

Imaginary Verma Modules and Kashiwara Algebras for $U_q(\widehat{\mathfrak{sl}(2)})$.

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ABSTRACT. We consider imaginary Verma modules for quantum affine algebra $U_q(\widehat{\mathfrak{sl}(2)})$ and construct Kashiwara type operators and the Kashiwara algebra \mathcal{K}_q . We show that a certain quotient \mathcal{N}_q^- of $U_q(\widehat{\mathfrak{sl}(2)})$ is a simple \mathcal{K}_q -module.

1. Introduction

Corresponding to the standard partition of the root system of an affine Lie algebra into set of positive and negative roots we have a standard Borel subalgebra from which we may induce the standard Verma modules. However, unlike for finite dimensional semisimple Lie algebras for an affine Lie algebra there exists other closed partitions of the root system which are not equivalent to the usual partition of the root system under the Weyl group action. Corresponding to such non-standard partitions we have non-standard Borel subalgebras from which one may induce other non-standard Verma-type modules and these typically contain both finite and infinite dimensional weight spaces. The classification of closed subsets of the root system for affine Kac-Moody algebras was obtained by Jakobsen and Kac [JK85, JK89], and independently by Futorny [Fut90, Fut92]. A categorical setting for these modules was introduced in [Cox94], with certain restrictions, and generalized in [CFM96]. For the algebra $\widehat{\mathfrak{sl}(2)}$, the only non-standard modules of Verma-type are the *imaginary Verma modules* [Fut94].

Drinfeld [Dri85] and Jimbo [Jim85] independently introduced the quantum group $U_q(\mathfrak{g})$ as q -deformations of universal enveloping algebras of a symmetrizable Kac-Moody Lie algebra \mathfrak{g} . For generic q , Lusztig [Lus88] showed that integrable highest weight modules of symmetrizable Kac-Moody algebras can be deformed to

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those over the corresponding quantum groups in such a way that the dimensions of the weight spaces are invariant under the deformation. Following the framework of [Lus88] and [Kan95], *quantum imaginary Verma modules* for the quantum group $U_q(\widehat{\mathfrak{sl}(2)})$ were constructed in [CFKM97] and it was shown that these modules are deformations of those over the universal enveloping algebra $U(\widehat{\mathfrak{sl}(2)})$ in such a way that the weight multiplicities, both finite and infinite-dimensional, are preserved.

Kashiwara ([Kas90, Kas91]) from algebraic view point and Lusztig [Lus90] from geometric view point introduced global crystal base (equivalently, canonical base) for standard Verma modules $V_q(\lambda)$ and integrable highest weight modules $L_q(\lambda)$ independently. The crystal base ([Kas90, Kas91]) can be thought of as the $q = 0$ limit of the global crystal base or canonical base. An important ingredient in the construction of crystal base by Kashiwara in [Kas91], is a subalgebra \mathcal{B}_q of the quantum group $U_q(\mathfrak{g})$ which acts on the negative part $U_q^-(\mathfrak{g})$ of the quantum group $U_q(\mathfrak{g})$ by left multiplication. This subalgebra \mathcal{B}_q , which we call the Kashiwara algebra, played an important role in the definition of the Kashiwara operators which defines the crystal base.

In this paper we construct an analog of Kashiwara algebra \mathcal{K}_q for the imaginary Verma module $M_q(\lambda)$ for the quantum group $U_q(\widehat{\mathfrak{sl}(2)})$. Then we prove that certain quotient \mathcal{N}_q^- of $U_q(\widehat{\mathfrak{sl}(2)})$ is a simple \mathcal{K}_q -module. In Sections 2 and 3 we recall necessary definitions and some new results that we need. In Section 4 we define certain operators we call Ω -operators acting on \mathcal{N}_q^- and prove generalized commutation relations among them. We define the Kashiwara algebra \mathcal{K}_q in Section 5 and show that \mathcal{N}_q^- is a left \mathcal{K}_q -module and define a symmetric invariant bilinear form on \mathcal{N}_q^- . The main result in Section 6 is that for any weight λ of level zero the reduced imaginary Verma module $\tilde{M}_q(\lambda)$ is simple if and only if $\lambda(h) \neq 0$ which shows that Lusztig's deformation functor preserves module structure in the case of imaginary Verma modules (see [Fut94]). Finally, in Section 7 we prove that \mathcal{N}_q^- is simple as a \mathcal{K}_q -module and that the form defined in Section 5 is nondegenerate.

2. Imaginary Verma Modules for $A_1^{(1)}$

We begin by recalling some basic facts and constructions for the affine Kac-Moody algebra $A_1^{(1)}$ and its imaginary Verma modules. See [Kac90] for Kac-Moody algebra terminology and standard notations.

2.1. Let \mathbb{F} be a field of characteristic 0. The algebra $A_1^{(1)}$ is the affine Kac-Moody algebra over field \mathbb{F} with generalized Cartan matrix $A = (a_{ij})_{0 \leq i, j \leq 1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. The algebra $A_1^{(1)}$ has a Chevalley-Serre presentation with generators $e_0, e_1, f_0, f_1, h_0, h_1, d$ and relations

$$\begin{aligned} [h_i, h_j] &= 0, & [h_i, d] &= 0, \\ [e_i, f_j] &= \delta_{ij} h_i, \\ [h_i, e_j] &= a_{ij} e_j, & [h_i, f_j] &= -a_{ij} f_j, \\ [d, e_j] &= \delta_{0,j} e_j, & [d, f_j] &= -\delta_{0,j} f_j, \\ (\text{ad } e_i)^3 e_j &= (\text{ad } f_i)^3 f_j = 0, & i &\neq j. \end{aligned}$$

Alternatively, we may realize $A_1^{(1)}$ through the loop algebra construction

$$A_1^{(1)} \cong \mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}] \oplus \mathbb{F}c \oplus \mathbb{F}d$$

with Lie bracket relations

$$\begin{aligned} [x \otimes t^n, y \otimes t^m] &= [x, y] \otimes t^{n+m} + n\delta_{n+m,0}(x, y)c, \\ [x \otimes t^n, c] &= 0 = [d, c], \quad [d, x \otimes t^n] = nx \otimes t^n, \end{aligned}$$

for $x, y \in \mathfrak{sl}_2$, $n, m \in \mathbb{Z}$, where $(\ , \)$ denotes the Killing form on \mathfrak{sl}_2 . For $x \in \mathfrak{sl}_2$ and $n \in \mathbb{Z}$, we write $x(n)$ for $x \otimes t^n$.

Let Δ denote the root system of $A_1^{(1)}$, and let $\{\alpha_0, \alpha_1\}$ be a basis for Δ . Let $\delta = \alpha_0 + \alpha_1$, the minimal imaginary root. Then

$$\Delta = \{\pm\alpha_1 + n\delta \mid n \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\}.$$

2.2. The universal enveloping algebra $U(A_1^{(1)})$ of $A_1^{(1)}$ is the associative algebra over \mathbb{F} with 1 generated by the elements $h_0, h_1, d, e_0, e_1, f_0, f_1$ with defining relations

$$\begin{aligned} [h_0, h_1] &= [h_0, d] = [h_1, d] = 0, \\ h_i e_j - e_j h_i &= a_{ij} e_j, \quad h_i f_j - f_j h_i = -a_{ij} f_j, \\ d e_j - e_j d &= \delta_{0,j} e_j, \quad d f_j - f_j d = -\delta_{0,j} f_j, \\ e_i f_j - f_j e_i &= \delta_{ij} h_i, \\ e_j e_i^3 - 3e_i e_j e_i^2 + 3e_i^2 e_j e_i - e_i^3 e_j &= 0 \text{ for } i \neq j, \\ f_j f_i^3 - 3f_i f_j f_i^2 + 3f_i^2 f_j f_i - f_i^3 f_j &= 0 \text{ for } i \neq j. \end{aligned}$$

Corresponding to the loop algebra formulation of $A_1^{(1)}$ is an alternative description of $U(A_1^{(1)})$ as the associative algebra over \mathbb{F} with 1 generated by the elements $e(k), f(k)$ ($k \in \mathbb{Z}$), $h(l)$ ($l \in \mathbb{Z} \setminus \{0\}$), c, d, h , with relations

$$\begin{aligned} [c, u] &= 0 \text{ for all } u \in U(A_1^{(1)}), \\ [h(k), h(l)] &= 2k\delta_{k+l,0}c, \\ [h, d] &= 0, \quad [h, h(k)] = 0, \\ [d, h(l)] &= lh(l), \quad [d, e(k)] = ke(k), \quad [d, f(k)] = kf(k), \\ [h, e(k)] &= 2e(k), \quad [h, f(k)] = -2f(k), \\ [h(k), e(l)] &= 2e(k+l), \quad [h(k), f(l)] = -2f(k+l), \\ [e(k), f(l)] &= h(k+l) + k\delta_{k+l,0}c. \end{aligned}$$

2.3. A subset S of the root system Δ is called *closed* if $\alpha, \beta \in S$ and $\alpha + \beta \in \Delta$ implies $\alpha + \beta \in S$. The subset S is called a *closed partition* of the roots if S is closed, $S \cap (-S) = \emptyset$, and $S \cup -S = \Delta$ [JK85],[JK89],[Fut90],[Fut92]. The set

$$S = \{\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{l\delta \mid l \in \mathbb{Z}_{>0}\}$$

is a closed partition of Δ and is $W \times \{\pm 1\}$ -inequivalent to the standard partition of the root system into positive and negative roots [Fut94].

