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Structure of intermediate Wakimoto modules

Ben L. Cox [a], Vyacheslav Futorny [b,*]

[a] *Department of Mathematics, College of Charleston, 66 George Street, Charleston, SC 29424, USA*
[b] *Institute of Mathematics, University of São Paulo, Caixa Postal 66281, São Paulo, CEP 05315-970, Brazil*

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Abstract

We show that our construction of boson type realizations of affine $\mathfrak{sl}(n+1)$ in terms of intermediate Wakimoto modules gives representations that are generically isomorphic to certain Verma type modules. We then use this identification to obtain information about the submodule structure of intermediate Wakimoto modules.

Keywords: Affine Kac–Mody algebras; Wakimoto modules; Verma modules

1. Introduction

Modules induced from the *natural Borel subalgebra* were first introduced by H. Jakobsen and V. Kac in their study of unitarizable highest weight representations of affine Kac–Moody algebras (see [JK85]). They were studied in [Fut94] under the name of *imaginary Verma modules*. A Fock space realization of the imaginary Verma modules for $\hat{\mathfrak{sl}}(2)$ were constructed by D. Bernard and G. Felder in [BF90] and then extended in [Cox04] to the case of $\hat{\mathfrak{sl}}(n)$. These realizations are given generically by certain Wakimoto type modules.

In his effort to prove the Kac–Kazhdan conjecture on the characters of irreducible representations of affine Kac–Moody algebras at the critical level, M. Wakimoto discovered a remarkable boson realization of $\hat{\mathfrak{sl}}(2)$ on the Fock space $\mathbb{C}[x_i, y_j \mid i \in \mathbb{Z},\ j \in \mathbb{N}]$ [Wak86]. Wakimoto modules for general affine Lie algebras were introduced by B. Feĭgin and E. Frenkel in [FF88] by

* Corresponding author.
E-mail addresses: coxbl@cofc.edu (B.L. Cox), futorny@ime.usp.br (V. Futorny).

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a homological characterization, [FF90b] which play an important role in the conformal field theory providing a new bosonization rule for the Wess–Zumino–Witten models. Wakimoto modules have a geometric interpretation as certain sheaves on a semi-infinite flag manifold [FF90a]. They belong to the category $\mathcal{O}$ and generically are isomorphic to corresponding Verma modules. Explicit formulae for these realizations for $\hat{\mathfrak{sl}}(n)$ are given in [FF88] and for general affine Lie algebras they are given in [dBF97,PRY96].

Affine Lie algebras admit Verma type modules associated with non-standard Borel subalgebras, see [Cox94b,FS93,JK89]. In our work [CF04] the problem of finding suitable boson type realizations for all Verma type modules over $\hat{\mathfrak{sl}}(n+1)$ was solved. In Theorem 4.3 below we construct such realizations, *intermediate Wakimoto modules*, for a series of generic Verma type modules depending on the parameter $0 \leqslant r \leqslant n$. If $r = n$ this construction coincides with the boson realization of Wakimoto modules in [FF88], and when $r = 0$ this is a realization described in [Cox04]. One difficulty that arises in the study of Verma type modules that are not induced from a standard Borel subalgebra is that certain of their weight spaces are infinite dimensional. On the other hand, the structure of representations that have infinite-dimensional weight spaces is an important problem that appears naturally in other contexts. Besides appearing in the representation theory of infinite-dimensional Heisenberg Lie algebras such representations also arise in the work of [BC94,CP87,CF01,Cox02,CM01]. Intermediate Wakimoto modules form another family of representations with certain weight spaces being infinite dimensional. Using the realization of the intermediate Wakimoto module for $\hat{\mathfrak{sl}}(n+1,\mathbb{C})$ given in [CF04] and below, we show that generically Verma type modules and intermediate Wakimoto modules are isomorphic, which is an analog of the classical relation between Verma and Wakimoto modules (see Corollary 5.3 below). Moreover, when intermediate Wakimoto modules are in general position (but not necessarily isomorphic to Verma type module) we completely describe their submodule structure (6.3).

Verma type modules have a complicated structure when the center c acts by zero (see, for example, [Fut94]). The realization given in Theorem 4.3 yields information about the structure of these modules at least in the case of $\hat{\mathfrak{sl}}(2,\mathbb{C})$. In the last section we present the formulas for the singular elements in imaginary Verma modules recently obtained by B. Wilson [Wil05]. These formulas were inspired by the free field realization of imaginary Verma modules for $\hat{\mathfrak{sl}}(2,\mathbb{C})$.

2. Verma type modules

Fix a positive integer n, $0 \leqslant r \leqslant n$, $\gamma \in \mathbb{C}^*$. Set $k = \gamma^2 - (r+1)$. Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ and let E^{ij}, $i, j = 1, \dots, n+1$ be the standard basis for $\mathfrak{gl}(n+1,\mathbb{C})$. Set $H_i := E^{ii} - E^{i+1,i+1}$, $E_i := E^{i,i+1}$, $F_i := E^{i+1,i}$ which is a basis for $\mathfrak{sl}(n+1,\mathbb{C})$. Furthermore we denote the Killing form by $(X|Y) = \mathrm{tr}(XY)$ and $X_m = t^m \otimes X$ for $X, Y \in \mathfrak{g}$ and $m \in \mathbb{Z}$. Let $\{\alpha_1, \dots, \alpha_n\}$ be a base for Δ^+, the positive set of roots for $\mathfrak{g}$, such that $H_i = \check{\alpha}_i$ and let Δ_r be the root system with base $\{\alpha_1, \dots, \alpha_r\}$ ($\Delta_r = \emptyset$, if $r = 0$) of the Lie subalgebra $\mathfrak{g}_r = \mathfrak{sl}(r+1,\mathbb{C})$. A Cartan subalgebra $\mathfrak{H}$ (respectively $\mathfrak{H}_r$) of $\mathfrak{g}$ (respectively $\mathfrak{g}_r$) is spanned by H_i, $i = 1, \dots, n$ (respectively $i = 1, \dots, r$) and set $\mathfrak{H}_0 = 0$.

For any Lie algebra $\mathfrak{a}$, let $L(\mathfrak{a}) = \mathfrak{a} \oplus \mathbb{C}[t, t^{-1}]$ be the loop algebra of $\mathfrak{a}$. Then $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}(n+1,\mathbb{C}) = L(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{g}}_r = L(\mathfrak{g}_r) \oplus \mathbb{C}c \oplus \mathbb{C}d$ are the associated affine Kac–Moody algebras with $\hat{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{H}}_r = \mathfrak{H}_r \oplus \mathbb{C}c \oplus \mathbb{C}d$, respectively.

The algebra $\hat{\mathfrak{g}}$ has generators E_{im}, F_{im}, H_{im}, $i = 1, \dots, n$, $m \in \mathbb{Z}$, and central element c with the product

$$[X_m, Y_n] = t^{m+n}[X, Y] + \delta_{m+n,0}\, m(X|Y)c.$$

Let $\mathfrak{a}$ be a Lie algebra with a Cartan subalgebra H and root system Δ. Denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$. A closed subset $P \subset \Delta$ is called a partition if $P \cap (-P) = \emptyset$ and $P \cup (-P) = \Delta$. If $\mathfrak{a}$ is finite dimensional then every partition corresponds to a choice of positive roots in Δ and all partitions are conjugate by the Weyl group. The situation is different in the infinite-dimensional case. If $\mathfrak{a}$ is an affine Lie algebra then partitions are divided into a finite number of Weyl group orbits (cf. [JK89,Fut97]).

Given a partition P of Δ we define a *Borel* subalgebra $\mathfrak{b}_P \subset \mathfrak{a}$ generated by H and the root spaces $\mathfrak{a}_\alpha$ with $\alpha \in P$. All Borel subalgebras are conjugate in the finite-dimensional case. A *parabolic* subalgebra is a subalgebra that contains a Borel subalgebra. If $\mathfrak{p}$ is a parabolic subalgebra of a finite-dimensional $\mathfrak{a}$ then $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+$ where $\mathfrak{p}_0$ is a reductive Levi factor and $\mathfrak{p}_+$ is a nilpotent subalgebra. Parabolic subalgebras correspond to a choice of a basis π of the root system Δ and a subset $S \subset \pi$. A classification of all Borel subalgebras in the affine case was obtained in [Fut97]. In this case not all of them are conjugate but there exists a finite number of conjugacy classes. These conjugacy classes are parametrized by parabolic subalgebras of the underlined finite-dimensional Lie algebra. Namely, let $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+$ a parabolic subalgebra of $\mathfrak{g}$ containing a fixed Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Define

$$B_{\mathfrak{p}} = \left(\mathfrak{p}_+ \otimes \mathbb{C}\left[t, t^{-1}\right]\right) \oplus \left(\mathfrak{p}_0 \otimes t\mathbb{C}[t]\right) \oplus (\mathfrak{b} \setminus \mathfrak{p}_+) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

For any Borel subalgebra $\mathfrak{B}$ of $\hat{\mathfrak{g}}$ there exists a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{B}$ is conjugate to $B_{\mathfrak{p}}$.

When $\mathfrak{p}$ coincides with $\mathfrak{g}$, i.e. $\mathfrak{p}_+ = 0$, the corresponding Borel subalgebra $B_{\mathfrak{g}}$ is the *standard* Borel subalgebra defined by the choice of positive roots in $\hat{\mathfrak{g}}$. Another extreme case is when $\mathfrak{p}_0 = \mathcal{H}$. This corresponds to the *natural* Borel subalgebra B_{nat} of $\hat{\mathfrak{g}}$ considered in [JK89].

