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Fock Space Realizations of Imaginary Verma Modules

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Abstract. This work expands to the setting of $\widehat{\mathfrak{sl}_n(\mathbb{C})}$ the results of H. Jakobsen and V. Kac and independently D. Bernard and G. Felder on the realization of $\widehat{\mathfrak{sl}_2(\mathbb{C})}$, in terms of infinite sums of partial differential operators. We note in the paper that, in the generic case, these geometric constructions are just realizations of the imaginary Verma modules studied by V. Futorny.

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1. Introduction

The motivation for this paper goes back to at least the work of A. Borel, R. Bott and A. Weil on their geometric construction of irreducible representations of complex simple Lie groups, and the related work of B. Kostant [10] and J. Bernšteĭn, I. Gel′fand and S. Gel′fand [2]. Articles more closely related to ours are those of H. Jakobsen and V. Kac [9] and then presumably Wakimoto [11] and more generally B. Feĭgin and È. Frenkel [4–6]. These latter authors in particular gave geometric constructions of certain representations of affine Kac–Moody algebras in terms of infinite sums of partial differential operators acting on a Fock space. Our work expands on the last section of the paper of D. Bernard and G. Felder (see [1]) to the case of $\widehat{\mathfrak{sl}_n(\mathbb{C})}$ and then notes that, in the generic case, these geometric constructions are just realizations of the imaginary Verma modules studied by V. Futorny (see [8]).

We'll now describe in more detail the cited work of D. Bernard and G. Felder and its relationship to our main result. Fix a positive integer n, $\gamma \in \mathbb{C}^*$ and for each $1 \leqslant i \leqslant n$ fix $\lambda_i \in \mathbb{C}$. Set $2\not{c} = \gamma^2$. We will let

$$\mathbb{C}[x] := \mathbb{C}[x_{ij}(m) \mid i, j, m \in \mathbb{Z},\ 1 \leqslant i, j \leqslant n], \quad \text{and}$$
$$\mathbb{C}[y] := \mathbb{C}[y_i(m) \mid i, m \in \mathbb{N}^*,\ 1 \leqslant i \leqslant n]$$

denote the algebras over $\mathbb{C}$ generated by the indeterminates $x_{ij}(m)$ and $y_i(m)$ respectively. There are natural operators $a_{ij}(m)$ and $a^*_{ij}(m)$, $1 \leqslant i, j \leqslant n$ on $\mathbb{C}[x]$ defined by

$$a_{ij}(m) := -x_{ij}(m), \qquad a^*_{ij}(m) := \partial/\partial x_{ij}(-m).$$

Then $[a_{ij}(m), a^*_{kl}(p)] = \delta_{ik}\delta_{jl}\delta_{m+p,0}$. We also have operators $b_i(m)$, $1 \leqslant i \leqslant n$, on $\mathbb{C}[y]$ defined by

$$b_i(0) = -\gamma^{-1}\lambda_i, \qquad b_i(-m) := -\gamma^{-1}y_i(m), \qquad b_i(m) := -\gamma m \frac{\partial}{\partial y_i(m)}$$

for $m > 0$.

In this case $[b_i(m), b_j(p)] = m\delta_{ij}\delta_{m+p,0}$.

For any symbol A and sequence of vectors $\{A(m)\}_{m\in\mathbb{Z}}$ one can form the *fields*

$$\begin{aligned} A(z) &:= A^-(z) + A^+(z), \\ A^-(z) &:= \sum_{-m\in\mathbb{N}} A(m)z^{-m}, \quad \text{and} \\ A^+(z) &:= \sum_{m\in\mathbb{N}^*} A(m)z^{-m}, \end{aligned}$$

where $\mathbb{N}$ consists of the nonnegative integers. An important formal series that is quite useful is the delta function and its degree derivative

$$\delta(z) := \sum_{m\in\mathbb{Z}} z^m, \qquad \dot{\delta}(z) := \sum_{m\in\mathbb{Z}} mz^m.$$

Let $C = (C_{ij})$ denote the Cartan matrix of type A_n, E_{ij} the standard basis for the associated finite-dimensional Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ and define $H_i := E_{ii} - E_{i+1,i+1}$. Set $E_i = E_{i,i+1}$ and $F_i = E_{i+1,i}$. Furthermore, we denote the Killing form by $(X|Y) = \mathrm{tr}(XY)$ and $X(m) = t^m \otimes X$ for $X, Y \in \mathfrak{sl}(n, \mathbb{C})$ and $m \in \mathbb{Z}$. In this case the affine Lie algebra $\hat{\mathfrak{sl}}(n, \mathbb{C})$ has generators $E(m), F(m), H(m)$, $m \in \mathbb{Z}$, and central element c with defining relations written in the compact form

$$[H_i(z), H_j(w)] = -(H_i|H_j)c\dot{\delta}(z/w), \tag{R1}$$
$$[H_i(z), E_j(w)] = C_{ij}E_j(z)\delta(z/w), \tag{R2}$$
$$[H_i(z), F_j(w)] = -C_{ij}F_j(z)\delta(z/w), \tag{R3}$$
$$[E_i(z), F_j(w)] = \delta_{i,j}(H_i(z)\delta(z/w) - c\dot{\delta}(z/w)), \tag{R4}$$
$$[F_i(z), F_j(w)] = [E_i(z), E_j(w)] = 0 \quad \text{if } C_{ij} \neq -1, \tag{R5}$$
$$[F_i(z_1), F_i(z_2), F_j(w)] = [E_i(z_1), E_i(z_2), E_j(w)] = 0 \quad \text{if } C_{ij} = -1, \tag{R6}$$

where $[X, Y, Z] := [X, [Y, Z]]$ is the Engel bracket for any three operators X, Y, Z.

Before we extend the construction of Bernard and Felder [1] and Jakobsen and Kac [9] of a Fock space representation of $\widehat{\mathfrak{sl}(n, C)}$ on the algebra $\mathbb{C}[x_m \mid m \in \mathbb{Z}] \otimes$

$\mathbb{C}[y_k \mid k \in \mathbb{N}^*]$ we briefly review the construction given in these papers: Jakobsen and Kac give a general realization of the action of $\mathfrak{sl}(2, R)$, R a commutative ring, on an induced highest weight module M. When $R = \mathbb{C}[t, t^{-1}]$ is the ring of Laurent polynomials in one indeterminate t, and $M = \mathbb{C}[x_m \mid m \in \mathbb{Z}]$ the action of $\mathfrak{sl}(2, R)$ is given by

$$\begin{aligned} F(m) &\longmapsto x_m, \\ E(m) &\longmapsto -\left(\sum_i \lambda_{i+m} \frac{\partial}{\partial x_i} + \sum_{i,j\in\mathbb{Z}} x_{i+j+m} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}\right), \\ H(m) &\longmapsto -\left(\lambda_m + 2\sum_{j\in\mathbb{Z}} x_{j+m} \frac{\partial}{\partial x_j}\right). \end{aligned}$$

Here $\lambda_m = \int_{S^1} z^m \,\mathrm{d}\mu$ where μ denotes a finite measure on the unit circle S^1, not concentrated in a finite number of points. In particular if μ is the normalized Lebesgue measure, then $\lambda_k = \delta_{k,0}$ (see [9]). Observe that the above action is well defined as the infinite sums on the right collapse to a finite sum when acting on an element in M.

Independently, but at a later date, D. Bernard and G. Felder constructed a representation through the use of ideas that go back to the work of A. Borel and A. Weil. The *natural Borel subalgebra*, $\mathfrak{b}_+$, D. Bernard and G. Felder choose to induce from, is spanned by $E(m)$, $m \in \mathbb{Z}$, and $H(k)$, $k > 0$. They then stipulate that elements g_+ in the Borel subgroup $\hat{B}_+$ of the loop group $\widehat{SL_2}$ may be written in the form

$$g_+ = \exp\left(\sum_{m\in\mathbb{Z}} x_m E(m)\right) \exp\left(\sum_{m\in\mathbb{N}^*} y_m H(m)\right),$$

where x_m and y_m are coordinate functions. One can similarly define the Borel subalgebra $\hat{B}_-$.

Here the Cartan subgroup H is generated by the exponential of $H(0)$ and c. As is well known a character $\chi_{K,J}$ of H is uniquely determined by specifying the value of c and $H(0)/2$, by two numbers, the *central charge* K and the *spin* J respectively. Each character defines a one-dimensional representation $\mathbb{C}_{K;J}$ of $\hat{B}_-$ and from this one can construct the line bundle over $\widehat{SL_2}/\hat{B}_-$;

$$\mathcal{L}_{K;J} := \widehat{SL_2} \times_{\hat{B}_-} \mathbb{C}_{K;J}.$$

Sections of this line bundle can be identified with functions f on $\widehat{SL_2}$ satisfying

$$f(gg_-) = f(g) \quad \text{and} \quad f(gh) = f(g)\chi_{K;J}(h)$$

for $g \in \widehat{SL_2}$, $g_- \in \hat{B}_-$ and $h \in H$. The loop group $\widehat{SL_2}$, then acts on sections of $\mathcal{L}_{K;J}$ by

$$(g_1 \cdot f)(g_2) = f(g_1^{-1} g_2) \quad \text{for } g_1, g_2 \in \widehat{SL_2}.$$

This action can be differentiated to an action ϕ of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ on sections:

$$\phi(x)f := (x \cdot f) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{tx} \cdot f)|_{t=0} \quad \text{for } x \in \widehat{\mathfrak{sl}_2}.$$

In terms of the coordinate functions x_m and y_k, this representation can be explicitly written in the form of differential operators

$$\begin{aligned}
\phi(E(m)) &= -\frac{\partial}{\partial x_m},\\
\phi(H(m)) &= -2\sum_j x_{j-m}\frac{\partial}{\partial x_j} - \theta(m)\frac{\partial}{\partial y_m} + \theta(-m)2mKy_{-m} + 2\delta_{m,0}J,\\
\phi(F(m)) &= \sum_{l,j} x_l x_{j-l-m}\frac{\partial}{\partial x_j} + \sum_{j>0} x_{j-m}\frac{\partial}{\partial y_j} + 2K\sum_{j>0} jy_j x_{-j-m} +\\
&\quad + Kmx_{-m} - 2Jx_{-m}.
\end{aligned}$$

Here $\theta(m)$ is one if $m > 0$ and zero otherwise. Observe that if one views these partial differential operators as acting on the tensor product $R = \mathbb{C}[x_m \mid m \in \mathbb{Z}] \otimes \mathbb{C}[y_k \mid k \in \mathbb{N}^*]$, then the first and third summations for $F(n)$ would need to take values in some type of formal completion of R, rather than in R itself. As was noted in D. Bernard and G. Felder's paper this realization is the same as that of H. Jakobsen and V. Kac's after twisting the action by an automorphism, specializing to the case where $K = 0$ and quotienting out $\mathbb{C}[y_k \mid k \in \mathbb{N}^*]$. This can be seen as follows. Define the anti-automorphisms ρ_1, and ρ_2 on the generators of the algebras, $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ and $\mathbb{C}\langle a(m), a^*(m), b(k) \mid m \in \mathbb{Z},\ k \in \mathbb{N}^*\rangle$, respectively by

$$\begin{aligned}
&\rho_1(E(m)) = -F(-m), && \rho_1(H(m)) = H(-m),\\
&\rho_1(F(m)) = -E(-m), && \rho_1(c) = c,
\end{aligned}$$

$$\rho_2(x_{-m}) = \frac{\partial}{\partial x_m}, \qquad \rho_2\left(\frac{\partial}{\partial x_m}\right) = x_{-m}, \qquad \rho_2(y_k) = -\frac{\partial}{\partial y_k},$$
$$\rho_2\left(\frac{\partial}{\partial y_k}\right) = -y_k.$$

Applying these two anti-automorphism to the above representation ϕ of D. Bernard and G. Felder lead to the representation $\rho = \rho_2 \circ \phi \circ \rho_1$ of $\widehat{\mathfrak{sl}}(2)$, where

$$\begin{aligned}
\rho(F(m)) &= x_m,\\
\rho(H(m)) &= -2\sum_j x_{j+m}\frac{\partial}{\partial x_j} + \theta(-m)y_{-m} + \theta(m)2mK\frac{\partial}{\partial y_m} + 2\delta_{m,0}J,\\
\rho(E(m)) &= -\sum_{l,j} x_{j+l+m}\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_j} + \sum_{j>0} y_j\frac{\partial}{\partial x_{-j-m}} +\\
&\quad + 2K\sum_{j>0} j\frac{\partial}{\partial y_j}\frac{\partial}{\partial x_{j-m}} + (Km + 2J)\frac{\partial}{\partial x_{-m}}.
\end{aligned}$$

Now this action is well defined on the polynomial ring $\mathbb{C}[x_m, y_k \mid m \in \mathbb{Z},\ k \in \mathbb{N}^*]$.