For $\mathfrak{g} = A_1^{(1)}$, let $\mathfrak{g}_{\pm}^{(S)} = \sum_{\alpha \in S} \mathfrak{g}_{\pm\alpha}$. In the loop algebra formulation of \mathfrak{g} , we have that $\mathfrak{g}_{+}^{(S)}$ is the subalgebra generated by $e(k)$ ($k \in \mathbb{Z}$) and $h(l)$ ($l \in \mathbb{Z}_{>0}$) and

$\mathfrak{g}_-^{(S)}$ is the subalgebra generated by $f(k)$ ($k \in \mathbb{Z}$) and $h(-l)$ ($l \in \mathbb{Z}_{>0}$). Since S is a partition of the root system, the algebra has a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_-^{(S)} \oplus \mathfrak{h} \oplus \mathfrak{g}_+^{(S)}.$$

Let $U(\mathfrak{g}_\pm^{(S)})$ be the universal enveloping algebra of $\mathfrak{g}_\pm^{(S)}$. Then, by the PBW theorem, we have

$$U(\mathfrak{g}) \cong U(\mathfrak{g}_-^{(S)}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}_+^{(S)}),$$

where $U(\mathfrak{g}_+^{(S)})$ is generated by $e(k)$ ($k \in \mathbb{Z}$), $h(l)$ ($l \in \mathbb{Z}_{>0}$), $U(\mathfrak{g}_-^{(S)})$ is generated by $f(k)$ ($k \in \mathbb{Z}$), $h(-l)$ ($l \in \mathbb{Z}_{>0}$) and $U(\mathfrak{h})$, the universal enveloping algebra of \mathfrak{h} , is generated by h, c and d .

Let $\lambda \in P$, the weight lattice of $\mathfrak{g} = A_1^{(1)}$. A $U(\mathfrak{g})$ -module V is called a *weight module* if $V = \bigoplus_{\mu \in P} V_\mu$, where

$$V_\mu = \{v \in V \mid h \cdot v = \mu(h)v, c \cdot v = \mu(c)v, d \cdot v = \mu(d)v\}.$$

Any submodule of a weight module is a weight module. A $U(\mathfrak{g})$ -module V is called an *S -highest weight module* with highest weight λ if there is a non-zero $v_\lambda \in V$ such that (i) $u^+ \cdot v_\lambda = 0$ for all $u^+ \in U(\mathfrak{g}_+^{(S)}) \setminus \mathbb{F}^*$, (ii) $h \cdot v_\lambda = \lambda(h)v_\lambda$, $c \cdot v_\lambda = \lambda(c)v_\lambda$, $d \cdot v_\lambda = \lambda(d)v_\lambda$, (iii) $V = U(\mathfrak{g}) \cdot v_\lambda = U(\mathfrak{g}_-^{(S)}) \cdot v_\lambda$. An S -highest weight module is a weight module.

For $\lambda \in P$, let $I_S(\lambda)$ denote the ideal of $U(A_1^{(1)})$ generated by $e(k)$ ($k \in \mathbb{Z}$), $h(l)$ ($l > 0$), $h - \lambda(h)1$, $c - \lambda(c)1$, $d - \lambda(d)1$. Then we define $M(\lambda) = U(A_1^{(1)})/I_S(\lambda)$ to be the *imaginary Verma module* of $A_1^{(1)}$ with highest weight λ . Imaginary Verma modules have many structural features similar to those of standard Verma modules, with the exception of the infinite-dimensional weight spaces. Their properties were investigated in [Fut94], from which we recall the following proposition [Fut94, Proposition 1, Theorem 1].

PROPOSITION 2.3.1. (i) $M(\lambda)$ is a $U(\mathfrak{g}_-^{(S)})$ -free module of rank 1 generated by the S -highest weight vector $1 \otimes 1$ of weight λ .

(ii) $\dim M(\lambda)_\lambda = 1$; $0 < \dim M(\lambda)_{\lambda - k\delta} < \infty$ for any integer $k > 0$; if $\mu \neq \lambda - k\delta$ for any integer $k \geq 0$ and $M(\lambda)_\mu \neq 0$, then $\dim M(\lambda)_\mu = \infty$.

(iii) Let V be a $U(A_1^{(1)})$ -module generated by some S -highest weight vector v of weight λ . Then there exists a unique surjective homomorphism $\varphi : M(\lambda) \rightarrow V$ such that $\varphi(1 \otimes 1) = v$.

(iv) $M(\lambda)$ has a unique maximal submodule.

(v) Let $\lambda, \mu \in P$. Any non-zero element of $\text{Hom}_{U(A_1^{(1)})}(M(\lambda), M(\mu))$ is injective.

(vi) $M(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$. \square

3. The quantum group $U_q(A_1^{(1)})$

3.1. The quantum group $U_q(A_1^{(1)})$ is the $\mathbb{F}(q^{1/2})$ -algebra with 1 generated by

$$e_0, e_1, f_0, f_1, K_0^{\pm 1}, K_1^{\pm 1}, D^{\pm 1}$$

with defining relations:

$$\begin{aligned}
 DD^{-1} &= D^{-1}D = K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
 e_i f_j - f_j e_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
 K_i e_i K_i^{-1} &= q^2 e_i, \quad K_i f_i K_i^{-1} = q^{-2} f_i, \\
 K_i e_j K_i^{-1} &= q^{-2} e_j, \quad K_i f_j K_i^{-1} = q^2 f_j, \quad i \neq j, \\
 K_i K_j - K_j K_i &= 0, \quad K_i D - D K_i = 0, \\
 D e_i D^{-1} &= q^{\delta_{i,0}} e_i, \quad D f_i D^{-1} = q^{-\delta_{i,0}} f_i, \\
 e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 &= 0, \quad i \neq j, \\
 f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 &= 0, \quad i \neq j,
 \end{aligned}$$

where, $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

The quantum group $U_q(A_1^{(1)})$ can be given a Hopf algebra structure with a comultiplication given by

$$\begin{aligned}
 \Delta(K_i) &= K_i \otimes K_i, \\
 \Delta(D) &= D \otimes D, \\
 \Delta(e_i) &= e_i \otimes K_i^{-1} + 1 \otimes e_i, \\
 \Delta(f_i) &= f_i \otimes 1 + K_i \otimes f_i,
 \end{aligned}$$

and an antipode given by

$$\begin{aligned}
 s(e_i) &= -e_i K_i^{-1}, \\
 s(f_i) &= -K_i f_i, \\
 s(K_i) &= K_i^{-1}, \\
 s(D) &= D^{-1}.
 \end{aligned}$$