Given a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ let $\lambda : B_{\mathfrak{p}} \to \mathbb{C}$ be a 1-dimensional representation of $B_{\mathfrak{p}}$. Then one defines an induced *Verma type* $\hat{\mathfrak{g}}$-module

$$M_{\mathfrak{p}}(\lambda) = U(\hat{\mathfrak{g}}) \otimes_{U(B_{\mathfrak{p}})} \mathbb{C}.$$

The module $M_{\mathfrak{g}}(\lambda)$ is the classical Verma module with highest weight λ [Kac90]. In the case of natural Borel subalgebra we obtain *imaginary* Verma modules studied in [Fut94]. Note that the module $M_{\mathfrak{p}}(\lambda)$ is $U(\mathfrak{p}_-)$-free, where $\mathfrak{p}_-$ is the opposite subalgebra to $\mathfrak{p}_+$. The theory of Verma type modules was developed in [Fut97]. It follows immediately from the definition that, unless it is a classical Verma module, a Verma type module with highest weight λ has a unique maximal submodule, it has both finite and infinite-dimensional weight spaces and it can be obtained using the parabolic induction from a standard Verma module M with highest weight λ over a certain affine Lie subalgebra. Moreover, if the central element c acts non-trivially on such Verma type module, then the structure of this module is completely determined by the structure of module M, which is well known [Fut97,Cox94a].

Let $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\pm\alpha}$. Denote $\mathfrak{n}_r^\pm = \mathfrak{n}^\pm \cap \mathfrak{g}_r$, $\mathfrak{n}^\pm(r) = \mathfrak{n}^\pm \setminus \mathfrak{n}_r^\pm$,

$$\bar{B}_r = L\left(\mathfrak{n}^+(r)\right) \oplus \left(\mathfrak{n}_r^+ \otimes \mathbb{C}[t]\right) \oplus \left(\left(\mathfrak{n}_r^-\right) \oplus \mathfrak{H}\right) \otimes \mathbb{C}[t]t.$$

Then $B_r = \bar{B}_r \oplus \hat{\mathfrak{H}}$ is a Borel subalgebra of $\hat{\mathfrak{g}}$ for any $0 \leqslant r \leqslant n$.

Given $\tilde{\lambda} \in \hat{\mathfrak{H}}^*$ the corresponding Verma type module is

$$M_r(\tilde{\lambda}) = U(\hat{\mathfrak{g}}) \otimes_{U(B_r)} \mathbb{C}v_{\tilde{\lambda}}$$

where $\bar{B}_r v_{\tilde{\lambda}} = 0$ and $hv_{\tilde{\lambda}} = \tilde{\lambda}(h)v_{\tilde{\lambda}}$ for all $h \in \hat{\mathfrak{H}}$.

When $r = n$ it gives us the usual Verma module construction. If $r = 0$ we get an imaginary Verma module.

Let $\tilde{\lambda}_r = \tilde{\lambda}|_{\hat{\mathfrak{H}}_r}$. Verma type module $M_r(\tilde{\lambda})$ contains a $\hat{\mathfrak{g}}_r$-submodule $M(\tilde{\lambda}_r) = U(\hat{\mathfrak{g}}_r)(1 \otimes v_{\tilde{\lambda}})$ which is isomorphic to a usual Verma module for $\hat{\mathfrak{g}}_r$.

Theorem 2.1. [Cox94b,FS93] *Let $\tilde{\lambda}(c) \neq 0$. Then the submodule structure of $M_r(\tilde{\lambda})$ is completely determined by the submodule structure of $M(\tilde{\lambda}_r)$. In particular, $M_r(\tilde{\lambda})$ is irreducible if $M(\tilde{\lambda}_r)$ is irreducible.*

Let V be a weight $\mathfrak{a}$-module, i.e. $V = \bigoplus_{\mu \in H^*} V_\mu$, $V_\mu = \{v \in V \mid hv = \mu(h)v,\ \forall h \in H\}$. Suppose that $\dim V_\mu \leqslant \infty$ for all μ. If $\mathfrak{a}$ is a Kac–Moody Lie algebra (finite or affine) with Serre generators e_i's and f_i's, then denote by w an anti-involution on $\mathfrak{a}$ which permutes e_i with f_i for all i, and which is the identity on H. Consider the $\mathfrak{a}$-module

$$V^* = \bigoplus_{\mu \in H^*} V_\mu^*,$$

where V_μ^* is a dual subspace of V_μ and the structure of $\mathfrak{a}$-module is given by: $(xf)(v) = f(w(x)v)$. Such modules are called *contragradient*. The advantage of considering these modules versus the whole dual modules of V is that V and its contragradient module belong to the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ simultaneously.

3. Geometric realizations

In this section we recall certain geometric realizations of Verma and imaginary Verma modules which served as a motivation for our main construction in Section 4.

3.1. Finite-dimensional case

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $\mathfrak{b}$ a Borel subalgebra, $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{b} = \operatorname{Lie} B$. Then the group G acts transitively on the *flag variety* $X = G/B$. Let $\mathcal{D}_X$ be the sheaf of differential operators on X with regular coefficients. If $\mathcal{M}$ is a $\mathcal{D}_X$-module, then the global sections $\Gamma(X, \mathcal{M})$ have a structure of a $\mathfrak{g}$-module.

If $\lambda \in \mathcal{H}^*$, then the Verma module $M(\lambda)$ with the highest weight λ admits the central character which we denote by ξ_λ. Let $\xi : Z(\mathfrak{g}) \to \mathbb{C}$ and $\lambda \in \mathcal{H}^*$ be such that $\xi = \xi_\lambda$. Consider $U_\lambda = U(\mathfrak{g})/U(\mathfrak{g}) \operatorname{Ker} \xi$. Then there is an isomorphism of algebras

$$U_\lambda \simeq \Gamma(X, \mathcal{D}_\lambda),$$

where $\mathcal{D}_\lambda$ is a *twisted sheaf* of differential operators on X introduced by Beilinson and Bernšteĭn [BGG73]. Moreover, there is an equivalence between the category of U_λ-modules and the category of quasi-coherent $\mathcal{D}_\lambda$-modules on X [BGG73].

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathcal{H} \oplus \mathfrak{n}_+$, $\mathfrak{b} = \mathfrak{n}_- \oplus \mathcal{H}$, $\mathfrak{n}_\pm = \operatorname{Lie} N_\pm$. Then $X = G/B$ has a decomposition into open Schubert cells: $X = \bigcup_{w\in W} C(w)$, where $C(w) = B_+ w B_- / B_-$, $W = N(T)/T$ is the Weyl group and $T = B_+/N_+$. The subgroup N_+ acts on X, and the largest orbit $\mathcal{U}$ of this action can be identified with proper N_+. The Lie algebra $\mathfrak{g}$ can be mapped into vector fields on X and hence on $\mathcal{U}$. Thus $\mathfrak{g}$ can be embedded into the differential operators on $\mathcal{U}$ of degree $\leqslant 1$. Note that the ring of regular functions $\mathcal{O}_{\mathcal{U}}$ on $\mathcal{U}$ is a polynomial ring in $m = |\Delta_+|$ variables and hence $\mathfrak{g}$ has an embedding into the Weyl algebra $\mathcal{A}_m$ generated by $x_1, \dots, x_m$ and partial derivatives $\partial_1, \dots, \partial_m$. If $\xi : \mathbb{Z}(\mathfrak{g}) \to \mathbb{C}$ is the central character of $\mathfrak{g}$ then the quotient $U(\mathfrak{g})/(\operatorname{Ker}\xi)U(\mathfrak{g})$ can be embedded into $\mathcal{A}_m$, $m = (1/2)(\dim\mathfrak{g} - \operatorname{rank}\mathfrak{g})$ [Con74], providing a realization of $\mathfrak{g}$ as differential operators acting on the Fock space $\mathbb{C}[x_1, \dots, x_m]$. A different approach was suggested by Khomenko [Kh05], who showed that the quotient $U(\mathfrak{gl}(n))/(\operatorname{Ker}\xi)U(\mathfrak{gl}(n))$ can be embedded into a certain localization of $\mathcal{A}_m$, $m = n(n+1)/2$, using the theory of Gelfand–Zetlin modules [DFO94].

The embedding above of the Lie algebra $\mathfrak{g}$ into $\mathcal{A}_m$ induces the structure of a $\mathfrak{g}$-module on $\mathcal{O}_{\mathcal{U}}$. In fact, a $\mathfrak{g}$-module $\mathcal{O}_{\mathcal{U}}$ is isomorphic to a contragradient module $M^*(0)$ with trivial highest weight.

For a general $\lambda \in \mathcal{H}^*$, $\Gamma(\mathcal{U}, \mathcal{D}_\lambda) \simeq M^*(\lambda)$ is a $\mathfrak{g}$-module under the inclusion

$$\mathfrak{g} \subset \Gamma(G/B_-, \mathcal{D}_\lambda)$$

(Remark 10.2.7 in [FBZ01]). In order to obtain a geometric realization of Verma modules one needs to consider the minimal 1-point orbit of N_+ on X. Choosing another orbit of N_+ gives a *twisted* Verma module parametrized by the elements of the Weyl group. These modules have the same character as corresponding Verma modules.