In the main result of the paper below, we will use the notation $\rho(X(m)) := \rho(X)(m)$, for $X \in \mathfrak{sl}(n, C)$. In the next theorem, we extend the representation ρ of $\widehat{\mathfrak{sl}}(2)$ given above, to a representation of $\widehat{\mathfrak{sl}}(n)$. This extension will also be referred to by ρ.

THEOREM 1.1 (Realization). *The generating functions given below*

$$\begin{aligned}
\rho(F_r)(w) &= -a_{r,r+1}(w) + \sum_{j=1}^{r-1} a_{j,r+1}(w)a^*_{jr}(w),\\
\rho(H_r)(w) &= 2a_{r,r+1}(z)a^*_{r,r+1}(w) +\\
&\quad + \sum_{i=1}^{r-1}\big(a_{i,r+1}(w)a^*_{i,r+1}(w) - a_{ir}(w)a^*_{ir}(w)\big) +\\
&\quad + \sum_{j=r+2}^{n}\big(a_{rj}(w)a^*_{rj}(w) - a_{r+1,j}(w)a^*_{r+1,j}(w)\big) -\\
&\quad - \gamma b_r(w) + \frac{\gamma}{2}\big(b^+_{r-1}(w) + b^+_{r+1}(w)\big),\\
\rho(E_r)(w) &= a_{r,r+1}(w)a^*_{r,r+1}(w)a^*_{r,r+1}(w) -\\
&\quad - \sum_{j=r+2}^{n} a_{r+1,j}(w)a^*_{rj}(w) + \sum_{j=1}^{r-1} a_{jr}(w)a^*_{j,r+1}(w) +\\
&\quad + \sum_{j=r+2}^{n}\big(a_{rj}(w)a^*_{rj}(w) - a_{r+1,j}(w)a^*_{r+1,j}(w)\big)a^*_{r,r+1}(w) -\\
&\quad - \gamma a^*_{r,r+1}(w)b_r(w) + \frac{\gamma}{2}a^*_{r,r+1}(w)\big(b^+_{r-1}(w) + b^+_{r+1}(w)\big) -\\
&\quad - \frac{\gamma^2}{2}\dot{a}^*_{r,r+1}(w)
\end{aligned}$$

define an action of the generators $E_i(n)$, $F_i(n)$ and $H_i(n)$ on the Fock space $\mathbb{C}[x]\otimes\mathbb{C}[y]$ (notation given earlier). In addition c acts as left multiplication by ¢.

Remark 1.2. The construction of this realization was through the use of a generalization of Bernard and Felder's construction above, but in the setting of $\hat{\mathfrak{sl}}(n, \mathbb{C})$.

Remark 1.3. Let $\dot{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra of $\dot{\mathfrak{g}}$, $\dot{\Delta}$ its root system with respect to $\mathfrak{h}$, and Π a set of simple roots and $\dot{\Delta}_+$ (resp. $\dot{\Delta}_-$) the set of positive (resp. negative) roots determined by Π. Let $\dot{\mathfrak{n}}_\pm := \bigoplus_{\alpha\in\dot{\Delta}_\pm}\dot{\mathfrak{g}}_\alpha$ so that $\dot{\mathfrak{g}} = \dot{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \dot{\mathfrak{n}}_+$ is the triangular decomposition of $\dot{\mathfrak{g}}$. For any Lie algebra $\mathfrak{a}$, let $L(\mathfrak{a}) = \mathfrak{a}\otimes\mathbb{C}[t, t^{-1}]$ be the loop algebra of $\mathfrak{a}$ and let $\hat{\mathfrak{g}} = L(\dot{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the associated nontwisted affine Kac–Moody algebra of $\dot{\mathfrak{g}}$ ([11]). We let $\mathfrak{H} = \mathfrak{h} + \mathbb{C}c + \mathbb{C}d$ denote the Cartan subalgebra of $\hat{\mathfrak{g}}$. The *natural Borel subalgebra* of $\hat{\mathfrak{g}}$ is defined to be the subalgebra

$$\mathfrak{b}_+ = \mathfrak{H} \oplus L(\dot{\mathfrak{n}}_+) \oplus (\mathfrak{h} \otimes \mathbb{C}[t]t).$$

This algebra seems to have first been introduced in [9]. For $\lambda \in \mathfrak{H}^*$ define the *imaginary Verma module* by

$$M(\lambda) = U(\hat{\mathfrak{g}}) \bigotimes_{U(\mathfrak{b}_+)} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the usual one-dimensional $\mathfrak{b}_+$-module. Now if $\dot{\mathfrak{g}}$ is $\mathfrak{sl}(n, \mathbb{C})$, then the realization above has the property that $L(\dot{\mathfrak{n}}_+) \oplus (\mathfrak{h} \otimes \mathbb{C}[t]t)$ annihilates $1 \otimes 1 \in \mathbb{C}[x] \otimes \mathbb{C}[y]$. This in turn implies that it is a quotient of the Verma module $M(\lambda) = U(\hat{\mathfrak{g}}) \bigotimes_{U(\mathfrak{b}_+)} \mathbb{C}_\lambda$ for suitable λ. For generic λ this Verma module is irreducible (see [3] and [7]).

2. Preliminary Lemmas

As an example of a computation using fields we have the following

LEMMA 2.1.

$$\begin{aligned}
&[a_{ij}(z), a^*_{kl}(w)] = \delta_{i,k}\delta_{j,l}\delta(z/w),\\
&[a_{ij}(z)a^*_{ij}(z), a_{ij}(w)a^*_{ij}(w)] = 0,\\
&[a_{ij}(z), \dot{a}^*_{kl}(w)] = \delta_{i,k}\delta_{j,l}\dot{\delta}(w/z),\\
&\dot{a}^*_{ij}(z)\delta(w/z) = a^*_{ij}(z)\dot{\delta}(w/z) - a^*_{ij}(w)\dot{\delta}(w/z),\\
&[b_i(z), b^-_j(w)] = [b^+_i(z), b^-_j(w)] = \delta_{i,j}\dot{\delta}^+(w/z),\\
&[b_i(z), b_j(w)] = [b^+_i(z), b^-_j(w)] + [b^-_i(z), b^+_j(w)] = \delta_{i,j}\dot{\delta}(w/z).
\end{aligned}$$

Proof. We prove only the second relation and leave the others to the reader

$$\begin{aligned}
&[a_{ij}(z)a^*_{ij}(z),\ a_{ij}(w)a^*_{ij}(w)]\\
&\quad = a_{ij}(z)[a^*_{ij}(z), a_{ij}(w)]a^*_{ij}(w) + a_{ij}(w)[a_{ij}(z), a^*_{ij}(w)]a^*_{ij}(z)\\
&\quad = -a_{ij}(z)\delta(w/z)a^*_{ij}(w) + a_{ij}(w)\delta(z/w)a^*_{ij}(z) = 0. \qquad \square
\end{aligned}$$

The following result collects some other computations involving the free fields that will make future calculations less tedious.

LEMMA 2.2.

$$\sum_{j=r+2}^{n}\sum_{k=1}^{s-1}[a_{ks}(w)a^*_{k,s+1}(w), a_{rj}(z)a^*_{rj}(z)]$$
$$= -\delta_{s,r+1}a_{r,r+1}(w)a^*_{r,r+2}(z)\delta(w/z), \tag{a}$$

$$\sum_{j=1}^{r-1}\sum_{k=s+2}^{n}[a_{s+1,k}(w)a^*_{sk}(w), a_{jr}(z)a^*_{j,r+1}(z)] = 0, \tag{b}$$

$$\sum_{j=r+2}^{n}\sum_{k=1}^{s-1}[a_{ks}(w)a^*_{k,s+1}(w),\ a_{r+1,j}(z)a^*_{r+1,j}(z)] = 0, \tag{c}$$

$$\sum_{j=r+2}^{n}\sum_{k=s+2}^{n}[a_{s+1,k}(w)a^*_{sk}(w), a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z)]$$
$$= -2\delta_{r,s}\delta(w/z)\sum_{j=r+2}^{n}a_{r+1,j}(w)a^*_{rj}(z) +$$
$$+\delta_{r,s+1}\delta(w/z)\sum_{j=r+2}^{n}a_{rj}(z)a^*_{r-1,j}(w) +$$
$$+\delta_{s,r+1}\delta(w/z)\sum_{j=r+3}^{n}a_{r+2,k}(w)a^*_{r+1,j}(z), \tag{d}$$

$$\sum_{j=1}^{r-1}[a^*_{s,s+1}(w), a_{jr}(z)a^*_{j,r+1}(z)] = -\delta_{r,s+1}a^*_{r-1,r+1}(z)\delta(w/z), \tag{e}$$

$$\sum_{j=r+2}^{n}[a^*_{s,s+1}(w), a_{r+1,j}(z)a^*_{rj}(z)] = -\delta_{r+1,s}a^*_{r,r+2}(z)\delta(w/z). \tag{f}$$

Proof. Let us consider (a):

$$\sum_{j=r+2}^{n}\sum_{k=1}^{s-1}[a_{rj}(z)a^*_{rj}(z), a_{ks}(w)a^*_{k,s+1}(w)]$$
$$= -a_{r,s+1}(z)a^*_{rs}(w)\delta(w/z)\sum_{j=r+2}^{n}(\delta_{j,s+1}-\delta_{j,s})\sum_{k=1}^{s-1}\delta_{r,k}.$$

If $s+1>n$ then the above summation is zero as it has the factor $a^*_{r,s+1}(w)=0$. Thus $s+1\leqslant n$. The first summation on the right is nonzero only if $r+2=s+1$ and the second is nonzero only if $r\leqslant s-1$. This gives the result for (a).

Now we consider (b):

$$\sum_{j=1}^{r-1}\sum_{k=s+2}^{n}[a_{s+1,k}(w)a^*_{sk}(w),\, a_{jr}(z)a^*_{j,r+1}(z)]$$
$$= a_{s+1,r}(w)a^*_{s,r+1}(z)\delta(w/z)\sum_{j=1}^{r-1}\sum_{k=s+2}^{n}(\delta_{j,s+1}\delta_{k,r+1}-\delta_{j,s}\delta_{k,r})$$
$$= a_{s+1,r}(w)a^*_{s,r+1}(z)\delta(w/z)\Bigg(\sum_{j=1}^{r-1}(\delta_{j,s+1}-\delta_{j,s})\sum_{k=s+2}^{n}\delta_{k,r+1}+ + \sum_{j=1}^{r-1}\delta_{j,s}\sum_{k=s+2}^{n}(\delta_{k,r+1}-\delta_{k,r})\Bigg).$$

The first summation on the right is nonzero only if its two factors are nonzero. This occurs only if $r = s + 1$ and then this sum is equal to -1. The second summation is nonzero only if $r = s + 1$ and in this case it is equal to 1.

Part (c) becomes

$$\sum_{j=r+2}^{n}\sum_{k=1}^{s-1}[a_{ks}(w)a^*_{k,s+1}(w),\, a_{r+1,j}(z)a^*_{r+1,j}(z)]$$
$$= -a_{r+1,s}(w)a^*_{r+1,s+1}(z)\delta(w/z)\sum_{j=r+2}^{n}(\delta_{j,s}-\delta_{j,s+1})\sum_{k=1}^{s-1}\delta_{k,r+1} = 0$$

since the first summation is nonzero only if $r + 2 = s + 1$ which means $r + 1 = s > s - 1$.

Next we have

$$\sum_{j=r+2}^{n}\sum_{k=s+2}^{n}[a_{s+1,k}(w)a^*_{sk}(w),\, a_{rj}(z)a^*_{rj}(z)]$$
$$= -\sum_{j=r+2}^{n}\sum_{k=s+2}^{n}\delta_{r,s}\delta_{j,k}\delta(w/z)a_{s+1,k}(w)a^*_{rj}(z) + + \sum_{j=r+2}^{n}\sum_{k=s+2}^{n}\delta_{r,s+1}\delta_{jk}\delta(w/z)a_{rj}(z)a^*_{sk}(w)$$
$$= -\delta_{r,s}\delta(w/z)\sum_{j=r+2}^{n}a_{r+1,j}(w)a^*_{rj}(z) + + \delta_{r,s+1}\delta(w/z)\sum_{j=r+2}^{n}a_{rj}(z)a^*_{r-1,j}(w).$$

On the other hand

$$\sum_{j=r+2}^{n}\sum_{k=s+2}^{n}[a_{s+1,k}(w)a^*_{sk}(w), a_{r+1,j}(z)a^*_{r+1,j}(z)]$$
$$= -\delta_{s,r+1}\delta(w/z)\sum_{j=r+3}^{n}a_{s+1,k}(w)a^*_{r+1,j}(z)+$$
$$+\delta_{r,s}\delta(w/z)\sum_{j=r+2}^{n}a_{r+1,j}(z)a^*_{rj}(w)$$

which is (d). Now parts (e) and (f) are straightforward. □

3. The Relations

We can now check

LEMMA 3.1 (R1).

$$[H_r(z), H_s(w)] = -(H_r|H_s)c\dot{\delta}(z/w).$$

Proof. First we calculate for $r = s$. We use Lemma 2.1 repeatedly in the following calculations:

$$2[a_{r,r+1}(z)a^*_{r,r+1}(z), H_r(w)] = 2[a_{r,r+1}(z)a^*_{r,r+1}(z), a_{r,r+1}(w)a^*_{r,r+1}(w)] = 0,$$

$$\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), H_r(w)]$$
$$= \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z), a_{i,r+1}(w)a^*_{i,r+1}(w)]+$$
$$+\sum_{i=1}^{r-1}[a_{ir}(z)a^*_{ir}(z), a_{ir}(w)a^*_{ir}(w)] = 0,$$

and similarly

$$\sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z), H_r(w)] = 0.$$

The last term in $H_r(z)$ contributes

$$\left[-\gamma b_r(z) + \frac{\gamma}{2}(b^+_{r-1}(z) + b^+_{r+1}(z)), H_r(w)\right] = \gamma^2[b_r(z), b_r(w)]$$
$$= 2\not{c}\dot{\delta}(w/z).$$

These equations above combine to give the result for $r = s$.