There is an alternative realization for $U_q(A_1^{(1)})$, due to Drinfeld [Dri85], which we shall also need. Let U_q be the associative algebra with 1 over $\mathbb{F}(q^{1/2})$ generated by the elements $x^\pm(k)$ ($k \in \mathbb{Z}$), $a(l)$ ($l \in \mathbb{Z} \setminus \{0\}$), $K^{\pm 1}$, $D^{\pm 1}$, and $\gamma^{\pm \frac{1}{2}}$ with the

following defining relations:

$$(3.1) \quad DD^{-1} = D^{-1}D = KK^{-1} = K^{-1}K = 1,$$

$$(3.2) \quad [\gamma^{\pm\frac{1}{2}}, u] = 0 \quad \forall u \in U,$$

$$(3.3) \quad [a(k), a(l)] = \delta_{k+l,0} \frac{[2k]}{k} \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}},$$

$$(3.4) \quad [a(k), K] = 0, \quad [D, K] = 0,$$

$$(3.5) \quad Da(k)D^{-1} = q^k a(k),$$

$$(3.6) \quad Dx^{\pm}(k)D^{-1} = q^k x^{\pm}(k),$$

$$(3.7) \quad Kx^{\pm}(k)K^{-1} = q^{\pm 2} x^{\pm}(k),$$

$$(3.8) \quad [a(k), x^{\pm}(l)] = \pm \frac{[2k]}{k} \gamma^{\mp \frac{|k|}{2}} x^{\pm}(k+l),$$

$$(3.9) \quad \begin{aligned} x^{\pm}(k+1)x^{\pm}(l) - q^{\pm 2} x^{\pm}(l)x^{\pm}(k+1) \\ = q^{\pm 2} x^{\pm}(k)x^{\pm}(l+1) - x^{\pm}(l+1)x^{\pm}(k), \end{aligned}$$

$$(3.10) \quad [x^+(k), x^-(l)] = \frac{1}{q - q^{-1}} \left(\gamma^{\frac{k-l}{2}} \psi(k+l) - \gamma^{\frac{l-k}{2}} \phi(k+l) \right),$$

$$(3.11) \quad \text{where } \sum_{k=0}^{\infty} \psi(k)z^{-k} = K \exp \left((q - q^{-1}) \sum_{k=1}^{\infty} a(k)z^{-k} \right),$$

$$(3.12) \quad \sum_{k=0}^{\infty} \phi(-k)z^k = K^{-1} \exp \left(-(q - q^{-1}) \sum_{k=1}^{\infty} a(-k)z^k \right).$$

The algebras $U_q(A_1^{(1)})$ and U_q are isomorphic [Dri85]. The action of the isomorphism, which we shall call the *Drinfeld Isomorphism*, on the generators of $U_q(A_1^{(1)})$ is:

$$e_0 \mapsto x^-(1)K^{-1}, \quad f_0 \mapsto Kx^+(-1),$$

$$e_1 \mapsto x^+(0), \quad f_1 \mapsto x^-(0),$$

$$K_0 \mapsto \gamma K^{-1}, \quad K_1 \mapsto K, \quad D \mapsto D.$$

If one uses the formal sums

$$(3.13) \quad \phi(u) = \sum_{p \in \mathbb{Z}} \phi(p)u^{-p}, \quad \psi(u) = \sum_{p \in \mathbb{Z}} \psi(p)u^{-p}, \quad x^{\pm}(u) = \sum_{p \in \mathbb{Z}} x^{\pm}(p)u^{-p}$$

Drinfeld's relations (3), (8)-(10) can be written as

$$(3.14) \quad [\phi(u), \phi(v)] = 0 = [\psi(u), \psi(v)]$$

$$(3.15) \quad \phi(u)x^{\pm}(v)\phi(u)^{-1} = g(uv^{-1}\gamma^{\mp 1/2})^{\pm 1} x^{\pm}(v)$$

$$(3.16) \quad \psi(u)x^{\pm}(v)\psi(u)^{-1} = g(vu^{-1}\gamma^{\mp 1/2})^{\mp 1} x^{\pm}(v)$$

$$(3.17) \quad (u - q^{\pm 2}v)x^{\pm}(u)x^{\pm}(v) = (q^{\pm 2}u - v)x^{\pm}(v)x^{\pm}(u)$$

$$(3.18) \quad [x^+(u), x^-(v)] = (q - q^{-1})^{-1} (\delta(u/v\gamma)\psi(v\gamma^{1/2}) - \delta(u\gamma/v)\phi(u\gamma^{1/2}))$$

where $g(t) = g_q(t)$ is the Taylor series at $t = 0$ of the function $(q^2t - 1)/(t - q^2)$ and $\delta(z) = \sum_{k \in \mathbb{Z}} z^k$ is the formal Dirac delta function.

REMARK 3.1.1. Writing $g(t) = g_q(t) = \sum_{p \geq 0} g(p)t^p$ we have

$$g(0) = q^{-2}, \quad g(p) = (1 - q^4)q^{-2p-2}, \quad p > 0.$$

Note that $g_q(t)^{-1} = g_{q^{-1}}(t)$.

We will need the following identity later:

LEMMA 3.1.2.

$$(3.19) \quad \exp \left((q - q^{-1}) \sum_{k=1}^{\infty} \frac{-[2k]}{k} z^{-k} \right) = 1 + (1 - q^4) \sum_{r=1}^{\infty} (zq^2)^{-r} = q^2 g(1/z)$$

PROOF.

$$\begin{aligned} \exp \left((q - q^{-1}) \sum_{k=1}^{\infty} \frac{-[2k]}{k} z^{-k} \right) &= \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} (zq^2)^{-k} - \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z}{q^2} \right)^{-k} \right) \\ &= \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} (zq^2)^{-k} \right) \exp \left(- \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z}{q^2} \right)^{-k} \right) \\ &= \left(\frac{1}{1 - \frac{1}{zq^2}} \right) \left(1 - \frac{q^2}{z} \right) \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{zq^2} \right)^k - q^4 \sum_{k=1}^{\infty} \left(\frac{1}{zq^2} \right)^k \end{aligned}$$

□

3.2. Using the root partition $S = \{\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{l\delta \mid l \in \mathbb{Z}_{>0}\}$ from Section 2.3, we define:

$U_q^+(S)$ to be the subalgebra of U_q generated by $x^+(k)$ ($k \in \mathbb{Z}$) and $a(l)$ ($l > 0$);

$U_q^-(S)$ to be the subalgebra of U_q generated by $x^-(k)$ ($k \in \mathbb{Z}$) and $a(-l)$ ($l > 0$), and

$U_q^0(S)$ to be the subalgebra of U_q generated by $K^{\pm 1}$, $\gamma^{\pm 1/2}$, and $D^{\pm 1}$.

Then we have the following PBW theorem.

THEOREM 3.2.1 ([CFKM97]). *A basis for U_q is the set of monomials of the form*

$$x^- a^- K^\alpha D^\beta \gamma^{\mu/2} a^+ x^+$$

where

$$\begin{aligned} x^\pm &= x^\pm(m_1)^{n_1} \cdots x^\pm(m_k)^{n_k}, & m_i < m_{i+1}, & m_i \in \mathbb{Z}, \\ a^\pm &= a(r_1)^{s_1} \cdots a(r_l)^{s_l}, & \pm r_i < \pm r_{i+1}, & \pm r_i \in \mathbb{N}^*, \end{aligned}$$

and $\alpha, \beta, \mu \in \mathbb{Z}$, $n_i, s_i \in \mathbb{N}$. In particular, $U_q \cong U_q^-(S) \otimes U_q^0(S) \otimes U_q^+(S)$.

Thus we may order monomials in U_q in such a way that $-r_1 \leq -r_2 \leq \dots \leq -r_l$ when $r_i > 0$ and we compare elements lexicographically.

Considering Serre's relation (3.9) with $k = l$, we get

$$x^-(k+1)x^-(k) = q^{-2}x^-(k)x^-(k+1).$$

The product on the right side is in the correct order for a basis element. If $k+1 > l$ and $k \neq l$ in (3.9), then $k+1 > l+1$ so that $k \geq l+1$, and thus we can write

$$x^-(k+1)x^-(l) = q^{-2}x^-(l)x^-(k+1) + q^{-2}x^-(k)x^-(l+1) - x^-(l+1)x^-(k)$$

and then after repeating the above identity (for example the next step is to replace $k + 1$ by k and l by $l + 1$ on the left), we will eventually arrive at terms that are in the correct order. In particular if $k + 1 > l$ and $k \neq l$ note that $x^-(l)x^-(k + 1) < x^-(l + 1)x^-(k)$.

4. Ω -operators and their relations

Let $\mathbb{N}^{\mathbb{N}^*}$ denote the set of all functions from $\{k\delta \mid k \in \mathbb{N}^*\}$ to \mathbb{N} with finite support. Then we can write

$$a^+ = a_+^{(s_k)} := a(r_1)^{s_1} \cdots a(r_l)^{s_l}, \quad a^- := a_-^{(s_k)} = a(-r_1)^{s_1} \cdots a(-r_l)^{s_l}$$

for $f = (s_k) \in \mathbb{N}^{\mathbb{N}^*}$ whereby $f(r_k) = s_k$ and $f(t) = 0$ for $t \neq r_i, 1 \leq i \leq l$.