3.2. Affine case

Geometric realizations for affine Lie algebras are less rigorous even in the case of classical Verma modules since they deal with certain infinite-dimensional varieties. But they provide a motivation for the construction of boson realizations. Our main references here are [FBZ01, K02].

Let $\hat{\mathfrak{g}}$ be the non-twisted affine Lie algebra associated with $\mathfrak{g}$. Consider a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathcal{H} \oplus \mathfrak{n}_+$ and a Borel subalgebra $\mathfrak{b}_\pm = \mathfrak{n}_\pm \oplus \mathcal{H}$. Denote

$$\hat{\mathfrak{n}}_\pm = (\mathfrak{n}_\pm \otimes 1) \oplus \big(\mathfrak{g} \otimes t^{\pm}\mathbb{C}[t^{\pm}]\big),$$

$\hat{\mathfrak{b}}_\pm = \hat{\mathfrak{n}}_\pm \oplus \mathcal{H} \otimes \mathbb{C}[t]$. Let $\hat{G}$, $\hat{N}_\pm$ and $\hat{B}_\pm$ be Lie groups corresponding to $\hat{\mathfrak{g}}$, $\hat{\mathfrak{n}}_\pm$ and $\hat{\mathfrak{b}}_\pm$, respectively. Then a scheme of infinite type $X = \hat{G}/\hat{B}_-$ splits into $\hat{N}_+$-orbits of finite codimension, parametrized by the affine Weyl group. There is an analogue of a big cell $\hat{\mathcal{U}}$ in X which is a projective limit of affine spaces, and hence, the ring of regular functions $\mathcal{O}_{\hat{\mathcal{U}}}$ on $\hat{\mathcal{U}}$ is a polynomial ring in infinitely many variables. Thus $\hat{\mathfrak{g}}$ acts on it by differential operators providing a realization for the contragradient Verma module with zero highest weight. Global sections of more general $\hat{N}_+$-equivariant sheaves on X will produce an arbitrary highest weight. Other $\hat{N}_+$-orbits in X correspond to twisted contragradient Verma modules. Standard Verma modules can be obtained by considering $\hat{N}_+$-orbits on $\hat{G}/\hat{B}_+$.

3.3. First free field realization

In the previous section we considered the case of classical Verma modules for affine Lie algebras. Their geometric realization is based on the flag variety associated with the standard Borel subalgebra.

Consider now the natural Borel subalgebra $\mathfrak{b}_{\text{nat}} = \mathfrak{n}_- \otimes \mathbb{C}[t, t^{-1}] \oplus \mathcal{H} \otimes \mathbb{C}[t^{-1}]$ of $\hat{\mathfrak{g}}$ and the corresponding Borel subgroup $\mathfrak{B}_{\text{nat}}$. The associated "flag variety" $X = \hat{G}/\mathfrak{B}_{\text{nat}}$ is a *semi-infinite* manifold [FBZ01,Vor93,Vor99]. We can consider the $\hat{N}_+$-orbits on it and, in particular, $\hat{N}_+$ can be viewed as an analogue of the big cell $\mathcal{U}$ in G/B_-.

Applying the same argument as in the previous section one can obtain an embedding of $\hat{\mathfrak{g}}$ into the Weyl algebra in infinitely many variables and hence a realization of our algebra in the Fock space $\mathbb{C}[x_n, n \in \mathbb{Z}]$. For example, consider the case $\mathfrak{g} = \mathfrak{sl}(2)$. Then

$$e_n = e \otimes t^n, \qquad h_n = h \otimes t^n, \qquad f_n = f \otimes t^n, \quad n \in \mathbb{Z},$$

form a basis of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Then we have the following embedding:

$$e_n \mapsto \frac{\partial}{\partial x_n}, \qquad h_n \mapsto -2 \sum_{m \in \mathbb{Z}} x_m \frac{\partial}{\partial x_{n+m}}, \qquad f_n \mapsto -\sum_{m,k \in \mathbb{Z}} x_m x_k \frac{\partial}{\partial x_{n+m+k}}.$$

Note that the differential operators corresponding to f_n are not well defined on $\mathbb{C}[x_n, n \in \mathbb{Z}]$ (they take values in some formal completion of $\mathbb{C}[x_n, n \in \mathbb{Z}]$). One way to deal with this problem is to apply the anti-involutions:

$$e_n \leftrightarrow f_n, \qquad h_n \leftrightarrow h_n; \qquad x_n \leftrightarrow \partial x_n, \quad n \in \mathbb{Z},$$

which gives the following formulas:

$$f_n \mapsto \frac{\partial}{\partial x_n}, \qquad h_n \mapsto -2 \sum_{m \in \mathbb{Z}} x_{n+m} \frac{\partial}{\partial x_m}, \qquad e_n \mapsto -\sum_{m,k \in \mathbb{Z}} x_{n+m+k} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_k}.$$

These formulas define the *first free field realization* of $\hat{sl}(2)$ in the polynomial ring $\mathbb{C}[x_m, m \in \mathbb{Z}]$. This module is, in fact, a quotient $M(0)$ of the imaginary Verma module with trivial highest weight by a submodule generated by the elements $h_n \otimes 1$, $n < 0$. Similar formulas for an arbitrary highest weight with a trivial action of the central element were obtained by Jakobsen and Kac [JK89] using analytic approach. Namely, if μ denotes a finite measure on the circle S^1, not concentrated in a finite number of points, and $\lambda_m = \int_{S^1} z^m \, d\mu$ then

$$f_n \mapsto \frac{\partial}{\partial x_n}, \qquad h_n \mapsto -\lambda_n - 2 \sum_{m \in \mathbb{Z}} x_{n+m} \frac{\partial}{\partial x_m},$$
$$e_n \mapsto -\sum_{m,k \in \mathbb{Z}} x_{n+m+k} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_k} - \sum_{m \in \mathbb{Z}} \lambda_{n+m} \frac{\partial}{\partial x_m}$$

gives a boson type realization of $M(\lambda)$, where $\lambda(c) = 0$ and $\lambda(h_0) = -\lambda_0$. This module is irreducible if $\lambda_0 \neq 0$ [Fut94].

To get a realization of the imaginary Verma module for $\hat{\mathfrak{sl}}(2)$ with a non-trivial central action Bernard and Felder used the Borel–Weil construction [BF90]. Let $\hat{B}_-$ be the Borel subgroup of the loop group $\hat{SL}(2)$ corresponding to a Borel subalgebra $\mathfrak{b}_{\mathrm{nat}}$. Then we presume $\hat{B}_-$ consists of the elements of the form

$$\exp\Bigl(\sum_{n\in\mathbb{Z}} x_n e_n\Bigr)\exp\Bigl(\sum_{m>0} y_m h_m\Bigr),$$

where x_n, y_m are coordinate functions. Consider a one-dimensional representation $\chi : \hat{B}_- \to \mathbb{C}$, where c acts by scalar K, h_0 acts by scalar J and all other elements act trivially. Then the group $\hat{SL}(2)$ acts on the sections of the line bundle $(\mathcal{L}_\chi, g)$ over $\hat{SL}(2)/\hat{B}_-$ by

$$(g_1 f)(g_2) = f\bigl(g_1^{-1} g_2\bigr),$$

$g_i \in \hat{SL}(2)$, where

$$\mathcal{L}_\chi = \hat{SL}(2) \times_{\hat{B}_-} \mathbb{C}$$

and $g : \hat{SL}(2) \times_{\hat{B}_-} \mathbb{C} \to \hat{SL}(2)/\hat{B}_-$ such that $(x, z) \mapsto x\hat{B}_-$.

Differentiating this action to an action of the Lie algebra $\hat{\mathfrak{g}}$ and applying two anti-involutions

$$e_n \leftrightarrow -f_{-n}, \qquad h_n \leftrightarrow h_{-n}, \qquad c \leftrightarrow c$$

and

$$x_{-n} \leftrightarrow \partial/\partial x_n, \qquad y_k \leftrightarrow -\partial/\partial y_k.$$

we obtain the following boson realization of $\hat{\mathfrak{g}}$ in the Fock space $\mathbb{C}[x_m, m \in \mathbb{Z}] \otimes \mathbb{C}[y_n, n > 0]$:

$$f_n \mapsto x_n, \qquad h_n \mapsto -2\sum_{m\in\mathbb{Z}} x_{m+n}\partial/\partial x_m + \delta_{n<0} y_{-n} + \delta_{n>0} 2nK\partial/\partial y_n + \delta_{n,0} J,$$

$$e_n \mapsto -\sum_{m,k\in\mathbb{Z}} x_{k+m+n}\frac{\partial^2}{\partial x_k \partial x_m} + \sum_{k>0} y_k \frac{\partial}{\partial x_{-k-n}} + 2K\sum_{m>0} m\frac{\partial^2}{\partial y_m \partial x_{m-n}} + (Kn + J)\frac{\partial}{\partial x_{-n}}.$$

This module is irreducible if and only if $K \neq 0$. If we let $K = 0$ and quotient out the submodule generated by $y_m, m > 0$ then the factor module is irreducible if and only if $J \neq 0$ (cf. [Fut94]). This construction has been generalized for $\hat{\mathfrak{sl}}(n)$ in [Cox04] providing a realization of imaginary Verma modules.