Now we assume that $s = r + 1$.

$$
\begin{aligned}
&2[a_{r,r+1}(z)a^*_{r,r+1}(z), H_{r+1}(w)] \\
&\quad = 4[a_{r,r+1}(z)a^*_{r,r+1}(z),\ a_{r,r+1}(w)a^*_{r,r+1}(w)] = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), H_{r+1}(w)] \\
&\quad = \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z), a_{i,r+1}(w)a^*_{i,r+1}(w)] = 0,
\end{aligned}
$$

and similarly

$$
\sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z), H_{r+1}(w)] = 0.
$$

Last but not least we calculate the contribution due to the Heisenberg Lie subalgebra:

$$
\begin{aligned}
&\left[-\gamma b_r(z) + \frac{\gamma}{2}(b^+_{r-1}(z) + b^+_{r+1}(z)), H_{r+1}(w)\right] \\
&\quad = -\frac{\gamma^2}{2}[b^+_{r+1}(z), b_{r+1}(w)] - \frac{\gamma^2}{2}[b_r(z), b^+_r(w)] \\
&\quad = -\frac{\gamma^2}{2}\dot{\delta}(w/z) = -\not{c}\dot{\delta}(w/z).
\end{aligned}
$$

Note that the case $s = r - 1$ is obtained above by replacing s and r by s' and r' respectively in the above argument where $s' = r$ and $r' = s$.

Finally, observe that if $|r - s| > 1$ then the indices of $a_{ij}(z)$ and $a_{kl}(w)$ appearing in $H_r(z)$ and $H_s(w)$, respectively, are disjoint and it is straightforward to see that the terms with $b^{\pm}_k(z)$ and $b^{\pm}_l(w)$ in them have bracket zero. Thus $[H_r(z), H_s(w)] = 0$ in this last case. □

LEMMA 3.2 (R2).

$$
[\rho(H_r)(z), \rho(E_s)(w)] = C_{rs}\rho(E_s)(z)\delta(z/w).
$$

Proof. We first assume $r = s$. In this case $\rho(E_r)(w)$ is equal to

$$
\begin{aligned}
&a_{r,r+1}(w)a^*_{r,r+1}(w)a^*_{r,r+1}(w) + \\
&\quad + \sum_{j=r+2}^{n}\big(a_{rj}(w)a^*_{rj}(w) - a_{r+1,j}(w)a^*_{r+1,j}(w)\big)a^*_{r,r+1}(w) +
\end{aligned}
$$

$$+\sum_{j=1}^{r-1} a_{jr}(w)a^*_{j,r+1}(w) - \sum_{j=r+2}^{n} a_{r+1,j}(w)a^*_{rj}(w) -$$
$$-\gamma a^*_{r,r+1}(w)b_r(w) + \frac{\gamma}{2}a^*_{r,r+1}(w)\big(b^+_{r-1}(w) - b^+_{r+1}(w)\big) - \frac{\gamma^2}{2}\dot{a}^*_{r,r+1}(w)$$

and $\rho(H_r)(z)$ expands to

$$2a_{r,r+1}(z)a^*_{r,r+1}(z) + \sum_{i=1}^{r-1}\big(a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z)\big) +$$
$$+\sum_{j=r+2}^{n}\big(a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z)\big) - \gamma b_r(z) +$$
$$+\frac{\gamma}{2}\big(b^+_{r-1}(z) + b^+_{r+1}(z)\big).$$

Now

$$2[a_{r,r+1}(z)a^*_{r,r+1}(z), \rho(E_r)(w)]$$
$$= \Bigg(2a_{r,r+1}(w)a^*_{r,r+1}(z)a^*_{r,r+1}(w) +$$
$$+2\sum_{j=r+2}^{n}\big(a_{r,j}(w)a^*_{r,j}(w) - a_{r+1,j}(w)a^*_{r+1,j}(w)\big)a^*_{r,r+1}(z) -$$
$$-2\gamma a^*_{r,r+1}(z)b_r(w) + \gamma a^*_{r,r+1}(z)\big(b^+_{r-1}(w) + b^+_{r+1}(w)\big)\Bigg)\delta(w/z) -$$
$$-\gamma^2 a^*_{r,r+1}(z)\dot{\delta}(w/z).$$

The second summation in $H_r(z)$ contributes

$$\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), \rho(E_r)(w)]$$
$$= \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), a_{ir}(w)a^*_{i,r+1}(w)]$$
$$= 2\sum_{i=1}^{r-1} a_{ir}(w)a^*_{i,r+1}(w)\delta(w/z).$$

Now the third summation in $\rho(H_r)(z)$ contains indices where at least one index is greater than or equal to $r+2$ and thus

$$\sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z), \rho(E_r)(w)]$$

$$
\begin{aligned}
&= \sum_{j=r+2}^{n} \big([a_{rj}(z)a^*_{rj}(z), a_{rj}(w)a^*_{rj}(w)] + \\
&\qquad\qquad + [a_{r+1,j}(z)a^*_{r+1,j}(z), a_{r+1,j}(w)a^*_{r+1,j}(w)]\big) + \\
&\quad + 2\sum_{j=r+2}^{n} a_{r+1,j}(w)a^*_{rj}(z)\delta(w/z) \\
&= 2\sum_{j=r+2}^{n} a_{r+1,j}(w)a^*_{rj}(z)\delta(w/z)
\end{aligned}
$$

by Lemma 2.1. The last terms in $\rho(H_r)(w)$ contribute

$$
-\gamma[b_r(z), \rho(E_r)(w)] = \gamma^2[b_r(z), a^*_{r,r+1}(w)b_r(w)] = \gamma^2 a^*_{r,r+1}(w)\dot{\delta}(w/z).
$$

The previous four calculations sum up to give us the desired result $[\rho(H_r)(z), \rho(E_r)(w)] = 2\rho(E_r)(z)\delta(w/z)$.

Now suppose $s = r+1$ so that $\rho(E_{r+1})(w)$ is equal to

$$
\begin{aligned}
&a_{r+1,r+2}(w)a^*_{r+1,r+2}(w)^2 + \\
&\quad + a^*_{r+1,r+2}(w)\sum_{j=r+3}^{n}\big(a_{r+1,j}(w)a^*_{r+1,j}(w) - a_{r+2,j}(w)a^*_{r+2,j}(w)\big) + \\
&\quad + \sum_{i=1}^{r} a_{i,r+1}(w)a^*_{i,r+2}(w) - \sum_{j=r+3}^{n} a_{r+2,j}(w)a^*_{r+1,j}(w) - \\
&\quad - \gamma b_{r+1}(w)a^*_{r+1,r+2}(w) + \frac{\gamma}{2}\big(a^*_{r+1,r+2}(w)b^+_r(w) + \\
&\quad + a^*_{r+1,r+2}(w)b^+_{r+2}(w)\big) - \frac{\gamma^2}{2}\dot{a}^*_{r+1,r+2}(w).
\end{aligned}
$$

Then

$$
2[a_{r,r+1}(z)a^*_{r,r+1}(z),\ \rho(E_{r+1})(w)] = -2a_{r,r+1}(z)a^*_{r,r+2}(w)\delta(w/z).
$$

The second summation in $H_r(z)$ contributes

$$
\begin{aligned}
&\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), \rho(E_{r+1})] \\
&\quad = \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), a_{i,r+1}(w)a^*_{i,r+2}(w)] \\
&\quad = -\sum_{i=1}^{r-1} a_{i,r+1}(w)a^*_{i,r+2}(w)\delta(w/z).
\end{aligned}
$$

The third summand contributes by Lemma 1

$$\sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z)-a_{r+1,j}(z)a^*_{r+1,j}(z),\,\rho(E_{r+1})]$$
$$\begin{aligned}=-\Bigg(&a_{r+1,r+2}(w)a^*_{r+1,r+2}(z)a^*_{r+1,r+2}(w)+\\&+\sum_{j=r+3}^{n}\big(a_{r+1,j}(w)a^*_{r+1,j}(w)-a_{r+2,j}(w)a^*_{r+2,j}(w)\big)a^*_{r+1,r+2}(z)-\\&-a_{r,r+1}(w)a^*_{r,r+2}(z)-\sum_{j=r+3}^{n}a_{r+2,j}(w)a^*_{r+1,j}(w)-\\&-\gamma a^*_{r+1,r+2}(w)b_{r+1}(w)+\\&+\frac{\gamma}{2}a^*_{r+1,r+2}(w)\big(b^+_r(w)+b^+_{r+2}(w)\big)\Bigg)\delta(w/z)+\\&+\frac{\gamma^2}{2}a^*_{r+1,r+2}(z)\dot{\delta}(w/z).\end{aligned}$$

The last summand in $\rho(H_r)(z)$ has commutator with $\rho(E_{r+1})(w)$ equal to

$$\left[-\gamma b_r(z)+\frac{\gamma}{2}b^+_{r+1}(z),\,\rho(E_{r+1})(w)\right]$$
$$=-\frac{\gamma^2}{2}a^*_{r+1,r+2}(w)\big([b_r(z),b^+_r(w)]+[b^+_{r+1}(z),b_{r+1}(w)]\big)$$
$$=-\frac{\gamma^2}{2}a^*_{r+1,r+2}(w)\dot{\delta}(w/z).$$

Adding the previous four equations up we get $[\rho(H_r)(z),\rho(E_{r+1})(w)]=-\rho(E_{r+1})\delta(w/z)$.

The final nontrivial case to consider is when $s=r-1$ so that $\rho(E_{r-1})(w)$ is equal to

$$\begin{aligned}&a_{r-1,r}(w)a^*_{r-1,r}(w)^2\\&\quad+\sum_{j=r+1}^{n}\big(a_{r-1,j}(w)a^*_{r-1,j}(w)-a_{rj}(w)a^*_{rj}(w)\big)a^*_{r-1,r}(w)+\\&\quad+\sum_{j=1}^{r-2}a_{j,r-1}(w)a^*_{jr}(w)-\sum_{j=r+1}^{n}a_{r,j}(w)a^*_{r-1,j}(w)-\\&\quad-\gamma a^*_{r-1,r}(w)b_{r-1}(w)+\frac{\gamma}{2}\big(a^*_{r-1,r}(w)b^+_{r-2}(w)+a^*_{r-1,r}(w)b^+_r(w)\big)-\\&\quad-\frac{\gamma^2}{2}\dot{a}^*_{r-1,r}(w).\end{aligned}$$

Then

$$2[a_{r,r+1}(z)a^*_{r,r+1}(z),\, \rho(E_{r-1})(w)] = -2a^*_{r-1,r+1}(z)a_{r,r+1}(w)\delta(w/z).$$

The second summation in $\rho(H_r)(z)$ contributes by Lemma 1

$$\begin{aligned}
&\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z),\, \rho(E_{r-1})(w)] \\
&\quad = -\Bigg(a_{r-1,r}(w)a^*_{r-1,r}(z)a^*_{r-1,r}(w) + \\
&\qquad + \sum_{j=r+1}^{n}\big(a_{r-1,j}(w)a^*_{r-1,j}(w) - a_{rj}(w)a^*_{rj}(w)\big)a^*_{r-1,r}(z) + \\
&\qquad + \sum_{j=1}^{r-2} a_{j,r-1}(w)a^*_{jr}(z) - a_{r,r+1}(w)a^*_{r-1,r+1}(z) - \gamma a^*_{r-1,r}(w)b_{r-1}(w) + \\
&\qquad + \frac{\gamma}{2}\big(a^*_{r-1,r}(w)b^+_{r-2}(w) - a^*_{r-1,r}(w)b^+_r(w)\big)\Bigg)\delta(w/z) + \\
&\qquad + \frac{\gamma^2}{2}a^*_{r-1,r}(z)\dot{\delta}(w/z).
\end{aligned}$$

The third summand contributes

$$\begin{aligned}
&\sum_{j=r+2}^{n}[a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z),\, \rho(E_{r-1})(w)] \\
&\quad = \sum_{j=r+2}^{n} a_{r,j}(w)a^*_{r-1,j}(w)\delta(w/z).
\end{aligned}$$

The last summation in $\rho(H_r)(z)$ has commutator with $\rho(E_{r-1})(w)$ equal to

$$\begin{aligned}
&\left[\gamma b_r(z) - \frac{\gamma}{2}b^+_{r-1}(z),\ \rho(E_{r-1})(w)\right] \\
&\quad = \frac{\gamma^2}{2}a^*_{r-1,r}(w)\big([b_r(z), b^+_r(w)] + [b^+_{r-1}(z), b_{r-1}(w)]\big) \\
&\quad = \frac{\gamma^2}{2}a^*_{r-1,r}(w)\dot{\delta}(w/z).
\end{aligned}$$

Adding the previous four equations up we get $[\rho(H_r)(z), \rho(E_{r-1})(w)] = -\rho(E_{r-1})(w)\delta(w/z)$.