Consider now the subalgebra \mathcal{N}_q^- , generated by $\gamma^{\pm 1/2}$, and $x^-(l)$, $l \in \mathbb{Z}$. Note that the corresponding relations (9) hold in \mathcal{N}_q^- .

LEMMA 4.0.2. *Fix $k \in \mathbb{Z}$. Then for any $P \in \mathcal{N}_q^-$, there exists unique*

$$Q(a, (q_k)), R(c, (r_l)) \in \mathcal{N}_q^-, \quad a, b \in \mathbb{Z}, (q_l), (r_m) \in \mathbb{N}^{\mathbb{N}^*},$$

such that

$$(4.1) \quad [x^+(k), P] = \sum \frac{a_+^{(q_l)} K^a Q(a, (q_l))}{q - q^{-1}} + \sum \frac{a_-^{(r_m)} K^b R(b, (r_m))}{q - q^{-1}}.$$

PROOF. The uniqueness follows from Theorem 3.2.1 above. Now any element in \mathcal{N}_q^- is a sum of products of elements of the form

$$P = \gamma^{l/2} x^-(m_1) \cdots x^-(m_k), \quad \text{where } m_i \in \mathbb{Z}, m_1 \leq m_2 \leq \cdots \leq m_k, k \geq 0, l \in \mathbb{Z}$$

and such a product is a summand of

$$P = P(v_1, \dots, v_k) := \gamma^{l/2} x^-(v_1) \cdots x^-(v_k)$$

Set $\bar{P} = x^-(v_1) \cdots x^-(v_k)$ and $\bar{P}_l = x^-(v_1) \cdots x^-(v_{l-1}) x^-(v_{l+1}) \cdots x^-(v_k)$.

Then we have by (3.15) and (3.16),

$$\begin{aligned} x^-(v_1) \cdots x^-(v_{l-1}) \psi(v_l \gamma^{1/2}) &= \prod_{j=1}^{l-1} g(v_j v_l^{-1})^{-1} \psi(v_l \gamma^{1/2}) x^-(v_1) \cdots x^-(v_{l-1}) \\ x^-(v_1) \cdots x^-(v_{l-1}) \phi(u \gamma^{1/2}) &= \prod_{j=1}^{l-1} g(u \gamma v_j^{-1}) \phi(u \gamma^{1/2}) x^-(v_1) \cdots x^-(v_{l-1}), \end{aligned}$$

so that by (3.18)

$$\begin{aligned}
 [x^+(u), x^-(v_1) \cdots x^-(v_k)] &= \sum_{l=1}^k x^-(v_1) \cdots [x^+(u), x^-(v_l)] \cdots x^-(v_k) \\
 &= \sum_{l=1}^k x^-(v_1) \cdots \left(\frac{\delta(u/v_l \gamma) \psi(v_l \gamma^{1/2}) - \delta(u \gamma/v_l) \phi(u \gamma^{1/2})}{q - q^{-1}} \right) \cdots x^-(v_k) \\
 &= \sum_{l=1}^k x^-(v_1) \cdots x^-(v_{l-1}) \psi(v_l \gamma^{1/2}) x^-(v_{l+1}) \cdots x^-(v_k) \frac{\delta(u/v_l \gamma)}{q - q^{-1}} \\
 &\quad - \sum_{l=1}^k x^-(v_1) \cdots x^-(v_{l-1}) \phi(u \gamma^{1/2}) x^-(v_{l+1}) \cdots x^-(v_k) \frac{\delta(u \gamma/v_l)}{q - q^{-1}} \\
 &= \sum_{l=1}^k \prod_{j=1}^{l-1} g(v_j v_l^{-1})^{-1} \frac{\psi(v_l \gamma^{1/2}) \delta(u/v_l \gamma)}{q - q^{-1}} \bar{P}_l \\
 &\quad - \sum_{l=1}^k \prod_{j=1}^{l-1} g(u \gamma v_j^{-1}) \frac{\phi(u \gamma^{1/2}) \delta(u \gamma/v_l)}{q - q^{-1}} \bar{P}_l \\
 &= \frac{\psi(u \gamma^{-1/2})}{q - q^{-1}} \sum_{l=1}^k \prod_{j=1}^{l-1} g_{q^{-1}}(v_j/v_l) \bar{P}_l \delta(u/v_l \gamma) \\
 &\quad - \frac{\phi(u \gamma^{1/2})}{q - q^{-1}} \sum_{l=1}^k \prod_{j=1}^{l-1} g(v_l/v_j) \bar{P}_l \delta(u \gamma/v_l)
 \end{aligned}$$

□

Lemma 4.0.2 motivates the definition of a family of operators as follows. Set

$$G_l = G_l^{1/q} := \prod_{j=1}^{l-1} g_{q^{-1}}(v_j/v_l), \quad G_l^q = \prod_{j=1}^{l-1} g(v_l/v_j)$$

where $G_1 := 1$. Now define a collection of operators $\Omega_\psi(k), \Omega_\phi(k) : \mathcal{N}_q^- \rightarrow \mathcal{N}_q^-$, $k \in \mathbb{Z}$, in terms of the generating functions

$$\Omega_\psi(u) = \sum_{l \in \mathbb{Z}} \Omega_\psi(l) u^{-l}, \quad \Omega_\phi(u) = \sum_{l \in \mathbb{Z}} \Omega_\phi(l) u^{-l}$$

by

$$(4.2) \quad \Omega_\psi(u)(\bar{P}) := \gamma^m \sum_{l=1}^k G_l \bar{P}_l \delta(u/v_l \gamma)$$

$$(4.3) \quad \Omega_\phi(u)(\bar{P}) := \gamma^m \sum_{l=1}^k G_l^q \bar{P}_l \delta(u \gamma/v_l).$$

Then we can write the above computation in the proof of Lemma 4.0.2 as

$$(4.4) \quad [x^+(u), \bar{P}] = (q - q^{-1})^{-1} \left(\psi(u\gamma^{-1/2})\Omega_\psi(u)(\bar{P}) - \phi(u\gamma^{1/2})\Omega_\phi(u)(\bar{P}) \right),$$

Note that $\Omega_\psi(u)(1) = \Omega_\phi(u)(1) = 0$. More generally let us write

$$\bar{P} = x^-(v_1) \cdots x^-(v_k) = \sum_{n \in \mathbb{Z}} \sum_{\substack{n_1, n_2, \dots, n_k \in \mathbb{Z} \\ n_1 + \dots + n_k = n}} x^-(n_1) \cdots x^-(n_k) v_1^{-n_1} \cdots v_k^{-n_k}$$

Then

$$\begin{aligned} & \psi(u\gamma^{-1/2})\Omega_\psi(u)(\bar{P}) \\ &= \sum_{k \geq 0} \sum_{p \in \mathbb{Z}} \sum_{n_i \in \mathbb{Z}} \gamma^{k/2} \psi(k) \Omega_\psi(p) (x^-(n_1) \cdots x^-(n_k)) v_1^{-n_1} \cdots v_k^{-n_k} u^{-k-p} \\ &= \sum_{n_i \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \sum_{k \geq 0} \gamma^{k/2} \psi(k) \Omega_\psi(m-k) (x^-(n_1) \cdots x^-(n_k)) v_1^{-n_1} \cdots v_k^{-n_k} u^{-m} \end{aligned}$$

while

$$[x^+(u), \bar{P}] = \sum_{m \in \mathbb{Z}} \sum_{n_1, n_2, \dots, n_k \in \mathbb{Z}} [x^+(m), x^-(n_1) \cdots x^-(n_k)] v_1^{-n_1} \cdots v_k^{-n_k} u^{-m}.$$

Thus for a fixed m and k -tuple (n_1, \dots, n_k) the sum

$$\sum_{l \geq 0} \gamma^{l/2} \psi(l) \Omega_\psi(m-l) (x^-(n_1) \cdots x^-(n_k))$$

must be finite. Hence

$$(4.5) \quad \Omega_\psi(m-l) (x^-(n_1) \cdots x^-(n_k)) = 0,$$

for l sufficiently large.