4. Boson type realizations

The generators of Weyl algebras are called *free bosons*. Hence any embedding of the affine Lie algebra $\hat{\mathfrak{g}}$ into a Weyl algebra leads to a boson type realization.

4.1. Second free field realization

There is another way to correct the formulas above using the construction of Wakimoto modules [Wak86].

Let $a_n = \partial/\partial x_n$, $a_n^* = x_{-n}$ and consider formal power series

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad a^*(z) = \sum_{n\in\mathbb{Z}} a_n^* z^{-n}.$$

Series $a(z)$ and $a^*(z)$ are called *formal distributions*. It is easy to see that $[a_n, a_m^*] = \delta_{n+m,0}$ and all other products are zero. The formulas in the $sl(2)$ case can be rewritten as follows:

$$e(z) \mapsto a(z), \qquad h(z) \mapsto -2a^*(z)a(z), \qquad f(z) \mapsto -a^*(z)^2 a(z),$$

where $g(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n-1}$ for $g \in \{e, f, h\}$. This realization is not well defined since $-a^*(z)^2 a(z)$ when expanded out and applied to a polynomial in the variables x_m is not a finite sum. It becomes well defined after the application of two anti-involutions described above. Then the formulas read:

$$f(z) \mapsto a(z), \qquad h(z) \mapsto 2a(z)a^*(z), \qquad f(z) \mapsto -a(z)a^*(z)^2,$$

where a_n and a_n^* have the following meaning now $a_n = x_n$, $a_n^* = -\partial/\partial x_{-n}$. Now the above formula for $f(z)$ when expanded out and applied to a polynomial in the variables x_m does give a finite sum. This is our quotient of the imaginary Verma module.

A different approach was suggested by Wakimoto [Wak86] who used the technique of *normal ordering*. Denote

$$a(z)_- = \sum_{n<0} a_n z^{-n-1}, \qquad a(z)_+ = \sum_{n\geqslant 0} a_n z^{-n-1}$$

and define the normal ordering as follows

$$:a(z)b(z): = a(z)_- b(z) + b(z)a_+(z).$$

Let now

$$a_n = \begin{cases} x_n, & n < 0, \\ \frac{\partial}{\partial x_n}, & n \geqslant 0, \end{cases} \qquad a_n^* = \begin{cases} x_{-n}, & n \leqslant 0, \\ -\frac{\partial}{\partial x_{-n}}, & n > 0, \end{cases} \qquad b_m = \begin{cases} m\frac{\partial}{\partial y_m}, & m \geqslant 0, \\ y_{-m}, & m < 0. \end{cases}$$

Here $[a_n, a_m^*] = [b_n, b_m] = \delta_{n+m,0}$.

It was shown in [Wak86] that the formulas

$$c \mapsto K, \qquad e(z) \mapsto a(z), \qquad h(z) \mapsto -2{:}a^*(z)a(z){:} + b(z),$$
$$f(z) \mapsto -{:}a^*(z)^2 a(z){:} + K\partial_z a^*(z) + a^*(z)b(z)$$

define the action of the affine $\hat{\mathfrak{sl}}(2)$ on the space $\mathbb{C}[x_n, n \in \mathbb{Z}] \otimes \mathbb{C}[y_m, m > 0]$. This boson type realization is called the *second free field realization* giving the celebrated *Wakimoto modules*.

For an arbitrary affine Lie algebra Wakimoto modules were constructed by Feĭgin and Frenkel [FF90a,FF90b]. Generically they are isomorphic to Verma modules. Wakimoto modules can be viewed as "infinite twistings" of Verma modules.

Remark 4.1. We see that the semi-infinite variety $\hat{G}/\mathfrak{B}_{\mathrm{nat}}$ gives rise to boson type realizations of imaginary Verma modules (the first free field realization) and of Wakimoto modules (the second free field realization). In fact, one can construct the whole family of "other" free field realizations [CF04].

4.2. Oscillator algebras

Let $\hat{\mathfrak{a}}$ be the infinite-dimensional Heisenberg algebra with generators $a_{ij,m}$, $a^*_{ij,m}$, and $\mathbf{1}$, $1 \leqslant i \leqslant j \leqslant n$ and $m \in \mathbb{Z}$, subject to the relations

$$\begin{aligned}
[a_{ij,m}, a_{kl,n}] &= \big[a^*_{ij,m}, a^*_{kl,n}\big] = 0,\\
\big[a_{ij,m}, a^*_{kl,n}\big] &= \delta_{ik}\delta_{jl}\delta_{m+n,0}\mathbf{1},\\
[a_{ij,m}, \mathbf{1}] &= \big[a^*_{ij,m}, \mathbf{1}\big] = 0.
\end{aligned}$$

Such an algebra has a representation $\tilde{\rho}: \hat{\mathfrak{a}} \to \mathfrak{gl}(\mathbb{C}[\mathbf{x}])$ where

$$\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_{ij,m} \mid i, j, m \in \mathbb{Z},\ 1 \leqslant i \leqslant j \leqslant n]$$

denotes the algebra over $\mathbb{C}$ generated by the indeterminates $x_{ij,m}$ and $\tilde{\rho}$ is defined by

$$\begin{aligned}
\tilde{\rho}(a_{ij,m}) &:= \begin{cases} \partial/\partial x_{ij,m} & \text{if } m \geqslant 0 \text{ and } j \leqslant r,\\ x_{ij,m} & \text{otherwise,} \end{cases}\\
\tilde{\rho}\big(a^*_{ij,m}\big) &:= \begin{cases} x_{ij,-m} & \text{if } m \leqslant 0 \text{ and } j \leqslant r,\\ -\partial/\partial x_{ij,-m} & \text{otherwise,} \end{cases}
\end{aligned}$$

and $\tilde{\rho}(\mathbf{1}) = 1$. In this case $\mathbb{C}[\mathbf{x}]$ is an $\hat{\mathfrak{a}}$-module generated by $1 =: |0\rangle$, where

$$a_{ij,m}|0\rangle = 0, \quad m \geqslant 0 \text{ and } j \leqslant r, \qquad a^*_{ij,m}|0\rangle = 0, \quad m > 0 \text{ or } j > r.$$

Let $\hat{\mathfrak{a}}_r$ denote the subalgebra generated by $a_{ij,m}$ and $a^*_{ij,m}$ and $\mathbf{1}$, where $1 \leqslant i \leqslant j \leqslant r$ and $m \in \mathbb{Z}$. If $r = 0$, we set $\hat{\mathfrak{a}}_r = 0$.

Let $A_n = ((\alpha_i|\alpha_j))$ be the Cartan matrix for $\mathfrak{sl}(n+1, \mathbb{C})$ and let $\mathfrak{B}$ be the matrix whose entries are

$$\mathfrak{B}_{ij} := (\alpha_i|\alpha_j)\left(\gamma^2 - \delta_{i>r}\delta_{j>r}(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\right)$$

where

$$\delta_{i>r} = \begin{cases} 1 & \text{if } i > r,\\ 0 & \text{otherwise.} \end{cases}$$

In other words

$$\mathfrak{B} := \gamma^2 A_n - (r+1)\begin{pmatrix} 0 & 0 \\ 0 & A_{n-r} \end{pmatrix} + rE_{r+1,r+1}.$$

Using cofactor expansion along the $r+1$st row one can show

$$\det \mathfrak{B} = (n+1)\gamma^{2r}\left(\gamma^2 - r - 1\right)^{n-r}$$

where we define $\gamma^{2r} = 1$ when $r = 0$ and $\gamma = 0$. Thus $\mathfrak{B}$ is degenerate if and only if either $\gamma = 0$ or $\gamma^2 = r+1$. We also have the Heisenberg Lie algebra $\hat{\mathfrak{b}}$ with generators b_{im}, $1 \leqslant i \leqslant n$, $m \in \mathbb{Z}$, $\mathbf{1}$, and relations $[b_{im}, b_{jp}] = m\mathfrak{B}_{ij}\delta_{m+p,0}\mathbf{1}$ and $[b_{im}, \mathbf{1}] = 0$.

For each $1 \leqslant i \leqslant n$ fix $\lambda_i \in \mathbb{C}$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then the algebra $\hat{\mathfrak{b}}$ has a representation $\rho_\lambda : \hat{\mathfrak{b}} \to \operatorname{End}(\mathbb{C}[\mathbf{y}]_\lambda)$ where

$$\mathbb{C}[\mathbf{y}] := \mathbb{C}\left[y_{i,m} \;\middle|\; i, m \in \mathbb{N}^*,\ 1 \leqslant i \leqslant n\right]$$

and ρ_λ is defined on $\mathbb{C}[\mathbf{y}]$ defined by

$$\rho_\lambda(b_{i0}) = \lambda_i, \qquad \rho_\lambda(b_{i,-m}) = \mathbf{e}_i \cdot \mathbf{y}_m, \qquad \rho_\lambda(b_{im}) = m\mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m} \quad \text{for } m > 0$$

and $\rho_\lambda(\mathbf{1}) = 1$. Here

$$\mathbf{y}_m = (y_{1m}, \ldots, y_{nm}), \qquad \frac{\partial}{\partial \mathbf{y}_m} = \left(\frac{\partial}{\partial y_{1m}}, \ldots, \frac{\partial}{\partial y_{nm}}\right)$$

and $\mathbf{e}_i$ are vectors in $\mathbb{C}^n$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$ where $\cdot$ means the usual dot product.