Lastly if $|r-s| > 1$ then observe the indices of a_{ij} and a^*_{ij} in $\rho(H_r)(z)$ and $\rho(E_s)(w)$ are disjoint and thus contribute nothing to the commutator $[\rho(H_r)(z),\ \rho(E_s)(w)]$. The remaining terms coming from the b_j have trivial commutator and thus $[\rho(H_r)(z), \rho(E_s)(w)] = 0$. □

LEMMA 3.3 (R3).

$$[\rho(H_r)(z), \rho(F_s)(w)] = -C_{rs}\rho(F_s)(z)\delta(z/w).$$

Proof. First we calculate for $r = s$. Recall the expansion for $\rho(F_r)(w)$ is

$$-a_{r,r+1}(w) + \sum_{j=1}^{r-1} a_{j,r+1}(w)a^*_{jr}(w).$$

The first summand in $\rho(H_r)(z)$ gives us

$$2[a_{r,r+1}(z)a^*_{r,r+1}(z), \rho(F_r)(w)] = 2a_{r,r+1}(z)\delta(z/w)$$

and the second

$$\begin{aligned}
&\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), \rho(F_r)(w)]\\
&\quad = \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z), a_{i,r+1}(w)a^*_{ir}(w)] -\\
&\qquad - \sum_{i=1}^{r-1}[a_{ir}(z)a^*_{ir}(z), a_{i,r+1}(w)a^*_{ir}(w)]\\
&\quad = -2\sum_{i=1}^{r-1} a_{i,r+1}(z)a^*_{ir}(w)\delta(z/w).
\end{aligned}$$

Since the second indices j of $a_{ij}(w)$ and $a^*_{ij}(w)$ in $\rho(F_r)(w)$ are less than or equal to $r + 1$ we get

$$\sum_{j=r+2}^{n} [a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z), \rho(F_r)(w)] = 0.$$

The last term in $\rho(H_r)(z)$ contributes

$$\left[-\gamma b_r(z) + \frac{\gamma}{2}\big(b^+_{r-1}(z) + b^+_{r+1}(z)\big), \rho(F_r)(w)\right] = 0.$$

These equations above combine to give the result for $r = s$.

Now we assume that $s = r + 1$. The expansion for $\rho(F_{r+1})(w)$ is

$$-a_{r+1,r+2}(w) + \sum_{j=1}^{r} a_{j,r+2}(w)a^*_{j,r+1}(w).$$

The first summand in $\rho(H_r)(z)$ gives us

$$\begin{aligned}
&2[a_{r,r+1}(z)a^*_{r,r+1}(z), \rho(F_{r+1})(w)] \\
&\quad = 2[a_{r,r+1}(z)a^*_{r,r+1}(z), a_{r,r+2}(w)a^*_{r,r+1}(w)] \\
&\quad = 2a_{r,r+2}(w)a^*_{r,r+1}(z)\delta(w/z).
\end{aligned}$$

Next using the observation that the second indices in $\rho(F_{r+1})(w)$ are either an $r+1$ or an $r+2$ we get

$$\begin{aligned}
&\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), \rho(F_{r+1})(w)] \\
&\quad = \sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z), \rho(F_{r+1})(w)] \\
&\quad = \sum_{i=1}^{r-1}\sum_{j=1}^{r}[a_{i,r+1}(z)a^*_{i,r+1}(z), a_{j,r+2}(w)a^*_{j,r+1}(w)] \\
&\quad = \sum_{i=1}^{r-1} a_{i,r+2}(w)a^*_{i,r+1}(z)\delta(z/w).
\end{aligned}$$

The second summation in $\rho(H_r)(z)$ contributes

$$\begin{aligned}
&\sum_{j=r+2}^{n} [a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z), \rho(F_{r+1})(w)] \\
&\quad = \sum_{j=r+2}^{n}\sum_{k=1}^{r}[a_{rj}(z)a^*_{rj}(z), a_{k,r+2}(w)a^*_{k,r+1}(w)] + \\
&\qquad + \sum_{j=r+2}^{n} [a_{r+1,j}(z)a^*_{r+1,j}(z), a_{r+1,r+2}(w)] \\
&\quad = -a_{r,r+2}(z)a^*_{r,r+1}(w)\delta(z/w) - a_{r+1,r+2}(z)\delta(z/w).
\end{aligned}$$

The last summands in $\rho(H_r)(z)$ contribute zero to the commutator;

$$\left[-\gamma b_r(z) + \frac{\gamma}{2}(b^+_{r-1}(z) + b^+_{r+1}(z)), \rho(F_{r+1})(w)\right] = 0.$$

These four calculations above combine to give

$$[\rho(H_r)(z), \rho(F_{r+1})(w)] = -\left(a_{r+1,r+2}(z) + \sum_{i=1}^{r} a_{i,r+2}(w)a^*_{i,r+1}(z)\right)\delta(z/w).$$

Next we consider $s = r - 1$. The expansion for $\rho(F_{r-1})(w)$ is

$$-a_{r-1,r}(w) + \sum_{j=1}^{r-2} a_{j,r}(w)a^*_{j,r-1}(w).$$

The first summand in $\rho(H_r)(z)$ gives us

$$2[a_{r,r+1}(z)a^*_{r,r+1}(z),\, \rho(F_{r-1})(w)] = 0.$$

The second summation in $\rho(H_r)(z)$ gives

$$\begin{aligned}
&\sum_{i=1}^{r-1}[a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z),\, \rho(F_{r-1})(w)] \\
&\quad = [a_{r-1,r}(z)a^*_{r-1,r}(z),\, a_{r-1,r}(w)] - \sum_{i=1}^{r-2} a_{ir}(z)[a^*_{ir}(z),\, a_{ir}(w)]a^*_{i,r-1}(w) \\
&\quad = \left(-a_{r-1,r}(z) + \sum_{i=1}^{r-2} a_{ir}(z)a^*_{i,r-1}(w)\right)\delta(z/w).
\end{aligned}$$

Since the first indices j of $a_{ij}(w)$ and $a^*_{ij}(w)$ in $\rho(F_{r-1})(w)$ are less than or equal to $r-1$ we get

$$\sum_{j=r+2}^{n} [a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z),\, \rho(F_r)(w)] = 0.$$

As above the last term in $\rho(H_r)(z)$ contributes zero to the commutator. These equations above combine to give the result for $s = r-1$.

Finally observe that if $|r-s| > 1$ then the indices of a_{ij} and a^*_{ij} appearing in $\rho(H_r)(z)$ and $\rho(F_s)(w)$ are disjoint and thus $[\rho(H_r)(z), \rho(F_s)(w)] = 0$ in this last case. □

LEMMA 3.4 (R4).

$$[\rho(E_r)(z),\, \rho(F_s)(w)] = \delta_{r,s}(\rho(H_r)(z))\delta(z/w) - \not{c}\dot{\delta}(z/w)).$$

Proof. First we take $r = s$. Now for the convenience of the reader we recall that $\rho(E_r)(z)$ is equal to

$$\begin{aligned}
&a_{r,r+1}(z)a^*_{r,r+1}(z)^2 + \sum_{j=r+2}^{n}\big(a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z)\big)a^*_{r,r+1}(z) + \\
&\quad + \sum_{j=1}^{r-1} a_{jr}(z)a^*_{j,r+1}(z) - \sum_{j=r+2}^{n} a_{r+1,j}(z)a^*_{rj}(z) - \\
&\quad - \gamma a^*_{r,r+1}(z)b_r(z) + \frac{\gamma}{2}a^*_{r,r+1}(z)\big(b^+_{r-1}(z) - b^+_{r+1}(z)\big) - \frac{\gamma^2}{2}\dot{a}^*_{r,r+1}(z)
\end{aligned}$$

and thus the first summand $-a_{r,r+1}(z)$ of $\rho(F_r)(w)$ brackets with $\rho(E_r)(z)$ to give us

$$\begin{aligned}
&\Bigg(2a_{r,r+1}(z)a^*_{r,r+1}(z) + \sum_{j=r+2}^{n}\big(a^*_{rj}(z)a_{rj}(z) - a^*_{r+1,j}(z)a_{r+1,j}(z)\big) - \\
&\quad - \gamma b_r(z) + \frac{\gamma}{2}\big(b^+_{r-1}(z) + b^+_{r+1}(z)\big)\Bigg)\delta(z/z) - \frac{\gamma^2}{2}\dot{\delta}(z/w).
\end{aligned}$$

The second summation in $\rho(F_r)(w)$ contributes

$$\sum_{j=1}^{r-1}[\rho(E_r)(z), a_{j,r+1}(w)a^*_{jr}(w)]$$
$$= \sum_{j=1}^{r-1}[a_{jr}(z)a^*_{j,r+1}(z), a_{j,r+1}(w)a^*_{jr}(w)]$$
$$= \sum_{j=1}^{r-1}\big(a_{j,r+1}(z)a^*_{j,r+1}(z) - a_{jr}(z)a^*_{jr}(z)\big)\delta(z/w).$$

Adding these two summations up we arrive at the desired result.

Now consider the case $|r-s| > 1$. Then $\rho(F_s)(w)$ acts by

$$-a_{s,s+1}(w) + \sum_{j=1}^{s-1} a_{j,s+1}(w)a^*_{js}(w)$$

and $\rho(E_r)(z)$ is equal to

$$a_{r,r+1}(z)a^*_{r,r+1}(z)a^*_{r,r+1}(z) +$$
$$+ \sum_{j=r+2}^{n}\big(a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z)\big)a^*_{r,r+1}(z) +$$
$$+ \sum_{j=1}^{r-1} a_{jr}(z)a^*_{j,r+1}(z) - \sum_{j=r+2}^{n} a_{r+1,j}(z)a^*_{rj}(z) -$$
$$- a^*_{r,r+1}(z)\gamma b_r(z) + \frac{\gamma}{2}a^*_{r,r+1}(z)\big(b^+_{r-1}(z) - b^+_{r+1}(z)\big) -$$
$$- \frac{\gamma^2}{2}\dot{a}^*_{r,r+1}(z).$$

In this case the first summand of $\rho(E_r)(z)$ has bracket with $\rho(F_s)(w)$ equal to zero as the difference in the indices of $a_{r,r+1}(w)$ is one and so can only interact nontrivially with $a^*_{s,s+1}(z)$ or $a^*_{s+1,s+2}(z)$. But this would force $s = r$ or $s+1 = r$.

The third summation of $\rho(E_r)(z)$ brackets with the second summation of $\rho(F_s)(w)$ to provide us with

$$\sum_{j=1}^{r-1}\sum_{k=1}^{s-1}[a_{kr}(z)a^*_{k,r+1}(z), a_{j,s+1}(w)a^*_{js}(w)] = 0$$

as $r \neq s$.