PROPOSITION 4.0.3. *Consider $x^-(v) = \sum_m x^-(m)v^{-m}$ as a formal power series of left multiplication operators $x^-(m) : \mathcal{N}_q^- \rightarrow \mathcal{N}_q^-$. Then*

$$(4.6) \quad \Omega_\psi(u)x^-(v) = \delta(v\gamma/u) + g_{q^{-1}}(v\gamma/u)x^-(v)\Omega_\psi(u),$$

$$(4.7) \quad \Omega_\phi(u)x^-(v) = \delta(u\gamma/v) + g(u\gamma/v)x^-(v)\Omega_\phi(u)$$

$$(4.8) \quad (q^2u_1 - u_2)\Omega_\psi(u_1)\Omega_\psi(u_2) = (u_1 - q^2u_2)\Omega_\psi(u_2)\Omega_\psi(u_1)$$

$$(4.9) \quad (q^2u_1 - u_2)\Omega_\phi(u_1)\Omega_\phi(u_2) = (u_1 - q^2u_2)\Omega_\phi(u_2)\Omega_\phi(u_1)$$

$$(4.10) \quad (q^2\gamma^2u_1 - u_2)\Omega_\phi(u_1)\Omega_\psi(u_2) = (\gamma^2u_1 - q^2u_2)\Omega_\psi(u_2)\Omega_\phi(u_1)$$

PROOF. Setting $\bar{P} = x^-(v_1) \cdots x^-(v_k)$ we get

$$\begin{aligned} \Omega_\psi(u)x^-(v)(\bar{P}) &= x^-(v_1) \cdots x^-(v_k) \delta(u/v\gamma) \\ &\quad + x^-(v) \sum_{l=1}^k g_{q^{-1}}(v/v_l) G_l \bar{P}_l \delta(u/v_l\gamma) \\ &= \bar{P} \delta(u/v\gamma) + x^-(v) g_{q^{-1}}(v\gamma/u) \Omega_\psi(u) \bar{P}. \end{aligned}$$

Similarly

$$\begin{aligned}\Omega_\phi(u)x^-(v)(\bar{P}) &= x^-(v_1) \cdots x^-(v_k)\delta(u\gamma/v) \\ &\quad + x^-(v) \sum_{l=1}^k g(v_l/v)G_l^q \bar{P}_l \delta(u\gamma/v_l) \\ &= \bar{P}\delta(v/u\gamma) + x^-(v)g(u\gamma/v)\Omega_\phi(u)\bar{P}.\end{aligned}$$

One can prove (4.8) and (4.9) directly from their definitions, (4.2) and (4.3), but there is another way to prove this identity and it goes as follows:

$$\begin{aligned}\Omega_\psi(u_1)\Omega_\psi(u_2)x^-(v) &= \Omega_\psi(u_1)\delta(v\gamma/u_2) + \Omega_\psi(u_1)x^-(v)g_{q^{-1}}(v\gamma/u_2)\Omega_\psi(u_2) \\ &= \Omega_\psi(u_1)\delta(v\gamma/u_2) + g_{q^{-1}}(v\gamma/u_2)\Omega_\psi(u_2)\delta(v\gamma/u_1) \\ &\quad + g_{q^{-1}}(v\gamma/u_2)g_{q^{-1}}(v\gamma/u_1)x^-(v)\Omega_\psi(u_1)\Omega_\psi(u_2)\end{aligned}$$

and on the other hand

$$\begin{aligned}\Omega_\psi(u_2)\Omega_\psi(u_1)x^-(v) &= \Omega_\psi(u_2)\delta(v\gamma/u_1) + \Omega_\psi(u_2)x^-(v)g_{q^{-1}}(v\gamma/u_1)\Omega_\psi(u_1) \\ &= \Omega_\psi(u_2)\delta(v\gamma/u_1) + g_{q^{-1}}(v\gamma/u_1)\Omega_\psi(u_1)\delta(v\gamma/u_2) \\ &\quad + g_{q^{-1}}(v\gamma/u_1)g_{q^{-1}}(v\gamma/u_2)x^-(v)\Omega_\psi(u_2)\Omega_\psi(u_1)\end{aligned}$$

Thus setting $S = (u_1 - q^{-2}u_2)\Omega_\psi(u_1)\Omega_\psi(u_2) - (q^{-2}u_1 - u_2)\Omega_\psi(u_2)\Omega_\psi(u_1)$ we get

$$\begin{aligned}Sx^-(v) &= (u_1 - q^{-2}u_2)\Omega_\psi(u_1)\delta(v\gamma/u_2) + (u_1 - q^{-2}u_2)g_{q^{-1}}(v\gamma/u_2)\Omega_\psi(u_2)\delta(v\gamma/u_1) \\ &\quad + (u_1 - q^{-2}u_2)g_{q^{-1}}(v\gamma/u_2)g_{q^{-1}}(v\gamma/u_1)x^-(v)\Omega_\psi(u_1)\Omega_\psi(u_2) \\ &\quad - (q^{-2}u_1 - u_2)\Omega_\psi(u_2)\delta(v\gamma/u_1) - (q^{-2}u_1 - u_2)g_{q^{-1}}(v\gamma/u_1)\Omega_\psi(u_1)\delta(v\gamma/u_2) \\ &\quad - (q^{-2}u_1 - u_2)g_{q^{-1}}(v\gamma/u_1)g_{q^{-1}}(v\gamma/u_2)x^-(v)\Omega_\psi(u_2)\Omega_\psi(u_1) \\ &= ((u_1 - q^{-2}u_2) - (q^{-2}u_1 - u_2)g_{q^{-1}}(v\gamma/u_1))\Omega_\psi(u_1)\delta(v\gamma/u_2) \\ &\quad + ((u_1 - q^{-2}u_2)g_{q^{-1}}(v\gamma/u_2) - (q^{-2}u_1 - u_2))\Omega_\psi(u_2)\delta(v\gamma/u_1) \\ &\quad + g_{q^{-1}}(v\gamma/u_2)g_{q^{-1}}(v\gamma/u_1)x^-(v) \\ &\quad \times ((u_1 - q^{-2}u_2)\Omega_\psi(u_1)\Omega_\psi(u_2) - (q^{-2}u_1 - u_2)\Omega_\psi(u_2)\Omega_\psi(u_1)) \\ &= g_{q^{-1}}(v\gamma/u_2)g_{q^{-1}}(v\gamma/u_1)x^-(v)S\end{aligned}$$

Hence

$$Sx^-(v_1) \cdots x^-(v_n) = \prod_{i=1}^n g_{q^{-1}}(v_i\gamma/u_1)g_{q^{-1}}(v_i\gamma/u_2)x^-(v_1) \cdots x^-(v_n)S,$$

which implies, after applying this to 1 that $S = 0$.

Next we have

$$\begin{aligned}\Omega_\phi(u_1)\Omega_\phi(u_2)x^-(v) &= \Omega_\phi(u_1)\delta(v/u_2\gamma) + \Omega_\phi(u_1)x^-(v)g(u_2\gamma/v)\Omega_\phi(u_2) \\ &= \Omega_\phi(u_1)\delta(v/u_2\gamma) + g(u_2\gamma/v)\Omega_\phi(u_2)\delta(v/u_1\gamma) \\ &\quad + g(u_2\gamma/v)g(u_1\gamma/v)x^-(v)\Omega_\phi(u_1)\Omega_\phi(u_2)\end{aligned}$$

and on the other hand

$$\begin{aligned}\Omega_\phi(u_2)\Omega_\phi(u_1)x^-(v) &= \Omega_\phi(u_2)\delta(v/u_1\gamma) + \Omega_\phi(u_2)x^-(v)g(u_1\gamma/v)\Omega_\phi(u_1) \\ &= \Omega_\phi(u_2)\delta(v/u_1\gamma) + g(u_1\gamma/v)\Omega_\phi(u_1)\delta(v/u_2\gamma) \\ &\quad + g(u_1\gamma/v)g(u_2\gamma/v)x^-(v)\Omega_\phi(u_2)\Omega_\phi(u_1)\end{aligned}$$

So if we set $S = (u_1 - q^{-2}u_2)\Omega_\phi(u_1)\Omega_\phi(u_2) - (q^{-2}u_1 - u_2)\Omega_\phi(u_2)\Omega_\phi(u_1)$ we get

$$\begin{aligned}Sx^-(v) &= (u_1 - q^{-2}u_2)\Omega_\phi(u_1)\delta(v/u_2\gamma) + (u_1 - q^{-2}u_2)g(u_2\gamma/v)\Omega_\phi(u_2)\delta(v/u_1\gamma) \\ &\quad + (u_1 - q^{-2}u_2)g(u_2\gamma/v)g(u_1\gamma/v)x^-(v)\Omega_\phi(u_1)\Omega_\phi(u_2) \\ &\quad - (q^{-2}u_1 - u_2)\Omega_\phi(u_2)\delta(v/u_1\gamma) - (q^{-2}u_1 - u_2)g(u_1\gamma/v)\Omega_\phi(u_1)\delta(v/u_2\gamma) \\ &\quad - (q^{-2}u_1 - u_2)g(u_1\gamma/v)g(u_2\gamma/v)x^-(v)\Omega_\phi(u_2)\Omega_\phi(u_1) \\ &= ((u_1 - q^{-2}u_2) - (q^{-2}u_1 - u_2)g(u_1\gamma/v))\Omega_\phi(u_1)\delta(v/u_2\gamma) \\ &\quad + ((u_1 - q^{-2}u_2)g(u_2\gamma/v) - (q^{-2}u_1 - u_2))\Omega_\phi(u_2)\delta(v/u_1\gamma) \\ &\quad + g(u_2\gamma/v)g(u_1\gamma/v)x^-(v) \\ &\quad \quad \times ((u_1 - q^{-2}u_2)\Omega_\phi(u_1)\Omega_\phi(u_2) - (q^{-2}u_1 - u_2)\Omega_\phi(u_2)\Omega_\phi(u_1)) \\ &= g(u_2\gamma/v)g(u_1\gamma/v)x^-(v)S.\end{aligned}$$

As in the calculation for (4.8) we get $S = 0$.