Note that since $\mathfrak{B}_{ij}$ is symmetric, it is orthogonally diagonalizable (i.e. there exists an orthogonal matrix P such that $P^t\mathfrak{B}P$ is a diagonal matrix), and hence we can find vectors $\mathbf{e}_i$ in $\mathbb{C}^n$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$. In fact for $m > 0$ and $n < 0$ we get

$$[b_{im}, b_{jn}] = m\delta_{m+n,0}\mathfrak{B}_{ij}.$$

(See also [FF90b].)

We also define for $0 \leqslant r \leqslant n$

$$\mathbb{C}_r[\mathbf{y}] := \mathbb{C}\left[\mathbf{e}_i \cdot \mathbf{y}_m \;\middle|\; i, m \in \mathbb{N}^*,\ 1 \leqslant i \leqslant r\right].$$

If $\mathfrak{B}$ is non-degenerate, then $\mathbb{C}_n[\mathbf{y}] = \mathbb{C}[\mathbf{y}]$. On the other hand, if $\mathfrak{B}$ is degenerate, then there is some choice to be made for the $\mathbf{e}_i$. For example, if $\gamma = 0$, $n = 2$ and $r = 1$, then

$$\mathfrak{B} = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$$

and one can choose, for example, $\mathbf{e}_1 = (0, 0)$ and $\mathbf{e}_2 = (0, \sqrt{-3})$ or $\mathbf{e}_1 = (0, 0)$ and $\mathbf{e}_2 = (\sqrt{-3}, 0)$. In either case $\mathbb{C}_n[\mathbf{y}] \neq \mathbb{C}[\mathbf{y}]$.

4.3. Formal distributions

Although the following is standard notation that can be found in [Kac98,MN99], we need to point out a few important differences that will simplify some of the arguments later and that could lead to confusion if not brought to the reader's attention. For any sequence of elements $\{a_m\}_{m\in\mathbb{Z}}$ in the ring $\mathrm{End}(V)$, V a vector space, the formal distribution

$$a(z) := \sum_{m\in\mathbb{Z}} a_m z^{-m-1}$$

is called a *field*, if for any $v \in V$, $a_m v = 0$ for $m \gg 0$. If $a(z)$ is a field, then one sets

$$a(z)_- := \sum_{m\geqslant 0} a_m z^{-m-1} \quad \text{and} \quad a(z)_+ := \sum_{m<0} a_m z^{-m-1}.$$

The *normal ordered product* of two distributions $a(z)$ and $b(w)$ (and their coefficients) is defined by

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} {:}a_m b_n{:} z^{-m-1} w^{-n-1} = {:}a(z)b(w){:} = a(z)_+ b(w) + b(w)a(z)_-. \tag{4.1}$$

For any $1 \leqslant i \leqslant j \leqslant n$, we define

$$a^*_{ij}(z) = \sum_{n\in\mathbb{Z}} a^*_{ij,n} z^{-n}, \qquad a_{ij}(z) = \sum_{n\in\mathbb{Z}} a_{ij,n} z^{-n-1}$$

and

$$b_i(z) = \sum_{n\in\mathbb{Z}} b_{in} z^{-n-1}.$$

In this case

$$\big[b_i(z), b_j(w)\big] = \mathfrak{B}_{ij}\partial_w \delta(z-w),$$
$$\big[a_{ij}(z), a^*_{kl}(w)\big] = \delta_{ik}\delta_{jl}\mathbf{1}\delta(z-w).$$

Observe that $a_{ij}(z)$ for $j > r$ is not a field whereas $a^*_{ij}(z)$ is always a field. We will call $a_{ij}(z)$ (respectively $a^*_{ij}(z)$) a *pure creation* (respectively *annihilation*) *operator* if $j > r$. Set

$$a_{ij}(z)_+ = a_{ij}(z), \qquad a_{ij}(z)_- = 0,$$
$$a^*_{ij}(z)_+ = 0, \qquad a^*_{ij}(z)_- = a^*_{ij}(z),$$

if $j > r$.

Now we should point out that while $:a^1(z_1)\cdots a^m(z_m):$ is always defined as a formal series, we will only define $:a(z)b(z): := \lim_{w\to z}:a(z)b(w):$ for certain pairs $(a(z), b(w))$. For example,

$$:a_{ij}(z)a^*_{kl}(z): = \sum_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}}:a_{ij,n}a^*_{kl,m-n}:\right)z^{-m-1}$$

is well defined as an element in $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])[[z,z^{-1}]]$ for all $l>r$ (as $\tilde\rho(a^*_{kl,m}) := -\partial/\partial x_{kl,-m}$ for $l>r$) or if both $l\leqslant r$ and $j\leqslant r$ (see also the remarks after Theorem 4.3).

Then one defines recursively

$$:a^1(z_1)\cdots a^k(z_k): = :a^1(z_1)\big(:a^2(z_2)\big(:\cdots:a^{k-1}(z_{k-1})a^k(z_k):\big)\cdots:\big):,$$

while normal ordered product

$$:a^1(z)\cdots a^k(z): = \lim_{z_1,z_2,\ldots,z_k\to z}:a^1(z_1)\big(:a^2(z_2)\big(:\cdots:a^{k-1}(z_{k-1})a^k(z_k):\big)\cdots\big):$$

will only be defined for certain k-tuples $(a^1,\ldots,a^k)$.

Recall

$$\lfloor ab\rfloor = a(z)b(w) - :a(z)b(w): = \big[a(z)_-, b(w)\big] \tag{4.2}$$

(half of $[a(z), b(w)]$) denote the *contraction* of any two formal distributions $a(z)$ and $b(w)$ where $a(z)$, $b(z)$ are free fields or pure creation or annihilation operators. For example, if $j,l\leqslant r$, then

$$\lfloor a_{ij}a^*_{kl}\rfloor = \sum_{m\geqslant 0}\delta_{ik}\delta_{jl}z^{-m-1}w^m = \delta_{i,k}\delta_{j,l}\delta_-(z-w) = \delta_{ik}\delta_{jl}\iota_{z,w}\left(\frac{1}{z-w}\right), \tag{4.3}$$

$$\lfloor a^*_{kl}a_{ij}\rfloor = -\sum_{n<0}\delta_{ik}\delta_{jl}z^n w^{-n-1} = -\delta_{ik}\delta_{jl}\delta_+(w-z) = \delta_{ik}\delta_{jl}\iota_{z,w}\left(\frac{1}{w-z}\right). \tag{4.4}$$

If $l>r$, then

$$\lfloor a_{ij}a^*_{kl}\rfloor = \big[a_{ij}(z)_-, a^*_{kl}(w)\big] = 0, \tag{4.5}$$

$$\lfloor a^*_{kl}a_{ij}\rfloor = \big[a^*_{kl}(z)_-, a_{ij}(w)\big] = -\delta_{ik}\delta_{jl}\delta(w-z). \tag{4.6}$$

The following theorem dealing with certain formal distributions as opposed to fields, is a small generalization of the version of Wick's Theorem found in such texts as [FBZ01,BS83,Hua98, Kac98]. This theorem is a key tool in the proof of Theorem 4.3.

Theorem 4.2. [CF04] *Let $a^i(z)$ and $b^j(z)$ be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$*, satisfying*

(1) $[\lfloor a^i(z)b^j(w)\rfloor, c^k(x)_\pm] = [\lfloor a^ib^j\rfloor, c^k(x)_\pm] = 0$, *for all* i, j, k *and* $c^k(x) = a^k(z)$ *or* $c^k(x) = b^k(w)$.

(2) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ *for all* i *and* j.

(3) *The products*

$$\lfloor a^{i_1}b^{j_1}\rfloor\cdots\lfloor a^{i_s}b^{i_s}\rfloor :a^1(z)\cdots a^M(z)b^1(w)\cdots b^N(w):_{(i_1,\dots,i_s;j_1,\dots,j_s)}$$

have coefficients in $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$ *for all subsets* $\{i_1,\dots,i_s\}\subset\{1,\dots,M\}$, $\{j_1,\dots,j_s\}\subset\{1,\dots N\}$. *Here the subscript* $(i_1,\dots,i_s;j_1,\dots,j_s)$ *means that those factors* $a^i(z)$, $b^j(w)$ *with indices* $i\in\{i_1,\dots,i_s\}$, $j\in\{j_1,\dots,j_s\}$ *are to be omitted from the product* $:a^1\cdots a^M b^1\cdots b^N:$ *and when* $s=0$ *we do not omit any factors.*

Then

$$\begin{aligned}&:a^1(z)\cdots a^M(z)::b^1(w)\cdots b^N(w):\\&\quad=\sum_{s=0}^{\min(M,N)}\sum_{\substack{i_1<\dots<i_s\\ j_1\neq\dots\neq j_s}}\lfloor a^{i_1}b^{j_1}\rfloor\cdots\lfloor a^{i_s}b^{i_s}\rfloor :a^1(z)\cdots a^M(z)b^1(w)\cdots b^N(w):_{(i_1,\dots,i_s;j_1,\dots,j_s)}.\end{aligned}$$

4.4. Intermediate Wakimoto modules

The idea of boson type realizations given in this section is based on the constructions of Wakimoto modules and geometric realizations described in the previous sections.