The second summation of $\rho(F_s)(w)$ brackets with the remaining terms of $\rho(E_r)(z)$ to give

$$B := \sum_{j=1}^{s-1}\Bigg[\sum_{j=r+2}^{n}\big(a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z)\big)a^*_{r,r+1}(z) +$$

$$
\begin{aligned}
&\quad + \sum_{j=r+2}^{n} a_{r+1,j}(z)a^*_{rj}(z), a_{j,s+1}(w)a^*_{js}(w)\Bigg] \\
&= \sum_{j=1}^{s-1}\sum_{k=r+2}^{n} [a_{rk}(z)a^*_{rk}(z) - \\
&\qquad - a_{r+1,k}(z)a^*_{r+1,k}(z), a_{j,s+1}(w)a^*_{js}(w)]a^*_{r,r+1}(z) - \\
&\quad - \sum_{j=1}^{s-1}\sum_{k=r+2}^{n} [a_{r+1,k}(z)a^*_{rk}(z), a_{j,s+1}(w)a^*_{js}(w)] \\
&= \sum_{j=1}^{s-1}\sum_{k=r+2}^{n} \big(a_{j,s+1}(w)a^*_{rk}(z)\delta_{s,k}\delta_{j,r} - a_{rk}(z)a^*_{js}(w)\delta_{s+1,k}\delta_{j,r} - \\
&\qquad - a_{j,s+1}(w)a^*_{r+1,k}(z)\delta_{r+1,j}\delta_{k,s} + \\
&\qquad + a_{r+1,k}(z)a^*_{js}(w)\delta_{r+1,j}\delta_{k,s+1}\big)a^*_{r,r+1}(z)\delta(z/w) + \\
&\quad + \sum_{j=1}^{s-1}\sum_{k=r+2}^{n} \big(a_{r+1,k}(z)a^*_{js}(w)\delta_{r,j}\delta_{k,s+1} - \\
&\qquad - a_{j,s+1}(w)a^*_{rk}(z)\delta_{r+1,j}\delta_{k,s}\big)\delta(z/w).
\end{aligned}
$$

If $s+1 > n$ then the $a_{j,s+1}(w) = 0 = \delta_{k,s+1}$ ($k \leqslant n$) above are zero and so $B = 0$. Thus we may assume $s+1 \leqslant n$. If $s-1 < r$ then the indices j above satisfy $j \leqslant s-1 < r$ and $\delta_{jr} = \delta_{j,r+1} = 0$. Again we would have $B = 0$. Hence we may assume $s-1 \geqslant r$ and thus $s-1 > r$ (otherwise $|s-r| = 1$) so that $s-1 \geqslant r+1$ and $s \geqslant r+2$. This means that $j = r, r+1$ and $k = s, s+1$ appear in B. In this case the summation above simplifies to

$$
\begin{aligned}
B &= \big(a_{r,s+1}(w)a^*_{rs}(z) - a_{r,s+1}(z)a^*_{rs}(w) - \\
&\quad - a_{r+1,s+1}(w)a^*_{r+1,s}(z) + a_{r+1,s+1}(z)a^*_{r+1,s}(w)\big)a^*_{r,r+1}(z)\delta(z/w) + \\
&\quad + \big(a_{r+1,s+1}(z)a^*_{rs}(w) - a_{r+1,s+1}(w)a^*_{rs}(z)\big)\delta(z/w) = 0.
\end{aligned}
$$

Next consider the case $r = s-1$. We have

$$
\begin{aligned}
\rho(E_{s-1})(z) &= a_{s-1,s}(z)a^*_{s-1,s}(z)a^*_{s-1,s}(z) + \\
&\quad + \sum_{j=s+1}^{n} \big(a_{s-1,j}(z)a^*_{s-1,j}(z) - a_{s,j}(z)a^*_{s,j}(z)\big)a^*_{s-1,s}(z) + \\
&\quad + \sum_{j=1}^{s-2} a_{j,s-1}(z)a^*_{j,s}(z) - \sum_{j=s+1}^{n} a_{s,j}(z)a^*_{s-1,j}(z) - \\
&\quad - \gamma a^*_{s-1,s}(z)b_{s-1}(z) + \frac{\gamma}{2}a^*_{s-1,s}(z)\big(b^+_{s-2}(z) - b^+_s(z)\big) - \\
&\quad - \frac{\gamma^2}{2}\dot{a}^*_{s-1,s}(z).
\end{aligned}
$$

The first summand $-a_{s,s+1}(w)$ of $\rho(F_s)(w)$ has a commutator with $\rho(E_{s-1})(z)$ equal to

$$\begin{aligned}
&-\sum_{j=s+1}^{n}\Big[\big(a_{s-1,j}(z)a^*_{s-1,j}(z)-a_{s,j}(z)a^*_{s,j}(z)\big)a^*_{s-1,s}(z),\, a_{s,s+1}(w)\Big]\\
&=-\sum_{j=s+1}^{n}[a_{s,s+1}(w),\, a_{s,j}(z)a^*_{s-1,s}(z)a^*_{s,j}(z)]\\
&=-a_{s,s+1}(z)a^*_{s-1,s}(z)\delta(w/z).
\end{aligned}$$

The second summation of $\rho(F_s)(w)$ brackets with the third summation of $\rho(E_{s-1})(z)$ to provide us with

$$\sum_{j=1}^{s-1}\sum_{k=1}^{s-2}[a_{k,s-1}(z)a^*_{k,s}(z),\, a_{j,s+1}(w)a^*_{js}(w)]=0.$$

The second summation of $\rho(F_s)(w)$ also brackets with the remaining terms of $\rho(E_{s-1})(z)$ to give

$$\begin{aligned}
&a_{s-1,s+1}(w)a^*_{s-1,s}(z)a^*_{s-1,s}(z)\delta(w/z)+\\
&\quad+\sum_{j=1}^{s-1}\sum_{k=s+1}^{n}\big(a_{j,s+1}(z)a^*_{s-1,k}(z)\delta_{s,k}\delta_{j,s-1}-a_{s-1,k}(z)a^*_{js}(z)\delta_{s+1,k}\delta_{j,s-1}-\\
&\quad-a_{j,s+1}(z)a^*_{s,k}(z)\delta_{s,j}\delta_{k,s}+a_{s,k}(z)a^*_{js}(z)\delta_{s,j}\delta_{k,s+1}\big)a^*_{s-1,s}(z)\delta(w/z)+\\
&\quad+\sum_{j=1}^{s-1}\sum_{k=s+1}^{n}\big(a_{s,k}(z)a^*_{js}(z)\delta_{s-1,j}\delta_{k,s+1}-a_{j,s+1}(z)a^*_{s-1,k}(z)\delta_{s,j}\delta_{k,s}\big)\delta(w/z)\\
&=a_{s,s+1}(z)a^*_{s-1,s}(z)\delta(w/z).
\end{aligned}$$

Summing these up we get $[\rho(E_{s-1})(z),\rho(F_s)(w)]=0$.

Lastly we assume $s+1=r$. Then

$$\begin{aligned}
\rho(E_{s+1})(z) \;=\;& a_{s+1,s+2}(z)a^*_{s+1,s+2}(z)a^*_{s+1,s+2}(z)+\\
&+\sum_{j=s+3}^{n}\big(a_{s+1,j}(z)a^*_{s+1,j}(z)-\\
&\qquad -a_{s+2,j}(z)a^*_{s+2,j}(z)\big)a^*_{s+1,s+2}(z)+\\
&+\sum_{j=1}^{s}a_{j,s+1}(z)a^*_{j,s+2}(z)-\sum_{j=s+3}^{n}a_{s+2,j}(z)a^*_{s+1,j}(z)-\\
&-\gamma a^*_{s+1,s+2}(z)b_{s+1}(z)+\frac{\gamma}{2}a^*_{s+1,s+2}(z)\big(b^+_s(z)-b^+_{s+2}(z)\big)-\\
&-\frac{\gamma^2}{2}\dot{a}^*_{s+1,s+2}(z)
\end{aligned}$$

and the first summand $-a_{s,s+1}(w)$ of $\rho(F_s)(w)$ brackets with $\rho(E_{s+1})(z)$ to give us 0. The second summation

$$\sum_{j=1}^{s-1} a_{j,s+1}(w)a^*_{js}(w)$$

in $\rho(F_s)(w)$ brackets with $\rho(E_{r+1})(z)$ to give us zero also. □

We are now left with the Serre relations:

LEMMA 3.5 (R5/R6).

$$[\rho(F_r)(z), \rho(F_s)(w)] = [\rho(E_r)(z), \rho(E_s)(w)] = 0 \quad \text{if } C_{rs} \neq -1,$$
$$[\rho(F_r)(z_1), \rho(F_r)(z_2), \rho(F_s)(w)] = [\rho(E_r)(z_1), \rho(E_r)(z_2), \rho(E_s)(w)] = 0 \quad \text{if } C_{rs} = -1.$$

Proof. Let us check the relations for $\rho(F_r)(w)$. A straightforward calculation shows

$$\left[-a_{r,r+1}(z),\ -a_{s,s+1}(w) + \sum_{j=1}^{s-1} a_{j,s+1}(w)a^*_{js}(w)\right] = \delta_{s,r+1}a_{r,r+2}(w)\delta(w/z).$$

Moreover

$$\begin{aligned}\left[\sum_{j=1}^{r-1} a_{j,r+1}(w)a^*_{jr}(w),\ -a_{s,s+1}(w) + \sum_{j=1}^{s-1} a_{j,s+1}(w)a^*_{js}(w)\right]\\ = \delta_{r,s+1}a_{r-1,r+1}(w)\delta(w/z) -\\ - \delta_{r,s+1}\sum_{j=1}^{s-1} a_{j,r+1}(w)a^*_{js}(w)\delta(w/z) +\\ + \delta_{s,r+1}\sum_{j=1}^{r-1} a_{j,s+1}(w)a^*_{jr}(w)\delta(w/z)\end{aligned}$$

so that

$$\begin{aligned}[\rho(F_r)(z), \rho(F_s)(w)] = {} & \big(\delta_{s,r+1}a_{r,r+2}(w) + \delta_{r,s+1}a_{r-1,r+1}(w)\big)\delta(w/z) -\\ & - \delta_{r,s+1}\sum_{j=1}^{r-2} a_{j,r+1}(w)a^*_{j,r-1}(w)\delta(w/z) +\\ & + \delta_{s,r+1}\sum_{j=1}^{r-1} a_{j,r+2}(w)a^*_{jr}(w)\delta(w/z).\end{aligned}$$

As the second index in a_{ij}^* (resp. a_{ij}) in $\rho(F_r)(z)$ is r (resp. $r+1$) we get

$$[\rho(F_r)(z_1), \rho(F_r)(z_1), \rho(F_s)(w)] = 0.$$

This completes the proof of the relations R5 and R6 for $\rho(F_r)(z)$.

Now we break up $\rho(E_r)(z)$ into four summands

$$\begin{aligned}
\rho(E_r^1)(z) &:= a_{r,r+1}(z)a_{r,r+1}^*(z)^2,\\
\rho(E_r^2)(z) &:= \sum_{j=r+2}^{n}\big(a_{rj}(z)a_{rj}^*(z) - a_{r+1,j}(z)a_{r+1,j}^*(z)\big)a_{r,r+1}^*(z),\\
\rho(E_r^3)(z) &:= \sum_{j=1}^{r-1} a_{jr}(z)a_{j,r+1}^*(z) - \sum_{j=r+2}^{n} a_{r+1,j}(z)a_{rj}^*(z),\\
\rho(E_r^4)(z) &:= -\gamma a_{r,r+1}^*(z)\left(b_r(z) - \frac{1}{2}a_{r,r+1}^*(z)\big(b_{r-1}^+(z) + b_{r+1}^+(z)\big)\right) -\\
&\quad - \frac{\gamma^2}{2}\dot{a}_{r,r+1}^*(z).
\end{aligned}$$

Then

$$\begin{aligned}
&[\rho(E_r^1)(z), \rho(E_s^1)(w)] = 0,\\
&[\rho(E_r^1)(z), \rho(E_s^2)(w)]\\
&\quad = -\delta_{r,s}\sum_{j=r+2}^{n}\big(a_{rj}(w)a_{rj}^*(w) - a_{r+1,j}(w)a_{r+1,j}^*(w)\big)a_{r,r+1}^*(z)^2\delta(w/z) -\\
&\quad\quad - \delta_{r,s+1}a_{r,r+1}(w)a_{r-1,r}^*(w)a_{r,r+1}^*(z)^2\delta(w/z),
\end{aligned}$$

$$\begin{aligned}
[\rho(E_r^1)(z), \rho(E_s^3)(w)] &= 2\delta_{r,s-1}a_{r,r+1}(z)a_{r,r+1}^*(z)a_{r,r+2}^*(w)\delta(w/z) -\\
&\quad - 2\delta_{r,s+1}a_{r,r+1}(z)a_{r,r+1}^*(z)a_{r-1,r+1}^*(w)\delta(w/z),
\end{aligned}$$

$$\begin{aligned}
&[\rho(E_r^1)(z), \rho(E_s^4)(w)]\\
&\quad = -\delta_{r,s}\gamma a_{r,r+1}^*(z)^2\left(b_r(w) - \frac{1}{2}\big(b_{r-1}^+(w) + b_{r+1}^+(w)\big)\right)\delta(w/z) -\\
&\quad\quad - \delta_{r,s}\frac{\gamma^2}{2}a_{r,r+1}^*(z)^2\dot{\delta}(w/z).
\end{aligned}$$