Moreover

$$\begin{aligned}\Omega_\phi(u_1)\Omega_\psi(u_2)x^-(v) &= \Omega_\phi(u_1)\delta(v\gamma/u_2) + \Omega_\phi(u_1)x^-(v)g_{q^{-1}}(v\gamma/u_2)\Omega_\psi(u_2) \\ &= \Omega_\phi(u_1)\delta(v\gamma/u_2) + g_{q^{-1}}(v\gamma/u_2)\Omega_\psi(u_2)\delta(u_1\gamma/v) \\ &\quad + g_{q^{-1}}(v\gamma/u_2)g(u_1\gamma/v)x^-(v)\Omega_\phi(u_1)\Omega_\psi(u_2)\end{aligned}$$

and

$$\begin{aligned}\Omega_\psi(u_2)\Omega_\phi(u_1)x^-(v) &= \Omega_\psi(u_2)\delta(u_1\gamma/v) + \Omega_\psi(u_2)x^-(v)g(u_1\gamma/v)\Omega_\phi(u_1) \\ &= \Omega_\psi(u_2)\delta(u_1\gamma/v) + g(u_1\gamma/v)\Omega_\phi(u_1)\delta(v\gamma/u_2) \\ &\quad + g_{q^{-1}}(v\gamma/u_2)g(u_1\gamma/v)x^-(v)\Omega_\psi(u_2)\Omega_\phi(u_1)\end{aligned}$$

Set $S = (q^2\gamma^2u_1 - u_2)\Omega_\phi(u_1)\Omega_\psi(u_2) - (\gamma^2u_1 - q^2u_2)\Omega_\psi(u_2)\Omega_\phi(u_1)$. Then

$$\begin{aligned}
 Sx^-(v) &= (q^2\gamma^2u_1 - u_2)\Omega_\phi(u_1)\delta(v\gamma/u_2) + (q^2\gamma^2u_1 - u_2)g_{q^{-1}}(v\gamma/u_2)\Omega_\psi(u_2)\delta(u_1\gamma/v) \\
 &\quad + (q^2\gamma^2u_1 - u_2)g_{q^{-1}}(v\gamma/u_2)g(u_1\gamma/v)x^-(v)\Omega_\phi(u_1)\Omega_\psi(u_2) \\
 &\quad - (\gamma^2u_1 - q^2u_2)\Omega_\psi(u_2)\delta(u_1\gamma/v) - (\gamma^2u_1 - q^2u_2)g(u_1\gamma/v)\Omega_\phi(u_1)\delta(v\gamma/u_2) \\
 &\quad - (\gamma^2u_1 - q^2u_2)g_{q^{-1}}(v\gamma/u_2)g(u_1\gamma/v)x^-(v)\Omega_\psi(u_2)\Omega_\phi(u_1) \\
 &= ((q^2\gamma^2u_1 - u_2) - (\gamma^2u_1 - q^2u_2)g(u_1\gamma/v))\Omega_\phi(u_1)\delta(v\gamma/u_2) \\
 &\quad + ((q^2\gamma^2u_1 - u_2)g_{q^{-1}}(v\gamma/u_2) - (\gamma^2u_1 - q^2u_2))\Omega_\psi(u_2)\delta(u_1\gamma/v) \\
 &\quad + g_{q^{-1}}(v\gamma/u_2)g(u_1\gamma/v)x^-(v) \\
 &\quad \quad \times ((q^2\gamma^2u_1 - u_2)\Omega_\phi(u_1)\Omega_\psi(u_2) - (\gamma^2u_1 - q^2u_2)\Omega_\psi(u_2)\Omega_\phi(u_1)) \\
 &= g_{q^{-1}}(v\gamma/u_2)g(u_1\gamma/v)x^-(v)S.
 \end{aligned}$$

As in the previous calculations we get that $S = 0$ and thus the last statement of the proposition hold. \square

The identities in Proposition 4.0.3 imply the identities

$$(4.11) \quad (q^2v\gamma - u)\Omega_\psi(u)x^-(v) = (q^2v\gamma - u)\delta(v\gamma/u) + (\gamma v\gamma - q^2u)x^-(v)\Omega_\psi(u),$$

$$(4.12) \quad (q^2v - u\gamma)\Omega_\phi(u)x^-(v) = (q^2v - u\gamma)\delta(v/u\gamma) + (v - q^2u\gamma)x^-(v)\Omega_\phi(u)$$

which may be written out in terms of components as

$$(4.13) \quad q^2\gamma\Omega_\psi(m)x^-(n+1) - \Omega_\psi(m+1)x^-(n)$$

$$(4.14) \quad = (q^2 - 1)\gamma^{m+1}\delta_{m,-n-1} + \gamma x^-(n+1)\Omega_\psi(m) - q^2x^-(n)\Omega_\psi(m+1),$$

$$(4.15) \quad q^2\Omega_\phi(m)x^-(n+1) - \gamma\Omega_\phi(m+1)x^-(n)$$

$$(4.16) \quad = (q^2 - 1)\gamma^{-n-1}\delta_{m,-n-1} + x^-(n+1)\Omega_\psi(m) - q^2\gamma x^-(n)\Omega_\psi(m+1),$$

We can also write (4.6) in terms of components and as operators on \mathcal{N}_q^-

$$(4.17) \quad \Omega_\psi(k)x^-(m) = \delta_{k,-m}\gamma^k + \sum_{r \geq 0} g_{q^{-1}}(r)x^-(m+r)\Omega_\psi(k-r)\gamma^r.$$

The sum on the right hand side turns into a finite sum when applied to an element in \mathcal{N}_q^- , due to (4.5).

We also have by (4.10)

$$(4.18) \quad \Omega_\psi(k)\Omega_\phi(m) = \sum_{r \geq 0} g(r)\gamma^{2r}\Omega_\phi(r+m)\Omega_\psi(k-r),$$

as operators on \mathcal{N}_q^- .

5. The Kashiwara algebra \mathcal{K}_q

The Kashiwara algebra \mathcal{K}_q is defined to be the $\mathbb{F}(q^{1/2})$ -algebra with generators $\Omega_\psi(m), x^-(n), \gamma^{\pm 1/2}$, $m, n \in \mathbb{Z}$ where $\gamma^{\pm 1/2}$ are central and the defining relations are

$$(5.1) \quad \begin{aligned} q^2 \gamma \Omega_\psi(m) x^-(n+1) - \Omega_\psi(m+1) x^-(n) \\ = (q^2 - 1) \gamma^{m+1} \delta_{m, -n-1} + \gamma x^-(n+1) \Omega_\psi(m) - q^2 x^-(n) \Omega_\psi(m+1) \end{aligned}$$

$$(5.2) \quad q^2 \Omega_\psi(k+1) \Omega_\psi(l) - \Omega_\psi(l) \Omega_\psi(k+1) = \Omega_\psi(k) \Omega_\psi(l+1) - q^2 \Omega_\psi(l+1) \Omega_\psi(k)$$

(which comes from (4.6), (4.8) written out in terms of components),

$$(5.3) \quad x^-(k+1) x^-(l) - q^{-2} x^-(l) x^-(k+1) = q^{-2} x^-(k) x^-(l+1) - x^-(l+1) x^-(k)$$

together with

$$\gamma^{1/2} \gamma^{-1/2} = 1 = \gamma^{-1/2} \gamma^{1/2}.$$

LEMMA 5.0.4. *The $\mathbb{F}(q^{1/2})$ -linear map $\bar{\alpha} : \mathcal{K}_q \rightarrow \mathcal{K}_q$ given by*

$$\bar{\alpha}(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2}, \quad \bar{\alpha}(x^-(m)) = \Omega_\psi(-m), \quad \bar{\alpha}(\Omega_\psi(m)) = x^-(-m)$$

for all $m \in \mathbb{Z}$ is an involutive anti-automorphism.