Define

$$E_i(z)=\sum_{n\in\mathbb{Z}}E_{in}z^{-n-1},\qquad F_i(z)=\sum_{n\in\mathbb{Z}}F_{in}z^{-n-1},\qquad H_i(z)=\sum_{n\in\mathbb{Z}}H_{in}z^{-n-1},\quad 1\leqslant i\leqslant n.$$

The defining relations between the generators of $\hat{\mathfrak{g}}$ can be written as follows

$$\big[H_i(z),H_j(w)\big]=(\alpha_i|\alpha_j)c\partial_w\delta(w-z), \tag{R1}$$

$$\big[H_i(z),E_j(w)\big]=(\alpha_i|\alpha_j)E_j(z)\delta(w-z), \tag{R2}$$

$$\big[H_i(z),F_j(w)\big]=-(\alpha_i|\alpha_j)F_j(z)\delta(w-z), \tag{R3}$$

$$\big[E_i(z),F_j(w)\big]=\delta_{i,j}\big(H_i(z)\delta(w-z)+c\partial_w\delta(w-z)\big), \tag{R4}$$

$$\big[F_i(z),F_j(w)\big]=\big[E_i(z),E_j(w)\big]=0\quad\text{if }(\alpha_i|\alpha_j)\neq-1, \tag{R5}$$

$$\big[F_i(z_1),F_i(z_2),F_j(w)\big]=\big[E_i(z_1),E_i(z_2),E_j(w)\big]=0\quad\text{if }(\alpha_i|\alpha_j)=-1, \tag{R6}$$

where $[X,Y,Z]:=[X,[Y,Z]]$ is the Engel bracket for any three operators X, Y, Z.

Recall that $\mathbb{C}[\mathbf{x}]$ is an $\hat{\mathfrak{a}}$-module with respect to the representation $\tilde{\rho}$ and $\mathbb{C}[\mathbf{y}]$ is a $\hat{\mathfrak{b}}$-module with respect to ρ_λ. In [CF04] we define a representation

$$\rho:\hat{\mathfrak{g}}\to\mathfrak{gl}\big(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}]\big),$$

where we use the notation $\rho(X_m):=\rho(X)_m$, for $X\in\mathfrak{g}$. This is described in the following result:

Theorem 4.3. *Let $\lambda \in \mathfrak{H}^*$ and set $\lambda_i = \lambda(H_i)$. The generating functions*

$$\rho(F_i)(z) = a_{ii} + \sum_{j=i+1}^{n} a_{ij} a^*_{i+1,j}, \tag{4.7}$$

$$\rho(H_i)(z) = 2{:}a_{ii}a^*_{ii}{:} + \sum_{j=1}^{i-1}\left({:}a_{ji}a^*_{ji}{:} - {:}\,a_{j,i-1}a^*_{j,i-1}{:}\right) + \sum_{j=i+1}^{n}\left({:}a_{ij}a^*_{ij}{:} - {:}\,a_{i+1,j}a^*_{i+1,j}{:}\right) + b_i, \tag{4.8}$$

$$\rho(E_i)(z) = {:}a^*_{ii}\left(\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right){:} + \sum_{k=i+1}^{n} a_{i+1,k}a^*_{ik} - \sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki} - a^*_{ii}b_i - \left(\delta_{i>r}(r+1) + \delta_{i\leqslant r}(i+1) - \gamma^2\right)\partial a^*_{ii}, \tag{4.9}$$

$$\rho(c) = \gamma^2 - (r+1) \tag{4.10}$$

define an action of the generators E_{im}, F_{im}, H_{im}, $i = 1, \ldots, n$, $m \in \mathbb{Z}$ and c, on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$. *In the above a_{ij}, a^*_{ij} and b_i denotes $a_{ij}(z)$, $a^*_{ij}(z)$ and $b_i(z)$, respectively.*

Theorem 4.3 defines a boson type realization of $\hat{\mathfrak{sl}}(n+1,\mathbb{C})$ and a module structure on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$ that depends on the parameter $0 \leqslant r \leqslant n$. We called such modules *intermediate Wakimoto module* in [CF04]. One can easily see that Theorem 4.3 defines also a boson type realization of $\hat{\mathfrak{sl}}(n+1,\mathbb{C})$ on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}_n[\mathbf{y}]$ which is different from the one above if $\mathfrak{B}$ is non-degenerate. It is more convenient to work with such realization, and we will call this module structure on $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}_n[\mathbf{y}]$, the *intermediate Wakimoto module* and denote it by $W_{n,r}(\lambda,\gamma)$.

The intermediate Wakimoto modules $W_{n,r}(\lambda,\gamma)$ have the property that the subalgebra $\bar{B}_r$ annihilates the vector $1 \otimes 1 \in \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$, $h(1\otimes 1) = \lambda(h)(1\otimes 1)$ for all $h \in \mathfrak{H}$ and $c(1\otimes 1) = (\gamma^2 - (r+1))(1\otimes 1)$. Consider the $\hat{\mathfrak{g}}_r$-submodule $W = U(\hat{\mathfrak{g}}_r)(1\otimes 1) \simeq W_{r,r}(\lambda,\gamma)$ of $W_{n,r}(\lambda,\gamma)$.

Remark 4.4. If λ is generic then W is isomorphic to the Wakimoto module $W_{\lambda(r),\tilde{\gamma}}$ [FF90b] where $\lambda(r) = \lambda|_{\mathfrak{H}_r}$, $\tilde{\gamma} = \gamma^2 - (r+1)$.

Example 4.5. If $\mathfrak{B}$ is non-degenerate then $W \simeq W_{\lambda(r),\tilde{\gamma}}$. We also have an isomorphism in the case $n = 2$, $r = 1$, $e_1 = (0,0)$, $e_2 = (\sqrt{-3}, 0)$. But there is no isomorphism if we choose $e_2 = (0, \sqrt{-3})$.

In particular, we have the following corollary. Consider $\tilde{\lambda} \in \hat{\mathfrak{H}}^*$ such that $\tilde{\lambda}|_{\mathfrak{H}} = \lambda$, $\tilde{\lambda}(c) = \gamma^2 - (r+1)$, a Verma type module $M_r(\tilde{\lambda})$ and its $\hat{\mathfrak{g}}_r$-submodule $M(\tilde{\lambda}_r)$.

Corollary 4.6. *Suppose that $M(\tilde{\lambda}_r)$ is irreducible and assume that $\lambda(c) \neq 0$. Let $\tilde{W} = U(\hat{\mathfrak{g}})W$. Then $M_r(\tilde{\lambda})$ is isomorphic to $\tilde{W}$.*

Proof. Since $M(\tilde{\lambda}_r)$ is irreducible and $\lambda(c) \neq 0$ we conclude that $\mathfrak{B}$ is non-degenerate. Hence $W \simeq W_{\lambda(r),\tilde{\gamma}}$ is essentially the Wakimoto module for $\hat{\mathfrak{g}}_r$. Moreover, the Wakimoto module $W_{\lambda(r),\tilde{\gamma}}$ is isomorphic to $M(\tilde{\lambda}_r)$ in this case. Indeed, the Verma module $M(\tilde{\lambda}_r)$ is irreducible, so that the canonical map $M(\tilde{\lambda}_r) \to W$, given by $1 \mapsto 1 \otimes 1$, is injective. As $M(\tilde{\lambda}_r)$ and W have the same character formulae, this canonical map provides an isomorphism. We also have a canonical map $\phi : M_r(\tilde{\lambda}_r) \to \tilde{W}$. This map restricts to the canonical map $M(\tilde{\lambda}_r) \to W$, W is contained in the image of ϕ, hence ϕ is surjective. Since $\tilde{\lambda}$ is generic, $M_r(\tilde{\lambda})$ is irreducible by Theorem 2.1 and thus ϕ must be an isomorphism. Therefore $M_r(\tilde{\lambda}) \simeq \tilde{W}$. □

It follows that Theorem 4.3 provides a boson type realization for generic Verma type modules.

5. Generating intermediate Wakimoto modules

Let $\hat{\mathfrak{k}} = \hat{\mathfrak{sl}}(r+1)$ with $\hat{\mathfrak{a}} \subset \hat{\mathfrak{g}} := \hat{\mathfrak{sl}}(n+1)$. As above $M_r(\tilde{\lambda})$ will denote a Verma type module with highest weight $\tilde{\lambda}$ and set

$$W_{\mathfrak{g}}(\tilde{\lambda}) := \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}_n[\mathbf{y}]$$

with the action defined by Theorem 4.3 above. When $\mathfrak{B}$ is non-degenerate, $W_{\mathfrak{g}}(\tilde{\lambda})$ is the intermediate Wakimoto module $W_{n,r}(\lambda,\gamma)$. Sitting inside $W_{\mathfrak{g}}(\tilde{\lambda})$ is a copy of $W_{\mathfrak{k}}(\tilde{\lambda}_r)$: For $1 \leqslant k \leqslant n$, set $\mathbb{C}_k[\mathbf{x}] := \mathbb{C}[x_{ij,m} \mid 1 \leqslant i \leqslant j \leqslant k, m \in \mathbb{Z}]$ and

$$W_{\mathfrak{k}}(\tilde{\lambda}_r) := \mathbb{C}_r[\mathbf{x}] \otimes \mathbb{C}_r[\mathbf{y}].$$

Remark 5.1. $W_{\mathfrak{k}}(\tilde{\lambda}_r) \simeq W$ for a generic $\tilde{\lambda}$.