Next the second summation of $\rho(E_r)(z)$ contributes

$$\begin{aligned}
&[\rho(E_r^2)(z), \rho(E_s^1)(w)]\\
&\quad = \delta_{r,s}\sum_{j=r+2}^{n}\big(a_{rj}(w)a_{rj}^*(w) - a_{r+1,j}(w)a_{r+1,j}^*(w)\big)a_{r,r+1}^*(z)^2\delta(w/z) +\\
&\quad\quad + \delta_{s,r+1}a_{s,s+1}(w)a_{s,s+1}^*(z)^2a_{s,s+1}^*(w)\delta(w/z)
\end{aligned}$$

so that

$$
\begin{aligned}
&[\rho(E_r^1)(z), \rho(E_s^2)(w)] + [\rho(E_r^2)(z), \rho(E_s^1)(w)] \\
&\quad = -\delta_{r,s+1} a_{r,r+1}(w) a_{r,r+1}^*(z)^2 a_{r,r+1}^*(w)\delta(w/z) + \\
&\qquad + \delta_{s,r+1} a_{s,s+1}(w) a_{s,s+1}^*(z)^2 a_{s,s+1}^*(w)\delta(w/z).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&[\rho(E_r^1)(z)\rho(E_s^3)(w)] + [\rho(E_r^3)(z), \rho(E_s^1)(w)] \\
&\quad = 2\delta_{r+1,s} a_{r,r+1}(z) a_{r,r+1}^*(z) a_{r,r+2}^*(w)\delta(w/z) - \\
&\qquad - 2\delta_{s+1,r} a_{r,r+1}(z) a_{r,r+1}^*(z) a_{r-1,r+1}^*(w)\delta(w/z) - \\
&\qquad - 2\delta_{s+1,r} a_{s,s+1}(z) a_{s,s+1}^*(z) a_{s,s+2}^*(w)\delta(w/z) + \\
&\qquad + 2\delta_{r+1,s} a_{s,s+1}(z) a_{s,s+1}^*(z) a_{s-1,s+1}^*(w)\delta(w/z)
\end{aligned}
$$

and

$$
\begin{aligned}
&[\rho(E_r^1)(z), \rho(E_s^4)(w)] + [\rho(E_r^4)(z), \rho(E_s^1)(w)] \\
&\quad = -\delta_{r,s}\frac{\gamma^2}{2}\big(a_{r,r+1}^*(z)^2\dot{\delta}(w/z) + a_{r,r+1}^*(w)^2\dot{\delta}(z/w)\big),
\end{aligned}
$$

$$
\begin{aligned}
&[\rho(E_r^2)(z), \rho(E_s^2)(w)] \\
&\quad = \delta_{s,r+1}\sum_{k=r+3}^{n}\big(a_{r+1,k}(w)a_{r+1,k}^*(w) - a_{r+2,k}(w)a_{r+2,j}^*(w)\big)\times \\
&\qquad \times a_{r,r+1}^*(z) a_{r+1,r+2}^*(z)\delta(w/z) - \\
&\qquad - \delta_{r,s+1}\sum_{j=r+2}^{n}\big(a_{rj}(z)a_{rj}^*(z) - a_{r+1,j}(z)a_{r+1,j}^*(z)\big)\times \\
&\qquad \times a_{r-1,r}^*(w) a_{r,r+1}^*(w)\delta(w/z).
\end{aligned}
$$

Now we use Lemma 2.2 to simplify the tedious calculation

$$
\begin{aligned}
&[\rho(E_r^2)(z), \rho(E_s^3)(w)] \\
&\quad = \sum_{j=r+2}^{n}\big(a_{rj}(z)a_{rj}^*(z) - a_{r+1,j}(z)a_{r+1,j}^*(z)\big)\times \\
&\qquad \times\big(\delta_{s,r+1}a_{r,r+2}^*(w) - \delta_{r,s+1}a_{r-1,r+1}^*(w)\big)\delta(w/z) - \\
&\qquad - \delta_{s,r+1}\left(a_{r,r+1}(w)a_{r,r+2}^*(z) + \sum_{j=r+3}^{n} a_{r+2,k}(w)a_{r+1,j}^*(z)\right)\times \\
&\qquad \times a_{r,r+1}^*(z)\delta(w/z) + 2\delta_{r,s}\sum_{j=r+2}^{n} a_{r+1,j}(w)a_{rj}^*(z)a_{r,r+1}^*(z)\delta(w/z) - \\
&\qquad - \delta_{r,s+1}\sum_{j=r+2}^{n} a_{rj}(z)a_{r-1,j}^*(w)a_{r,r+1}^*(z)\delta(w/z),
\end{aligned}
$$

$$
\begin{aligned}
&[\rho(E_r^3)(z), \rho(E_s^2)(w)] \\
&= -\delta_{r,s+1} \sum_{j=s+2}^{n} \big(a_{sj}(z)a_{sj}^*(z) - a_{s+1,j}(z)a_{s+1,j}^*(z)\big)a_{s,s+2}^*(w)\delta(w/z) + \\
&\quad + \delta_{r,s+1}a_{s,s+1}(w)a_{s,s+1}^*(z)a_{s,s+2}^*(z)\delta(w/z) + \\
&\quad + \delta_{r,s+1} \sum_{j=s+3}^{n} a_{s+2,k}(w)a_{s+1,j}^*(z)a_{s,s+1}^*(z)\delta(w/z) + \\
&\quad + \delta_{s,r+1} \sum_{j=s+2}^{n} a_{sj}(z)a_{s-1,j}^*(w)a_{s,s+1}^*(z)\delta(w/z) + \\
&\quad + \delta_{s,r+1} \sum_{j=s+2}^{n} \big(a_{sj}(z)a_{sj}^*(z) - a_{s+1,j}(z)a_{s+1,j}^*(z)\big)a_{s-1,s+1}^*(w)\delta(w/z).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&[\rho(E_r^2)(z), \rho(E_s^3)(w)] + [\rho(E_r^3)(z), \rho(E_s^2)(w)] \\
&= \sum_{j=r+2}^{n} \big(a_{rj}(z)a_{rj}^*(z) - a_{r+1,j}(z)a_{r+1,j}^*(z)\big)\times \\
&\quad \times\big(\delta_{s,r+1}a_{r,r+2}^*(w) - \delta_{r,s+1}a_{r-1,r+1}^*(w)\big)\delta(w/z) - \\
&\quad - \sum_{j=s+2}^{n} \big(a_{sj}(w)a_{sj}^*(w) - a_{s+1,j}(w)a_{s+1,j}^*(w)\big)\times \\
&\quad \times\big(\delta_{r,s+1}a_{s,s+2}^*(z) - \delta_{s,r+1}a_{s-1,s+1}^*(z)\big)\delta(z/w) - \\
&\quad - \delta_{s,r+1}\left(a_{r,r+1}(w)a_{r,r+2}^*(z) + \right. \\
&\qquad\qquad \left. + \sum_{j=r+3}^{n} a_{r+2,k}(w)a_{r+1,j}^*(z)\right)a_{r,r+1}^*(z)\delta(w/z) + \\
&\quad + \delta_{r,s+1}\left(a_{s,s+1}(z)a_{s,s+2}^*(w) + \right. \\
&\qquad\qquad \left. + \sum_{j=s+3}^{n} a_{s+2,k}(z)a_{s+1,j}^*(w)\right)a_{s,s+1}^*(w)\delta(z/w) - \\
&\quad - \delta_{r,s+1} \sum_{j=r+2}^{n} a_{rj}(z)a_{r-1,j}^*(w)a_{r,r+1}^*(z)\delta(w/z) + \\
&\quad + \delta_{s,r+1} \sum_{j=s+2}^{n} a_{sj}(w)a_{s-1,j}^*(z)a_{s,s+1}^*(z)\delta(w/z).
\end{aligned}
$$

Our final computation for $\rho(E_r^2)(z)$ is

$$
\begin{aligned}
&[\rho(E_r^2)(z), \rho(E_s^4)(w)] \\
&= a^*_{r,r+1}(z)a^*_{r+1,r+2}(z)\delta_{s,r+1}\Bigg(\gamma b_s(w)\delta(w/z) - \\
&\quad - \frac{\gamma}{2}\big(b^+_{s-1}(w) + b^+_{s+1}(w)\big)\delta(w/z) + \frac{\gamma^2}{2}\dot{\delta}(w/z)\Bigg).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&[\rho(E_r^2)(z), \rho(E_s^4)(w)] + [\rho(E_r^4)(z), \rho(E_s^2)(w)] \\
&= a^*_{r,r+1}(z)a^*_{r+1,r+2}(z)\delta_{s,r+1}\Bigg(\gamma b_s(w)\delta(w/z) - \\
&\quad - \frac{\gamma}{2}\big(b^+_{s-1}(w) + b^+_{s+1}(w)\big)\delta(w/z) + \frac{\gamma^2}{2}\dot{\delta}(w/z)\Bigg) - \\
&\quad - a^*_{s,s+1}(w)a^*_{s+1,s+2}(w)\delta_{r,s+1}\Bigg(\gamma b_r(z)\delta(z/w) - \\
&\quad - \frac{\gamma}{2}\big(b^+_{r-1}(z) + b^+_{r+1}(z)\big)\delta(z/w) + \frac{\gamma^2}{2}\dot{\delta}(z/w)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&[\rho(E_r^3)(z), \rho(E_s^3)(w)] \\
&= \delta_{s+1,r}\Bigg(-\sum_{j=1}^{r-2} a_{j,r-1}(w)a^*_{j,r+1}(z) + \sum_{j=r+2}^{n} a_{r+1,j}(z)a^*_{r-1,j}(w)\Bigg)\delta(w/z) + \\
&\quad + \delta_{s,r+1}\Bigg(\sum_{j=1}^{r-1} a_{jr}(z)a^*_{j,r+2}(w) - \sum_{j=r+3}^{n} a_{r+2,j}(w)a^*_{rj}(z)\Bigg)\delta(w/z),
\end{aligned}
$$

$$
\begin{aligned}
&[\rho(E_r^3)(z), \rho(E_s^4)(w)] \\
&= \big(\delta_{r+1,s}a^*_{s-1,s+1}(z) - \delta_{r,s+1}a^*_{r-1,r+1}(z)\big)\times \\
&\quad \times\Bigg(\Bigg(\gamma b_s(z) - \frac{\gamma}{2}\big(b^+_{s-1}(z) + b^+_{s+1}(z)\big)\Bigg)\delta(w/z) + \frac{\gamma^2}{2}\dot{\delta}(w/z)\Bigg) - \\
&\quad - \frac{\gamma^2}{2}\delta_{r,s+1}\dot{a}^*_{r-1,r+1}(z)\delta(w/z)
\end{aligned}
$$

so that

$$
\begin{aligned}
&[\rho(E_r^3)(z), \rho(E_s^4)(w)] + [\rho(E_r^4)(z), \rho(E_s^3)(w)] \\
&= \big(\delta_{r+1,s}a^*_{s-1,s+1}(z) - \delta_{r,s+1}a^*_{r-1,r+1}(z)\big)\Bigg(\gamma(b_s(z) + b_r(z)) - \\
&\quad - \frac{\gamma}{2}\big(b^+_{s-1}(z) + b^+_{r-1}(z) + b^+_{s+1}(z) + b^+_{r+1}(z)\big)\Bigg)\delta(w/z) + \\
&\quad + \frac{\gamma^2}{2}\big(\delta_{r+1,s}\dot{a}^*_{s-1,s+1}(z) - \delta_{r,s+1}\dot{a}^*_{r-1,r+1}(z)\big)\delta(w/z).
\end{aligned}
$$

Before adding all of these up we need to calculate

$$
\begin{aligned}
&[\rho(E_r^4)(z), \rho(E_s^4)(w)] \\
&\quad = \gamma^2 a^*_{r,r+1}(z) a^*_{s,s+1}(w) \left(\delta_{r,s} - \frac{1}{2}(\delta_{r,s+1} + \delta_{s,r+1}) \right) \dot{\delta}(w/z).
\end{aligned}
$$