PROOF. We have

$$\begin{aligned} \bar{\alpha}(x^-(k+1) x^-(l) - q^{-2} x^-(l) x^-(k+1)) \\ = \Omega_\psi(-l) \Omega_\psi(-k-1) - q^{-2} \Omega_\psi(-k-1) \Omega_\psi(-l) \\ = q^{-2} \Omega_\psi(-l-1) \Omega_\psi(-k) - \Omega_\psi(-k) \Omega_\psi(-l-1) \\ = \bar{\alpha}(q^{-2} x^-(k) x^-(l+1) - x^-(l+1) x^-(k)) \end{aligned}$$

and

$$\begin{aligned} \bar{\alpha}(q^2 \gamma \Omega_\psi(m) x^-(n+1) - \Omega_\psi(m+1) x^-(n)) \\ = q^2 \gamma \Omega_\psi(-n-1) x^-(-m) - \Omega_\psi(-n) x^-(-m-1) \\ = (q^2 \gamma - 1) \delta_{-m, n+1} + \gamma x^-(-m) \Omega_\psi(-n-1) - q^2 x^-(-m-1) \Omega_\psi(-n) \\ = \bar{\alpha}((q^2 \gamma - 1) \delta_{m, -n-1} + \gamma x^-(n+1) \Omega_\psi(m) - q^2 x^-(n) \Omega_\psi(m+1)) \end{aligned}$$

□

LEMMA 5.0.5. *\mathcal{N}_q^- is a left \mathcal{K}_q -module.*

PROOF. This follows from (4.0.3) □

LEMMA 5.0.6. *Let C denote the subalgebra of \mathcal{K}_q generated by $x^-(m), \gamma^{\pm 1/2}$. Then $C \cong \mathcal{N}_q^-$.*

PROOF. We have an induced left \mathcal{K}_q -module epimorphism from \mathcal{K}_q to \mathcal{N}_q^- which sends 1 to 1. Since $\Omega_\psi(k)$ annihilates 1 for all k , we get an induced left \mathcal{K}_q -module epimorphism

$$\mathcal{K}_q / \sum_k \mathcal{K}_q \Omega_\psi(k) \xrightarrow{\eta} \mathcal{N}_q^-$$

Composing the inclusion $C \hookrightarrow \mathcal{K}_q$ with the canonical projection we have a homomorphism

$$C \xrightarrow{\mu} \mathcal{K}_q / \sum_k \mathcal{K}_q \Omega_\psi(k)$$

The composition $\eta \circ \mu$ is surjective and since \mathcal{N}_q^- is defined by generators $x^-(n), \gamma^{\pm 1/2}$ and relations (3.9) and (3.2), we get an induced surjective map $\nu : \mathcal{N}_q^- \rightarrow C$ such that the composition $\eta \circ \mu \circ \nu$ is the identity. Hence we get that ν is an isomorphism. \square

PROPOSITION 5.0.7. *There is a unique symmetric form $(\ , \)$ defined on \mathcal{N}_q^- satisfying*

$$(x^-(m)a, b) = (a, \Omega_\psi(-m)b), \quad (\gamma^m, 1) = 1.$$

PROOF. Using the anti-automorphism \bar{a} we can make $M = \text{Hom}(\mathcal{N}_q^-, \mathbb{F}(q^{1/2}))$ into a left \mathcal{K}_q -module by defining

$$\begin{aligned} (x^-(m)\phi)(a) &= \phi(\Omega_\psi(-m)a), \quad (\Omega_\psi(m)\phi)(a) = \phi(x^-(m)a), \\ (\gamma^{\pm 1/2}\phi)(a) &= \phi(\gamma^{\pm 1/2}a). \end{aligned}$$

for $a \in \mathcal{N}_q^-$ and $\phi \in M$.

Consider the element $\beta_0 \in M$ satisfying $\beta_0(\gamma^{m/2}) = 1$ for all $m \in \mathbb{Z}$ and

$$\beta_0 \left(\sum_{m \in \mathbb{Z}} x^-(m) \mathcal{N}_q^- \right) = 0.$$

Then $\Omega_\psi(m)\beta_0 = 0$ for any $m \in \mathbb{Z}$. Moreover we get an induced \mathcal{K}_q -module homomorphism

$$\mathcal{K}_q / \sum_{m \in \mathbb{Z}} \mathcal{K}_q \Omega_\psi(m) \rightarrow M.$$

Composing the inclusion map $\mathcal{N}_q^- \hookrightarrow \mathcal{K}_q$ of \mathcal{N}_q^- -modules with the canonical projection $\mathcal{K}_q \rightarrow \mathcal{K}_q / \sum_{m \in \mathbb{Z}} \mathcal{K}_q \Omega_\psi(m)$ we get an induced \mathcal{N}_q^- -module map

$$\bar{\beta} : \mathcal{N}_q^- \rightarrow M.$$

Here \mathcal{N}_q^- is viewed as a left \mathcal{N}_q^- -module under left multiplication. Since $\bar{\beta}(1) = \beta_0$, we get that $\bar{\beta}$ is non-zero.

Define the bilinear form $(\ , \) : \mathcal{N}_q \times \mathcal{N}_q \rightarrow \mathbb{F}(q^{1/2})$ by

$$(a, b) = (\bar{\beta}(a))(b)$$

This form satisfies $(\gamma^{m/2}, 1) = 1$ for all $m \in \mathbb{Z}$ and

$$(x^-(m)a, b) = (a, \Omega_\psi(-m)b), \quad (\gamma^{\pm 1/2}a, b) = (a, \gamma^{\pm 1/2}b).$$

Since \mathcal{N}_q^- is generated by $x^-(m)$ and $\gamma^{\pm 1/2}$ we get that the form is the unique form satisfying these two conditions. The form is symmetric as can be seen by the following inductive argument.

$$\begin{aligned} (x^-(m)\gamma^{r/2}, x^-(k)\gamma^{s/2}) &= \beta_0(\gamma^{(r+s)/2}\Omega_\psi(-m)x^-(k)^-) = \delta_{m,k} \\ &= (x^-(k)\gamma^{s/2}, x^-(m)\gamma^{r/2}) \end{aligned}$$

by (4.17). Then by using an induction hypothesis on the number of factors one has

$$\begin{aligned}
& (x^-(m_1) \cdots x^-(m_n) \gamma^{r/2}, x^-(k_1) \cdots x^-(k_n) \gamma^{s/2}) \\
&= (x^-(m_2) \cdots x^-(m_n) \gamma^{r/2}, \Omega_\psi(-m_1) x^-(k_1) \cdots x^-(k_n) \gamma^{s/2}) \\
&= \delta_{m_1, k_1} (x^-(m_2) \cdots x^-(m_n) \gamma^{r/2}, x^-(k_2) \cdots x^-(k_n) \gamma^{s/2}) \\
&\quad + \sum_{r \geq 0} g_{q^{-1}}(r) (x^-(m_2) \cdots x^-(m_n) \gamma^{r/2}, x^-(k_1 + r) \Omega_\psi(-m_1 - r) x^-(k_2) \cdots x^-(k_n) \gamma^{s/2}) \\
&= \delta_{m_1, k_1} (x^-(k_2) \cdots x^-(k_n) \gamma^{s/2}, x^-(m_2) \cdots x^-(m_n) \gamma^{r/2}) \\
&\quad + \sum_{r \geq 0} g_{q^{-1}}(r) (x^-(k_1 + r) \Omega_\psi(-m_1 - r) x^-(k_2) \cdots x^-(k_n) \gamma^{s/2}, x^-(m_2) \cdots x^-(m_n) \gamma^{r/2}) \\
&= (x^-(k_1) \cdots x^-(k_n) \gamma^{s/2}, x^-(m_1) \cdots x^-(m_n) \gamma^{r/2})
\end{aligned}$$

