_{fx}(z) z^{m+1}\partial_z a^*(z) - \frac{\mu}{2}(m+1)z^m\hat{\Phi}^W_{hx}(z) \\
&\quad + \frac{1}{4}\left[\sum_{n\in\mathbb{Z}} : b_n b_{m-n} :, \hat{\Phi}^W_x(z)\right] \\
&= \hat{\Phi}^W_{fx}(z) z^{m+1}\partial_z a^*(z) - \frac{\mu\alpha}{2}(m+1)z^m\hat{\Phi}^W_{x}(z) \\
&\quad + \frac{1}{4}\left[\sum_{n\in\mathbb{Z}} : b_n b_{m-n} :, \hat{\Phi}^W_x(z)\right].
\end{aligned}
$$

So, we only need to calculate $[\sum_{n\in\mathbb{Z}} : b_n b_{m-n} :, \hat{\Phi}^W_x(z)]$. Then we have

$$
\sum_{n\in\mathbb{Z}}[: b_n b_{m-n} :, \hat{\Phi}^W_x(z)] = 2\sum_{n>m}[b_{m-n}b_n, \hat{\Phi}^W_x(z)] + \sum_{i=0}^{m}[b_{m-i}b_i, \hat{\Phi}^W_x(z)]
$$

$$
\begin{aligned}
&= 2\sum_{n>m} b_{m-n}[b_n, \hat{\Phi}^W_x(z)] + 2\sum_{n>m}[b_{m-n}, \hat{\Phi}^W_x(z)]b_n \\
&\quad + \sum_{i=0}^{m} b_{m-i}[b_i, \hat{\Phi}^W_x(z)] + \sum_{i=0}^{m}[b_{m-i}, \hat{\Phi}^W_x(z)]b_i \\
&= 2z^{m+1}\left(\sum_{n>m} b_{m-n} z^{n-m-1}\hat{\Phi}^W_{hx}(z) + \hat{\Phi}^W_{hx}(z)\sum_{n>m} b_n z^{-n-1}\right) \\
&\quad + \left(\sum_{i=0}^{m} b_{m-i} z^i \hat{\Phi}^W_{hx}(z) + \hat{\Phi}^W_{hx}(z)\sum_{i=0}^{m} b_i z^{m-i}\right) \\
&= 2z^{m+1}\left(\sum_{n<0} b_n z^{-n-1}\hat{\Phi}^W_{hx}(z) + \hat{\Phi}^W_{hx}(z)\sum_{n\geq 0} b_n z^{-n-1}\right) - 2z^{m+1}\hat{\Phi}^W_{hx}(z)\sum_{n=0}^{m} b_n z^{-n-1} \\
&\quad + z^{m+1}\sum_{i=0}^{m} b_{m-i} z^{-i-1}\hat{\Phi}^W_{hx}(z) + z^{m+1}\hat{\Phi}^W_{hx}(z)\sum_{i=0}^{m} b_i z^{-i-1} \\
&= 2z^{m+1} : b(z)\hat{\Phi}^W_{hx}(z) : + \sum_{i=0}^{m} b_{m-i} z^{m-i}\hat{\Phi}^W_{hx}(z) - \hat{\Phi}^W_{hx}(z)\sum_{i=0}^{m} b_i z^{m-i} \\
&= 2z^{m+1} : b(z)\hat{\Phi}^W_{hx}(z) : + \sum_{i=0}^{m} z^{m-i}[b_i, \hat{\Phi}^W_{hx}(z)] \\
&= 2z^{m+1} : b(z)\hat{\Phi}^W_{hx}(z) : + \sum_{i=0}^{m} z^{m}\hat{\Phi}^W_{h^2x}(z) \\
&= 2z^{m+1} : b(z)\hat{\Phi}^W_{hx}(z) : + (m+1)z^m\hat{\Phi}^W_{h^2x}(z) \\
&= 2\alpha z^{m+1} : b(z)\hat{\Phi}^W_x(z) : + (m+1)\alpha^2 z^m\hat{\Phi}^W_x(z).
\end{aligned}
$$

Hence,

$$
[L_m, \hat{\Phi}^W_x(z)] = \hat{\Phi}^W_{fx}(z)z^{m+1}\partial_z a^*(z) + \frac{\alpha}{2}z^{m+1} : b(z)\hat{\Phi}^W_x(z) : + (m+1)\Delta(\mu,\alpha)z^m\hat{\Phi}^W_x(z).
$$

□

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