Now we add to obtain

$$
\begin{aligned}
&[\rho(E_r)(z), \rho(E_s)(w)] \\
&\quad = \big(-\delta_{r,s+1} a_{r,r+1}(w) a^*_{r,r+1}(z)^2 a^*_{r-1,r}(w) + \\
&\qquad + \delta_{s,r+1} a_{s,s+1}(w) a^*_{s-1,s}(z)^2 a^*_{s,s+1}(w) + \\
&\qquad + 2\delta_{r+1,s} a_{r,r+1}(z) a^*_{r,r+1}(z) a^*_{r,r+2}(w) - \\
&\qquad - 2\delta_{s+1,r} a_{r,r+1}(z) a^*_{r,r+1}(z) a^*_{r-1,r+1}(w) - \\
&\qquad - 2\delta_{s+1,r} a_{s,s+1}(z) a^*_{s,s+1}(z) a^*_{s,s+2}(w) + \\
&\qquad + 2\delta_{r+1,s} a_{s,s+1}(z) a^*_{s,s+1}(z) a^*_{s-1,s+1}(w) \big) \delta(w/z) + \\
&\qquad + \delta_{r,s} \frac{\gamma^2}{2} (a^*_{r,r+1}(w)^2 - a^*_{r,r+1}(z)^2) \dot{\delta}(w/z) + \\
&\qquad + \delta_{s,r+1} \sum_{k=r+3}^{n} \big(a_{r+1,k}(w) a^*_{r+1,k}(w) - \\
&\qquad\qquad - a_{r+2,k}(w) a^*_{r+2,j}(w) \big) a^*_{r,r+1}(z) a^*_{r+1,r+2}(z) \delta(w/z) - \\
&\qquad - \delta_{r,s+1} \sum_{j=r+2}^{n} \big(a_{rj}(z) a^*_{rj}(z) - \\
&\qquad\qquad - a_{r+1,j}(z) a^*_{r+1,j}(z) \big) a^*_{r-1,r}(w) a^*_{r,r+1}(w) \delta(w/z) + \\
&\qquad + \sum_{j=r+2}^{n} \big(a_{rj}(z) a^*_{rj}(z) - a_{r+1,j}(z) a^*_{r+1,j}(z) \big) \times \\
&\qquad \times \big(\delta_{s,r+1} a^*_{r,r+2}(w) - \delta_{r,s+1} a^*_{r-1,r+1}(w) \big) \delta(w/z) - \\
&\qquad - \sum_{j=s+2}^{n} \big(a_{sj}(w) a^*_{sj}(w) - a_{s+1,j}(w) a^*_{s+1,j}(w) \big) \times \\
&\qquad \times \big(\delta_{r,s+1} a^*_{s,s+2}(z) - \delta_{s,r+1} a^*_{s-1,s+1}(z) \big) \delta(z/w) - \\
&\qquad - \delta_{s,r+1} \Bigg(a_{r,r+1}(w) a^*_{r,r+2}(z) + \\
&\qquad\qquad + \sum_{j=r+3}^{n} a_{r+2,k}(w) a^*_{r+1,j}(z) \Bigg) a^*_{r,r+1}(z) \delta(w/z) + \\
&\qquad + \delta_{r,s+1} \Bigg(a_{s,s+1}(z) a^*_{s,s+2}(w) +
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\quad + \sum_{j=s+3}^{n} a_{s+2,k}(z)a^*_{s+1,j}(w)\Bigg)a^*_{s,s+1}(w)\delta(z/w) - \\
&- \delta_{r,s+1}\sum_{j=r+2}^{n} a_{rj}(z)a^*_{r-1,j}(w)a^*_{r,r+1}(z)\delta(w/z) + \\
&+ \delta_{s,r+1}\sum_{j=s+2}^{n} a_{sj}(w)a^*_{s-1,j}(z)a^*_{s,s+1}(z)\delta(w/z) + \\
&+ a^*_{r,r+1}(z)a^*_{r+1,r+2}(z)\delta_{s,r+1}\Bigg(\gamma b_s(w)\delta(w/z) - \\
&- \frac{\gamma}{2}\big(b^+_{s-1}(w) + b^+_{s+1}(w)\big)\delta(w/z) + \frac{\gamma^2}{2}\dot{\delta}(w/z)\Bigg) - \\
&- a^*_{s,s+1}(w)a^*_{s+1,s+2}(w)\delta_{r,s+1}\Bigg(\gamma b_r(z)\delta(z/w) - \\
&- \frac{\gamma}{2}\big(b^+_{r-1}(z) + b^+_{r+1}(z)\big)\delta(z/w) + \frac{\gamma^2}{2}\dot{\delta}(z/w)\Bigg) + \\
&+ \delta_{s+1,r}\Bigg(-\sum_{j=1}^{r-2} a_{j,r-1}(w)a^*_{j,r+1}(z) + \\
&\qquad\quad + \sum_{j=r+2}^{n} a_{r+1,j}(z)a^*_{r-1,j}(w)\Bigg)\delta(w/z) + \\
&+ \delta_{s,r+1}\Bigg(\sum_{j=1}^{r-1} a_{jr}(z)a^*_{j,r+2}(w) - \sum_{j=r+3}^{n} a_{r+2,j}(w)a^*_{rj}(z)\Bigg)\delta(w/z) + \\
&+ \big(\delta_{r+1,s}a^*_{s-1,s+1}(z) - \delta_{r,s+1}a^*_{r-1,r+1}(z)\big)\Bigg(\gamma(b_s(z) + b_r(z)) - \\
&- \frac{\gamma}{2}\big(b^+_{s-1}(z) + b^+_{r-1}(z) + b^+_{s+1}(z) + b^+_{r+1}(z)\big)\Bigg)\delta(w/z) + \\
&+ \gamma^2 a^*_{r,r+1}(z)a^*_{s,s+1}(w)\Bigg(\delta_{r,s} - \frac{1}{2}(\delta_{r,s+1} + \delta_{s,r+1})\Bigg)\dot{\delta}(w/z) + \\
&+ \frac{\gamma^2}{2}\big(\delta_{r+1,s}\dot{a}^*_{s-1,s+1}(z) - \delta_{r,s+1}\dot{a}^*_{r-1,r+1}(z)\big)\delta(w/z).
\end{aligned}
$$

This proves the result for $r \neq s \pm 1$. In particular, a straightforward calculation shows that the three terms above with $\delta_{r,s}$ in them cancel.

If $r = s + 1$ then we get

$$
\begin{aligned}
&[\rho(E_r)(z), \rho(E_{r-1})(w)] \\
&\quad = -a_{r,r+1}(w)a^*_{r,r+1}(z)^2 a^*_{r-1,r}(w)\delta(w/z) - \\
&\quad\quad - a_{r,r+1}(z)a^*_{r,r+1}(z)a^*_{r-1,r+1}(w)\delta(w/z) -
\end{aligned}
$$

$$
\begin{aligned}
&- a_{r-1,r}(z)a^*_{r-1,r}(z)a^*_{r-1,r+1}(w)\delta(w/z) - \\
&- a_{r-1,r+1}(w)a^*_{r-1,r+1}(z)a^*_{r-1,r+1}(w)\delta(z/w) - \\
&- \sum_{j=r+2}^{n} \big(a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z)\big)a^*_{r-1,r}(w)a^*_{r,r+1}(w)\delta(w/z) - \\
&- \sum_{j=r+2}^{n} \big(a_{r-1,j}(z)a^*_{r-1,j}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z)\big)a^*_{r-1,r+1}(w)\delta(w/z) + \\
&+ \sum_{j=r+2}^{n} a_{r+1,k}(z)a^*_{rj}(w)a^*_{r-1,r}(w)\delta(z/w) - \\
&- \sum_{j=r+2}^{n} a_{rj}(z)a^*_{r-1,j}(w)a^*_{r,r+1}(z)\delta(w/z) - \\
&- a^*_{r-1,r}(w)a^*_{r,r+1}(w)\bigg(\gamma b_r(z)\delta(z/w) - \\
&\qquad\qquad - \frac{\gamma}{2}\big(b^+_{r-1}(z) + b^+_{r+1}(z)\big)\delta(z/w)\bigg) + \\
&+ \Bigg(-\sum_{j=1}^{r-2} a^*_{j,r+1}(z)a_{j,r-1}(w) + \sum_{j=r+2}^{n} a^*_{r-1,j}(w)a_{r+1,j}(z)\Bigg)\delta(w/z) - \\
&- a^*_{r-1,r+1}(z)\bigg(\gamma(b_{r-1}(z) + b_r(z)) - \\
&\qquad - \frac{\gamma}{2}\big(b^+_{r-2}(z) + b^+_{r-1}(z) + b^+_r(z) + b^+_{r+1}(z)\big)\bigg)\delta(w/z) - \\
&- \frac{\gamma^2}{2}\big(\dot{a}^*_{r,r+1}(w)a^*_{r-1,r}(w) + \dot{a}^*_{r-1,r+1}(z)\big)\delta(w/z).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&[\rho(E^1_r)(z_1), \rho(E_r)(z_2), \rho(E_{r-1})(w)] \\
&\quad = -a_{r,r+1}(w)a^*_{r,r+1}(z_1)^2 a^*_{r-1,r+1}(w)\delta(z_1/z_2)\delta(w/z_2) + \\
&\quad + \sum_{j=r+2}^{n} \big(a_{rj}(z_2)a^*_{rj}(z_2) - \\
&\qquad\qquad - a_{r+1,j}(z_2)a^*_{r+1,j}(z_2)\big)a^*_{r,r+1}(z_1)^2 a^*_{r-1,r}(w)\delta(z_1/z_2)\delta(w/z_2) + \\
&\quad + \sum_{j=r+2}^{n} a_{rj}(z_2)a^*_{r-1,j}(w)a^*_{r,r+1}(z_1)^2\delta(z_1/z_2)\delta(w/z_2) + \\
&\quad + a^*_{r,r+1}(z_1)^2 a^*_{r-1,r}(w)\delta(z_1/z_2)\delta(w/z_2) \times \\
&\quad \times \bigg(\gamma b_r(z_2) - \frac{\gamma}{2}\big(b^+_{r-1}(z_2) + b^+_{r+1}(z_2)\big)\bigg) -
\end{aligned}
$$

$$- \frac{\gamma^2}{2} a^*_{r,r+1}(z_1)^2 a^*_{r-1,r}(w) \dot{\delta}(w/z_1) \delta(z_2/w).$$

Next we have

$$\begin{aligned}
&[\rho(E_r^2)(z_1), \rho(E_r)(z_2), \rho(E_{r-1})(w)] \\
&= - \sum_{j=r+2}^{n} \big(a_{rj}(z_1) a^*_{rj}(z_1) - \\
&\qquad - a_{r+1,j}(z_1) a^*_{r+1,j}(z_1) \big) a^*_{r,r+1}(z_1)^2 a^*_{r-1,r}(w) \delta(z_1/z_2) \delta(w/z_2) - \\
&\quad - \sum_{j=r+2}^{n} \big(a_{rj}(z_1) a^*_{rj}(z_1) - \\
&\qquad - a_{r+1,j}(z_1) a^*_{r+1,j}(z_1) \big) a^*_{r,r+1}(z_1) a^*_{r-1,r+1}(w) \delta(z_1/z_2) \delta(w/z_2) - \\
&\quad - 2 \sum_{j=r+2}^{n} a_{r+1,j}(z_1) a^*_{rj}(w) a^*_{r,r+1}(z_1) a^*_{r-1,r}(w) \delta(z_1/z_2) \delta(w/z_2) - \\
&\quad - \sum_{j=r+2}^{n} a_{rj}(z_2) a^*_{r-1,j}(w) a^*_{r,r+1}(z_1)^2 \delta(z_1/z_2) \delta(w/z_2) - \\
&\quad - \sum_{j=r+2}^{n} a_{r+1,j}(z_1) a^*_{r-1,j}(w) a^*_{r,r+1}(z_1) \delta(w/z_2) \delta(z_1/z_2).
\end{aligned}$$

The third summation contributes

$$\begin{aligned}
&[\rho(E_r^3)(z_1), \rho(E_r)(z_2), \rho(E_{r-1})(w)] \\
&= \Bigg(a^*_{r-1,r+1}(z_1) a^*_{r,r+1}(w) \sum_{j=r+2}^{n} \big(a^*_{rj}(z_2) a_{rj}(z_2) - a^*_{r+1,j}(z_2) a_{r+1,j}(z_2) \big) - \\
&\quad - a^*_{r-1,r+1}(z_1) a^*_{r,r+1}(w) \bigg(\gamma b_r(z_1) - \frac{\gamma}{2} \big(b^+_{r-1}(z_1) + b^+_{r+1}(z_1) \big) \bigg) + \\
&\quad + \frac{\gamma^2}{2} a^*_{r-1,r+1}(z) \dot{a}^*_{r,r+1}(w) + \\
&\quad + 2 \sum_{j=r+2}^{n} a_{r+1,j}(z_1) a^*_{rj}(z_1) a^*_{r-1,r}(w) a^*_{r,r+1}(w) + \\
&\quad + \sum_{j=r+2}^{n} a_{r+1,j}(z) a^*_{r-1,k}(w) a^*_{r,r+1}(z_2) \Bigg) \delta(z_1/z_2) \delta(w/z_2).
\end{aligned}$$