We will show that in fact $W_{\mathfrak{g}}(\tilde{\lambda})$ is generated by $W_{\mathfrak{k}}(\tilde{\lambda}_r)$. Namely we have

Theorem 5.2. $W_{\mathfrak{g}}(\tilde{\lambda}) = U(\hat{\mathfrak{g}})W_{\mathfrak{k}}(\tilde{\lambda}_r)$.

Proof. First of all note that $L(\mathfrak{n}^-(r))$ is generated by $F_{i,m}$ with $r < i \leqslant n$ and $W_{\mathfrak{k}}(\tilde{\lambda}_r)$ is generated by $x_{ij,m}$ and $\mathbf{e}_j \cdot \mathbf{y}_p$ with $1 \leqslant i \leqslant j \leqslant r$, $m \in \mathbb{Z}$ and $p \in \mathbb{N}$.

For $1 \leqslant k \leqslant n$, set $W_k := \mathbb{C}_k[\mathbf{x}] \otimes \mathbb{C}_n[\mathbf{y}]$. Let us first see that $W_r \subset U(\hat{\mathfrak{g}})W_{\mathfrak{k}}(\tilde{\lambda}_r)$. Now an arbitrary element in W_r has the form

$$\sum_j u_j \otimes v_j, \quad u_j \in \mathbb{C}_r[\mathbf{x}],\ v_j \in \mathbb{C}_n[\mathbf{y}],$$

and so it suffices to show that any element of the form $u \otimes v$, with $u \in \mathbb{C}_r[\mathbf{x}]$ and $v \in \mathbb{C}_r[\mathbf{y}]$ monomials, is in $U(\hat{\mathfrak{g}})W_{\mathfrak{k}}(\tilde{\lambda}_r)$. Write $v = v^r v_r$ with $v_r \in \mathbb{C}[\mathbf{e}_j \cdot \mathbf{y}_m \mid 1 \leqslant j \leqslant r,\ m \in \mathbb{N}]$ and $v^r \in \mathbb{C}[\mathbf{e}_j \cdot \mathbf{y}_m \mid r < j \leqslant n,\ m \in \mathbb{N}]$ where we may assume $v^r = (\mathbf{e}_r \cdot \mathbf{y}_{i_1})^{p_{i_1}} \cdots (\mathbf{e}_n \cdot \mathbf{y}_{i_s})^{p_{i_s}}$ is a non-constant monomial (here $i_1, \ldots, i_s \in \mathbb{N}$). Recall

$$\rho(H_i)(z) = 2{:}a_{ii}a_{ii}^*{:} + \sum_{j=1}^{i-1} \left({:}a_{ji}a_{ji}^*{:} - {:}a_{j,i-1}a_{j,i-1}^*{:}\right) + \sum_{j=i+1}^{n} \left({:}a_{ij}a_{ij}^*{:} - {:}a_{i+1,j}a_{i+1,j}^*{:}\right) + b_i.$$

For $i > r+1$ and $m \in \mathbb{N}$ those summands above having factors with $a^*_{ik,m}$, $k \geqslant r+1$ act as zero on $W_{\mathfrak{k}}(\tilde{\lambda}_r)$ as these factors act by $-\partial/\partial_{x_{ik,-m}}$. Thus we get when restricted to $W_{\mathfrak{k}}(\tilde{\lambda}_r)$ that $\rho(H_{im}) = b_{im}$ and this is given by left multiplication by $\mathbf{e}_i \cdot \mathbf{y}_m$. For $i = r+1$ and $m \in \mathbb{N}$, we have when restricted to $W_{\mathfrak{k}}(\tilde{\lambda}_r)$,

$$\rho(H_{r+1,m}) = -\sum_{j=1}^{r} \sum_{p \in \mathbb{Z}} {:}a_{jr,p} a^*_{jr,m-p}{:} + b_{r+1,m}.$$

Altogether for $i \geqslant r+1$ and $m \in \mathbb{N}$, we can write

$$\rho(H_{im}) = X_{im} + b_{im},$$

where the first summand maps $\mathbb{C}_r[\mathbf{x}]$ back into itself and the second summand is left multiplication by $\mathbf{e}_i \cdot \mathbf{y}_m$ (X_{im} would be defined to be zero for $i > r+1$). As a consequence we get

$$\begin{aligned} u \otimes v &= u \otimes (\mathbf{e}_r \cdot \mathbf{y}_{i_1})^{p_{i_1}} \cdots (\mathbf{e}_n \cdot \mathbf{y}_{i_s})^{p_{i_s}} v_r \\ &= b^{p_{i_1}}_{r+1,i_1} \cdots b^{p_{i_s}}_{n,i_s} (u \otimes v_r) \\ &= \big(\rho(H_{r+1,i_1}) - X_{r,i_1}\big)^{p_{i_1}} \cdots \big(\rho(H_{n,i_s}) - X_{n,i_s}\big)^{p_{i_s}} (u \otimes v_r). \end{aligned}$$

Hence if we let $\mathcal{H}$ denote the abelian Lie algebra generated by H_{im} with $1 \leqslant i \leqslant n$, $m \in \mathbb{N}$, then

$$u \otimes v \in U(\mathcal{H}) W_{\mathfrak{k}}(\tilde{\lambda}_r), \tag{5.1}$$

and thus $W_r \subset U(\hat{\mathfrak{g}}) W_{\mathfrak{k}}(\tilde{\lambda}_r)$.

Fix $r \leqslant k < n$. Consider $u \otimes v \in W_{\mathfrak{g}}(\tilde{\lambda})$ with

$$u = x^p_{k+1,k+1,m_1} u_k, \quad u_k \in \mathbb{C}_k[\mathbf{x}],\ m_1 \in \mathbb{Z},\ p \in \mathbb{N},$$

and $v \in \mathbb{C}_n[\mathbf{y}]$ and assume the induction hypothesis that $W_k \subset U(\hat{\mathfrak{g}}) W_{\mathfrak{k}}(\tilde{\lambda}_r)$. (This was shown to be true for $k = r$ above.)

From (4.7) for $i > k$, we get when restricted to W_k

$$\rho(F_{i,m}) = x_{ii,m} - \sum_{p \in \mathbb{Z}} \sum_{j=i+1}^{n} x_{ij,p} \partial_{x_{i+1,j,p-m}} \tag{5.2}$$

and as a consequence

$$\rho\big(F^p_{k+1,m}\big)(u_k \otimes v) = x^p_{k+1,k+1,m} u_k \otimes v = u \otimes v$$

for any $m \in \mathbb{Z}$, $p \in \mathbb{N}$. Since we assume $u_k \otimes v \in U(\hat{\mathfrak{g}}) W_{\mathfrak{k}}(\tilde{\lambda}_r)$, the above equation tells us that $u \otimes v \in U(\hat{\mathfrak{g}}) W_{\mathfrak{k}}(\tilde{\lambda}_r)$.

Now recall from the proof of Theorem 4.3 (see [CF04])

$$\begin{aligned}&\bigl[\rho(F_i)(z),\rho(F_j)(w)\bigr]\\&\quad=\bigl(\delta_{i,j+1}a_{j,j+1}(w)-\delta_{j,i+1}a_{i,i+1}(z)\bigr)\delta(z-w)\\&\qquad+\left(\delta_{i,j+1}\sum_{q=j+2}^{n}a_{jq}(z)a^*_{j+2,q}(z)-\delta_{j,i+1}\sum_{q=i+2}^{n}a_{iq}(z)a^*_{i+2,q}(z)\right)\delta(z-w).\end{aligned}\tag{5.3}$$

We can show by induction that for $1\leqslant j<i$,

$$\bigl[\rho(F_i)(z_i),\dots,\rho(F_j)(z_j)\bigr]=\left(a_{ji}(z_j)+\sum_{q=i+1}^{n}a_{jq}(z_j)a^*_{i+1,q}(z_j)\right)\prod_{q=j}^{i-1}\delta(z_q-z_{q+1}).\tag{5.4}$$

Indeed this is true for $j=i-1$ by (5.3) and the induction step is given by

$$\begin{aligned}&\bigl[\bigl[\rho(F_i)(z_i),\dots,\rho(F_{j+1})(z_{j+1})\bigr],F_j(z_j)\bigr]\\&\quad=\left[a_{j+1,i}(z_{j+1})+\sum_{q=i+1}^{n}a_{j+1,q}(z_{j+1})a^*_{i+1,q}(z_{j+1}),a_{jj}(z_j)\right.\\&\qquad\left.+\sum_{l=j+1}^{n}a_{jl}(z_j)a^*_{j+1,l}(z_j)\right]\times\prod_{q=j+1}^{i-1}\delta(z_q-z_{q+1})\\&\quad=\left(a_{ji}(z_j)+\sum_{q=i+1}^{n}a_{jq}(z_j)a^*_{i+1,q}(z_j)\right)\prod_{q=j}^{i-1}\delta(z_q-z_{q+1}).\end{aligned}$$

(There are no double contractions occurring in the calculation.)