The last summation contributes

$$[\rho(E_r^4)(z_1), \rho(E_r)(z_2), \rho(E_{r-1})(w)]$$

$$= -a^*_{r,r+1}(z_2)^2 a^*_{r-1,r}(w)\delta(z_1/z_2)\delta(w/z_2)\Big(\gamma b_r(z_1) - \frac{\gamma}{2}\big(b^+_{r-1}(z_1) + b^+_{r+1}(z_1)\big)\Big) -$$
$$- \frac{\gamma^2}{2} a^*_{r,r+1}(z_2)^2 a^*_{r-1,r}(w)\dot{\delta}(z_1/w)\delta(w/z_2) +$$
$$+ a^*_{r,r+1}(z_2) a^*_{r-1,r+1}(w)\delta(z_1/z_2)\delta(w/z_2)\Big(\gamma b_r(z_1) - \frac{\gamma}{2}\big(b^+_{r-1}(z_1) + b^+_{r+1}(z_1)\big)\Big) -$$
$$- \frac{\gamma^2}{2} a^*_{r,r+1}(z_2) a^*_{r-1,r+1}(w)\dot{\delta}(z_1/z_2)\delta(w/z_2) -$$
$$- \gamma^2 a^*_{r,r+1}(z_1) a^*_{r-1,r}(w) a^*_{r,r+1}(w)\delta(z_2/w)\dot{\delta}(z_2/z_1) +$$
$$+ \frac{\gamma^2}{2} a^*_{r,r+1}(z_1) a^*_{r-1,r+1}(z_2)\dot{\delta}(z_1/z_2)\delta(w/z_2).$$

Consequently

$$[\rho(E_r)(z_1), \rho(E_r)(z_2), \rho(E_{r-1})(w)] = 0.$$

Now we turn to the last case $s = r + 1$. In this case

$$[\rho(E_r)(z), \rho(E_{r+1})(w)]$$
$$= a_{r+1,r+2}(w) a^*_{r+1,r+2}(z)^2 a^*_{r,r+1}(w)\delta(w/z) +$$
$$+ a_{r,r+1}(z) a^*_{r,r+1}(z) a^*_{r,r+2}(w)\delta(w/z) +$$
$$+ a_{r+1,r+2}(z) a^*_{r+1,r+2}(z) a^*_{r,r+2}(w)\delta(w/z) +$$
$$+ a_{r,r+2}(z) a^*_{r,r+2}(z) a^*_{r,r+2}(w)\delta(w/z) +$$
$$+ \sum_{j=r+3}^{n} \big(a_{r+1,j}(w) a^*_{r+1,j}(w) - a_{r+2,j}(w) a^*_{r+2,j}(w)\big) a^*_{r,r+1}(z) a^*_{r+1,r+2}(z)\delta(w/z) +$$
$$+ \sum_{j=r+3}^{n} \big(a_{rj}(z) a^*_{rj}(z) - a_{r+2,j}(w) a^*_{r+2,j}(w)\big) a^*_{r,r+2}(w)\delta(w/z) -$$
$$- \sum_{j=r+3}^{n} a_{r+2,j}(w) a^*_{r+1,j}(z) a^*_{r,r+1}(z)\delta(w/z) +$$
$$+ a_{r+1,j}(w) \sum_{j=r+3}^{n} a^*_{r,j}(z) a^*_{r+1,r+2}(z)\delta(w/z) +$$
$$+ a^*_{r,r+1}(z) a^*_{r+1,r+2}(z)\Big(\gamma b_{r+1}(w)\delta(w/z) -$$

$$
\begin{aligned}
&\qquad\qquad - \frac{\gamma}{2}\big(b_r^+(w) + b_{r+2}^+(w)\big)\delta(w/z)\Bigg) + \\
&+ \left(\sum_{j=1}^{r-1} a_{jr}(z)a_{j,r+2}^*(w) - \sum_{j=r+3}^{n} a_{r+2,j}(w)a_{rj}^*(z)\right)\delta(w/z) + \\
&+ a_{r,r+2}^*(z)\Bigg(\gamma(b_{r+1}(z) + b_r(z)) - \\
&\qquad - \frac{\gamma}{2}\big(b_r^+(z) + b_{r-1}^+(z) + b_{r+2}^+(z) + b_{r+1}^+(z)\big)\Bigg)\delta(w/z) + \\
&+ \frac{\gamma^2}{2}(a_{r,r+1}^*(z)\dot{a}_{r+1,r+2}^*(w) + \dot{a}_{r,r+2}^*(z))\delta(w/z).
\end{aligned}
$$

Our first commutator is

$$
\begin{aligned}
&[\rho(E_r^1)(z_1), \rho(E_r)(z_2), \rho(E_{r+1})(w)] \\
&= -\Bigg(a_{r+1,r+2}(w)a_{r,r+1}^*(z_1)^2 a_{r+1,r+2}^*(z_2)^2 - \\
&\quad - a_{r,r+1}(z_2)a_{r,r+1}^*(z_1)^2 a_{r,r+2}^*(w) + \\
&\quad + \sum_{j=r+3}^{n}\big(a_{r+1,j}(w)a_{r+1,j}^*(w) - \\
&\qquad\quad - a_{r+2,j}(w)a_{r+2,j}^*(w)\big)a_{r,r+1}^*(z_1)^2 a_{r+1,r+2}^*(z_2) - \\
&\quad - \left(\sum_{j=r+3}^{n} a_{r+2,j}(w)a_{r+1,j}^*(z_2)\right)a_{r,r+1}^*(z_1)^2 + \\
&\quad + a_{r,r+1}^*(z_1)^2 a_{r+1,r+2}^*(z_2)\left(\gamma b_{r+1}(w) - \frac{\gamma}{2}\big(b_r^+(w) + b_{r+2}^+(w)\big)\right) + \\
&\quad + \frac{\gamma^2}{2}a_{r,r+1}^*(z_1)^2\dot{a}_{r+1,r+2}^*(w)\Bigg)\delta(w/z_2)\delta(z_1/z_2).
\end{aligned}
$$

The second commutator that we need to calculate is

$$
\begin{aligned}
&[\rho(E_r^2)(z_1), \rho(E_r)(z_2), \rho(E_{r+1})(w)] \\
&\quad = a_{r+1,r+2}(w)a_{r,r+1}^*(z_1)^2 a_{r+1,r+2}^*(z_1)^2\delta(w/z_2)\delta(z_1/z_2) - \\
&\quad - \big(a_{r+1,r+2}(z_1)a_{r,r+1}^*(z_2)a_{r,r+2}^*(w)a_{r+1,r+2}^*(z_1) - \\
&\qquad - a_{r,r+2}(w)a_{r,r+1}^*(z_2)a_{r,r+2}^*(w)^2\big)\delta(w/z_2)\delta(z_1/z_2) + \\
&\quad + \sum_{j=r+3}^{n}\big(a_{r,j}(z_1)a_{r,j}^*(z_1) - \\
&\qquad\qquad - a_{r+1,j}(w)a_{r+1,j}^*(w)\big)a_{r,r+1}^*(z_2)a_{r,r+2}^*(w)\delta(w/z_2)\delta(z_1/z_2) -
\end{aligned}
$$

$$
\begin{aligned}
&- a_{r,r+1}(z_2)a^*_{r,r+1}(z_1)^2a^*_{r,r+2}(w)\delta(z_1/z_2)\delta(w/z_2) - \\
&- a_{r+1,r+2}(z_2)a^*_{r,r+1}(z_1)a^*_{r+1,r+2}(z_2)a^*_{r,r+2}(w)\delta(z_1/z_2)\delta(w/z_2) - \\
&- a_{r,r+2}(w)a^*_{r,r+1}(z_2)a^*_{r,r+2}(w)^2\delta(w/z_2)\delta(z_1/z_2) + \\
&+ \sum_{j=r+3}^{n} \big(a_{r+1,j}(w)a^*_{r+1,j}(w) - \\
&- a_{r+2,j}(w)a^*_{r+2,j}(w)\big)a^*_{r,r+1}(z_1)^2a^*_{r+1,r+2}(z_2)\delta(w/z_2)\delta(z_1/z_2) - \\
&- \sum_{j=r+3}^{n} \big(a_{rj}(z_2)a^*_{rj}(z_2) - \\
&\qquad - a_{r+2,j}(w)a^*_{r+2,j}(w)\big)a^*_{r,r+1}(z_1)a^*_{r,r+2}(w)\delta(w/z_2) - \\
&- \sum_{j=r+3}^{n} a_{r+2,k}(w)a^*_{r+1,j}(z_2)a^*_{r,r+1}(z_1)^2\delta(w/z_2)\delta(z_1/z_2) - \\
&- \sum_{j=r+3}^{n} a_{r+1,j}(w)a^*_{r,j}(z_2)a^*_{r,r+1}(z_1)a^*_{r+1,r+2}(z_2)\delta(w/z_2)\delta(z_1/z_2) + \\
&+ a^*_{r,r+1}(z_1)^2a^*_{r+1,r+2}(z_2)\bigg(\gamma b_{r+1}(w) - \\
&\qquad\qquad - \frac{\gamma}{2}\big(b^+_r(w) + b^+_{r+2}(w)\big)\bigg)\delta(w/z_2)\delta(z_1/z_2) + \\
&+ \sum_{j=r+3}^{n} a_{r+2,j}(w)a^*_{rj}(z_2)a^*_{r,r+1}(z_1)\delta(w/z_2)\delta(z_1/z_2) - \\
&- a^*_{r,r+1}(z_1)a^*_{r,r+2}(z_2)\bigg(\gamma\big(b_{r+1}(z_2) + b_r(z_2)\big) - \\
&- \frac{\gamma}{2}\big(b^+_r(z_2) + b^+_{r-1}(z_2) + b^+_{r+2}(z_2) + b^+_{r+1}(z_2)\big)\bigg)\delta(w/z_2)\delta(z_1/z_2) + \\
&+ \frac{\gamma^2}{2}a^*_{r,r+1}(z_1)\big(a^*_{r,r+1}(z_2)\dot{a}^*_{r+1,r+2}(w)\delta(z_1/z_2) - a^*_{r,r+2}(z_1)\dot{\delta}(z_2/z_1) + \\
&\qquad\qquad + a^*_{r,r+1}(z_1)a^*_{r+1,r+2}(w)\dot{\delta}(w/z_1)\big)\delta(w/z_2),
\end{aligned}
$$

$$
\begin{aligned}
&[\rho(E^3_r)(z_1), \rho(E_r)(z_2), \rho(E_{r+1})(w)] \\
&\quad = 2a_{r+1,r+2}(w)a^*_{r,r+2}(z_1)a^*_{r+1,r+2}(z_2)a^*_{r,r+1}(w)\delta(w/z_2)\delta(z_1/z_2) + \\
&\quad + \sum_{j=r+3}^{n} \big(a_{r+1,j}(w)a^*_{r+1,j}(w) - \\
&\qquad\qquad - a_{r+2,j}(w)a^*_{r+2,j}(w)\big)a^*_{r,r+1}(z_2)a^*_{r,r+2}(z_2)\delta(z_1/z_2)\delta(w/z_2) + \\
&\quad + \bigg(\sum_{j=r+3}^{n} a_{r+1,j}(z_1)a^*_{rj}(z_1)\bigg)a^*_{r,r+1}(z_2)a^*_{r+1,r+2}(z_2)\delta(z_1/z_2)\delta(w/z_2) -
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\sum_{j=r+3}^{n} a_{r+2,j}(w)a_{rj}^{*}(z_1)\right)a_{r,r+1}^{*}(z_2)\delta(z_1/z_2)\delta(w/z_2)+\\
&+a_{r,r+2}^{*}(z_1)a_{r,r+1}^{*}(z_2)\left(\gamma b_{r+1}(w)-\frac{\gamma}{2}\big(b_r^{+}(w)+b_{r+2}^{+}(w)\big)\right)\times\\
&\times\delta(z_1/z_2)\delta(w/z_2)+\\
&+\frac{\gamma^2}{2}a_{r,r+2}^{*}(z_1)a_{r,r+1}^{*}(z_2)\dot{\delta}(w/z_1)\delta(w/z_2),
\end{aligned}
$$

$$
\begin{aligned}
&[\rho(E_r^4)(z_1),\rho(E_r)(z_2),\rho(E_{r+1})(w)]\\
&\quad=\left(\gamma a_{r,r+1}^{*}(z_2)a_{r,r+2}^{*}(w)\left(b_r(z_1)-\frac{1}{2}\big(b_{r-1}^{-}(z_1)+b_{r+1}^{-}(z_1)\big)\right)\delta(z_1/z_2)+\right.\\
&\qquad\left.+\frac{\gamma^2}{2}a_{r,r+1}^{*}(z_2)a_{r,r+2}^{*}(w)\dot{\delta}(z_1/z_2)\right)\delta(w/z_2)-\\
&\quad-\frac{\gamma^2}{2}a_{r,r+1}^{*}(z_1)a_{r,r+1}^{*}(z_2)a_{r+1,r+2}^{*}(z_2)\dot{\delta}(w/z_1)\delta(w/z_2)+\\
&\quad+\frac{\gamma^2}{2}a_{r,r+1}^{*}(z_1)a_{r,r+2}^{*}(z_2)\dot{\delta}(z_2/z_1)\delta(w/z_2).
\end{aligned}
$$

Adding these all up we get

$$[\rho(E_r)(z_1),\rho(E_r)(z_2),\rho(E_{r+1})(w)]=0. \qquad \square$$

References

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