Suppose we have shown $u^k_{i,\mathbf{m}}u_k\otimes v\in U(\hat{\mathfrak{g}})W_{\mathfrak{k}}(\tilde{\lambda}_r)$ for all $u^k_{i,\mathbf{m}}$ of the form $u^k_{i,\mathbf{m}}=x^{p_i}_{i,k+1,m_i}\cdots x^{p_{k+1}}_{k+1,k+1,m_{k+1}}$ and $u_k\in\mathbb{C}_k[\mathbf{x}]$, $m_j\in\mathbb{Z}$, $p_j\in\mathbb{N}$ and $v\in\mathbb{C}_n[\mathbf{y}]$. Then by (5.4) we get for $j<k+1$

$$\bigl[\rho(F_{k+1})(z_{k+1}),\dots,\rho(F_j)(z_j)\bigr]_m=a_{j,k+1,m}+\sum_{q=k+2}^{n}\sum_{p\in\mathbb{Z}}a_{jq,p}a^*_{k+2,q,m-p},\tag{5.5}$$

and thus

$$\begin{aligned}u^k_{i-1,\mathbf{m}}u_k\otimes v&=x^{p_{i-1}}_{i-1,k+1,m_{i-1}}x^{p_i}_{i,k+1,m_i}\cdots x^{p_{k+1}}_{k+1,k+1,m_{k+1}}u_k\otimes v\\&=\bigl[\rho(F_{k+1})(z_{k+1}),\dots,\rho(F_{i-1})(z_{i-1})\bigr]^{p_{i-1}}_{m_{i-1}}(u_k\otimes v)\\&=u^k_{i-1,\mathbf{m}}u_k\otimes v=u\otimes v.\end{aligned}$$

This proves that $W_{k+1}\subset U(\hat{\mathfrak{g}})W_{\mathfrak{k}}(\tilde{\lambda}_r)$. Hence by induction we have $W_{\mathfrak{g}}(\lambda)=\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}_n[\mathbf{y}]=U(\hat{\mathfrak{g}})W_{\mathfrak{k}}(\tilde{\lambda}_r)$. □

We immediately have

Corollary 5.3. *If $\tilde{\lambda} \in \mathcal{H}^*$ is generic so that $M_r(\tilde{\lambda})$ is irreducible, then*

$$M_r(\tilde{\lambda}) \cong W_{\mathfrak{g}}(\tilde{\lambda}).$$

Proof. For a generic $\tilde{\lambda}$, $W_{\mathfrak{k}}(\tilde{\lambda}_r) \simeq W$ and $W_{\mathfrak{g}}(\tilde{\lambda}) \simeq \tilde{W}$. It remains to apply Corollary 4.6. □

6. Submodule structure of intermediate Wakimoto modules

We saw in the previous section that the intermediate Wakimoto module $W_{\mathfrak{g}}(\tilde{\lambda})$ is generated by its $\mathfrak{k}$-submodule $W_{\mathfrak{k}}(\tilde{\lambda}_r)$. We will show in this section that in the generic case $W_{\mathfrak{k}}(\tilde{\lambda}_r)$ determines completely the structure of $W_{\mathfrak{g}}(\tilde{\lambda})$.

We assume in this case section that the intermediate Wakimoto module $W_{\mathfrak{g}}(\tilde{\lambda})$ is in general position, namely that $\mathfrak{B}$ is non-degenerate. Hence, in particular, $\tilde{\lambda}(c) \neq 0$. Note that it does not imply that $W_{\mathfrak{g}}(\tilde{\lambda})$ is isomorphic to a corresponding Verma type module, whose structure is known by Theorem 2.1.

Consider a parabolic subalgebra

$$\mathfrak{p} = B_r + \hat{\mathfrak{k}}$$

of $\hat{\mathfrak{g}}$ with the Levi factor $\tilde{\mathfrak{k}} = \hat{\mathfrak{k}} + \hat{\mathfrak{H}}$ and the radical $\mathfrak{R}$. Then $W_{\mathfrak{k}}(\tilde{\lambda}_r)$ belongs to the standard category $\mathcal{O}$ of $\tilde{\mathfrak{k}}$-modules. Moreover, we immediately have

Lemma 6.1. *$W_{\mathfrak{k}}(\tilde{\lambda}_r)$ is a $\mathfrak{p}$-submodule of $W_{\mathfrak{g}}(\tilde{\lambda})$ with a trivial action of the radical $\mathfrak{R}$.*

Hence we can consider $W_{\mathfrak{k}}(\tilde{\lambda}_r)$ as a $\mathfrak{p}$-module with a trivial action of $\mathfrak{R}$ and construct a generalized Verma module

$$M\big(W_{\mathfrak{k}}(\tilde{\lambda}_r)\big) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{p})} W_{\mathfrak{k}}(\tilde{\lambda}_r).$$

Lemma 6.1 and Theorem 5.2 immediately imply that there exists a canonical epimorphism

$$\phi : M\big(W_{\mathfrak{k}}(\tilde{\lambda}_r)\big) \to W_{\mathfrak{g}}(\tilde{\lambda}).$$

Hence, the module $W_{\mathfrak{g}}(\tilde{\lambda})$ is a homomorphic image of the generalized Verma module $M(W_{\mathfrak{k}}(\tilde{\lambda}_r))$.

Denote $M^f = 1 \otimes W_{\mathfrak{k}}(\tilde{\lambda}_r)$. The following result describes the structure of the module $M(W_{\mathfrak{k}}(\tilde{\lambda}_r))$.

Theorem 6.2. *Let $N \neq 0$ be a submodule of $M(W_{\mathfrak{k}}(\tilde{\lambda}_r))$ and $N^f = N \cap M^f$.*

(i) *$N^f \neq 0$.*
(ii) *If $\mathfrak{B}$ is non-degenerate then $N \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} N^f$, where $\mathfrak{R}$ acts trivially on N^f.*

Proof. The first statement follows from Lemma 17 in [FKM01], while the second statement follows from Theorem 8(i) in [FKM01]. □

Using Theorem 6.2 we obtain the following description of the submodule structure of the intermediate Wakimoto modules in the generic case.

Corollary 6.3. *Let $N \neq 0$ be a submodule of $W_{\mathfrak{g}}(\tilde{\lambda})$.*

(i) $N \cap W_{\mathfrak{k}}(\tilde{\lambda}_r) \neq 0$.
(ii) *If $\mathfrak{B}$ is non-degenerate then $N \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (N \cap W_{\mathfrak{k}}(\tilde{\lambda}_r))$.*

7. Conclusion

As it was mentioned above Wakimoto modules can be obtained from the classical Verma modules by an infinite number of twistings. The same twisting can be applied to a Verma type module $M_r(\tilde{\lambda})$ in the part $M(\tilde{\lambda}_r)$, i.e. only using the reflections corresponding to the roots of $\mathfrak{k}$. We will say that in this case the module is obtained by *real twisting*. Clearly, imaginary Verma modules do not admit any real twisting, while, on the other hand, any intermediate Wakimoto module is obtained from the corresponding Verma type module by an infinite number of real twistings. Hence, all boson type realizations associated with the natural Borel subalgebra correspond to infinite (or empty in the imaginary case) real twistings of corresponding Verma type modules. But we do not get realizations of Verma type modules this way. In order to construct boson type realizations for these modules one needs to start with the Borel subalgebra different from the standard one or the natural one and consider a corresponding "flag manifold." Its "cells" will produce boson type realizations for Verma type modules, their contragradient analogs and finite real twistings. We are going to address this question in a subsequent paper.

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Appendix A

In this section we present the formulas for the singular elements in imaginary Verma modules recently obtained by B. Wilson [Wil05]. These formulas were inspired by the free field realization of imaginary Verma modules for $\hat{sl}(2)$.

Let e_i, f_i, h_i be a basis of $\hat{\mathfrak{sl}}(2) = \mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}]$, $i \in \mathbb{Z}$ where $x_i := x \otimes t^i$, $x \in \mathfrak{sl}(2)$. Consider the imaginary Verma module $M_0(0)$ with a trivial highest weight. Then it has a submodule generated by $h_i \otimes 1, i < 0$. Denote by $V(0)$ the corresponding quotient. A non-zero element $v \in V(0)$ is called *singular* if $e_i v = 0$ and $h_j v = 0$ for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}^*$. Let $F = \sum_i \mathbb{C} f_i$. Then $V(0)$ is a free $U(F)$-module, where $U(F)$ is just a polynomial algebra and the elements

$$\prod_{i=0}^{r-1} f_{s_i}, \quad s_i \in \mathbb{Z},\ s_i \leqslant s_{i+1},\ r \geqslant 1,$$

form a basis of $V(0)$.

Denote by $M_r = \text{span}_{\mathbb{C}}\{\prod_{i=0}^{r-1} f_{s_i} \mid s_i \in \mathbb{Z}\}$. Let S_r be the set of singular vectors in M_r and let $\text{Sym}(r)$ be the symmetric group realized as permutations of $0, 1, \ldots, r-1$. For any $r \geqslant 1$ and

$\bar{s} \in \mathbb{Z}^r$ set

$$v_r(\bar{s}) = \sum_{\sigma \in \mathrm{Sym}(r)} (-1)^{\operatorname{sgn}\sigma} f_{s_0+\sigma(0)} f_{s_1+\sigma(1)} \cdots f_{s_{r-1}+\sigma(r-1)}.$$

Theorem 7.1. [Wil05] *Elements $v_r(\bar{s})$, $r \geqslant 1$, $\bar{s} \in \mathbb{Z}^r$, form a basis of S_r.*

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