□

6. Imaginary Verma modules

Let Λ denotes the weight lattice of $A_1^{(1)}$, $\lambda \in \Lambda$. Denote by $I^q(\lambda)$ the ideal of $U_q = U_q(\widehat{\mathfrak{sl}}(2))$ generated by $x^+(k)$, $k \in \mathbb{Z}$, $a(l)$, $l > 0$, $K^{\pm 1} - q^{\lambda(h)} 1$, $\gamma^{\pm \frac{1}{2}} - q^{\pm \frac{1}{2} \lambda(c)} 1$ and $D^{\pm 1} - q^{\pm \lambda(d)} 1$. The imaginary Verma module with highest weight λ is defined to be ([CFKM97])

$$M_q(\lambda) = U/I^q(\lambda).$$

THEOREM 6.0.8 ([CFKM97], Theorem 3.6). *Imaginary Verma module $M_q(\lambda)$ is simple if and only if $\lambda(c) \neq 0$.*

Suppose now that $\lambda(c) = 0$. Then $\gamma^{\pm \frac{1}{2}}$ acts on $M_q(\lambda)$ by 1. Consider an ideal $J^q(\lambda)$ of U_q generated by $I^q(\lambda)$ and $a(l)$ for all l . Denote

$$\tilde{M}_q(\lambda) = U_q/J^q(\lambda).$$

Then $\tilde{M}_q(\lambda)$ is a homomorphic image of $M_q(\lambda)$ which we call *reduced imaginary Verma module*. Module $\tilde{M}_q(\lambda)$ has a Λ -gradation:

$$\tilde{M}_q(\lambda) = \sum_{\xi \in \Lambda} \tilde{M}_q(\lambda)_\xi.$$

If α denotes a simple root of $\mathfrak{sl}(2)$ and δ denotes an indivisible imaginary root then $\tilde{M}_q(\lambda)_{\lambda - \xi} \neq 0$ if and only if $\xi = 0$ or $\xi = -n\alpha + m\delta$ with $n > 0$, $m \in \mathbb{Z}$.

If $\xi = -n\alpha + m\delta$ then we set $|\xi| = n$. Note that \mathcal{N}_q^- has also a Λ -grading: $x^-(n_1) x^-(n_2) \cdots x^-(n_k) \in (\mathcal{N}_q^-)_\xi$, where $\xi = -k\alpha + (n_1 + \cdots + n_k)\delta$, $|\xi| = k$.

In this section we discuss the properties of the reduced imaginary Verma modules.

LEMMA 6.0.9. *Let $\lambda \in \Lambda$ such that $\lambda(c) = \lambda(h) = 0$, $v \in \tilde{M}_q(\lambda)$ a nonzero element, $v = u\tilde{v}_\lambda$, where $u \in (\mathcal{N}_q^-)_\xi$, $|\xi| = 2$. Then there exists $s \in \mathbb{Z}$ such that $x^+(s)v \neq 0$.*

PROOF. Let $\xi = -2\alpha + m\delta$, $m \in \mathbb{Z}$. We may assume

$$u = \sum_l A_l x^-(l) x^-(m-l),$$

where all but finitely many of the $A_l \in \mathbb{C}(q)$ are nonzero. Then by (3.10), we have

$$x^+(s)v = [x^+, u]\tilde{v}_\lambda = \frac{1}{q - q^{-1}} \sum_l A_l \psi(s+l)x^-(m-l)\tilde{v}_\lambda$$

for $s \gg 0$ as $\phi(l+s) = 0$ for $l+s > 0$ and $\psi(m-l+s)\tilde{v}_\lambda = 0$ for $s \gg 0$.

Observe that

$$\begin{aligned} \psi(r) &= K(q - q^{-1}) \left(a(r) + \frac{1}{2}(q - q^{-1}) \sum_{\substack{k_1+k_2=r \\ k_i \geq 1}} a(k_1)a(k_2) \right. \\ &\quad \left. + \frac{1}{3!}(q - q^{-1})^2 \sum_{\substack{k_1+k_2+k_3=r \\ k_i \geq 1}} a(k_1)a(k_2)a(k_3) + \dots + \frac{1}{r!}(q - q^{-1})^{r-1}a(1)^r \right), \end{aligned}$$

and for $k_1 + \dots + k_n = s+l$

$$a(k_1) \cdots a(k_n)x^-(m-l)\tilde{v}_\lambda = (-1)^n \prod_{i=1}^n \frac{[2k_i]}{k_i} x^-(s+m)\tilde{v}_\lambda,$$

by (3.8). Thus we have

$$x^+(s)v = \sum_l A_l q^{2l+2s} f_l(s) K x^-(s+m)\tilde{v}_\lambda,$$

and where for $s+l \geq 1$ one has

$$\begin{aligned} f_l(s) &= \left(-\frac{[2(s+l)]}{s+l} + \frac{1}{2}t \sum_{k_1+k_2=s+l} \frac{[2k_1][2k_2]}{k_1k_2} \right. \\ &\quad \left. - \frac{1}{3!}t^2 \sum_{k_1+k_2+k_3=s+l} \frac{[2k_1][2k_2][2k_3]}{k_1k_2k_3} + \dots + \frac{(-1)^{s+l}[2]^{s+l}}{(s+l)!} t^{s+l-1} \right), \end{aligned}$$

where $t = q - q^{-1}$. Note by (3.19) we have

$$f_l(s) = \frac{(1 - q^4)}{q^{2(s+l)}(q - q^{-1})}.$$

Suppose $x^+(s)v = 0$ for any s . Then

$$\sum_l A_l q^{2l+2s} f_l(s) = 0$$

for any sufficiently large s and so

$$(6.1) \quad \sum_l A_l = 0.$$

Note that this equality does not depend on s .

We can assume by (3.9) that without loss of generality the monomials in u are ordered in such a way that $m-l \leq l$ for each l . Choose now the smallest among $m-l$, say r , with $A_l = A_{m-r} \neq 0$ and apply $x^+(-r)$ to \tilde{v}_λ noting that $r \leq l$ (so $l-r \geq 0$ and $-r+m-l \geq 0$):

$$\begin{aligned}
x^+(-r)v &= \frac{1}{q-q^{-1}} \sum_l A_l \psi(-r+l)x^-(m-l)\tilde{v}_\lambda + A_{m-r}x^-(m-r)x^+(-r)x^-(r)\tilde{v}_\lambda \\
&= \sum_l A_l q^{2l-2r} f_l(-r)Kx^-(m-l)\tilde{v}_\lambda + A_{m-r}x^-(m-r) \left(\frac{K-K^{-1}}{q-q^{-1}} \right) \tilde{v}_\lambda \\
&= A_{m-r}x^-(m-r) \left(\frac{K-K^{-1}}{q-q^{-1}} \right) \tilde{v}_\lambda,
\end{aligned}$$

due to (6.1).

This is a contradiction. It implies $v = 0$.

□

THEOREM 6.0.10. *Let $\lambda \in \Lambda$ such that $\lambda(c) = 0$. Then module $\tilde{M}_q(\lambda)$ is simple if and only if $\lambda(h) \neq 0$.*

PROOF. Suppose $\lambda(h) = 0$. Let $v = x^-(m)\tilde{v}_\lambda$. Then for any $s \neq -m$ we have $x^+(s)v = 0$. Similarly, $x^+(-m)v = [x^+(-m), x^-(m)]\tilde{v}_\lambda = \frac{1}{q-q^{-1}}(K-K^{-1})\tilde{v}_\lambda = 0$, since $K\tilde{v}_\lambda = q^{\lambda(h)}\tilde{v}_\lambda$. Hence, v generates a proper nonzero submodule of $\tilde{M}_q(\lambda)$.

Assume now $\lambda(h) \neq 0$. To show simplicity of $\tilde{M}_q(\lambda)$ consider an arbitrary homogeneous element $v \in \tilde{M}_q(\lambda)_{\lambda-\xi}$ such that $x^+(s)v = 0$ for any s . We need to show that v is a scalar multiple of \tilde{v}_λ . We will proceed by the induction in $|\xi|$ to show that if $|\xi| > 0$ then $v = 0$.

If $|\xi| = 1$ and $v \neq 0$ then $v = x^-(m)\tilde{v}_\lambda$ and $x^+(-m)v \neq 0$. Hence $v = 0$. The case $|\xi| = 2$ follows from Lemma 6.0.9. Note that this case does not depend on the value $\lambda(h)$.

Suppose now $|\xi| = k > 2$, $v = u\tilde{v}_\lambda$ and

$$u = \sum_{n_1, \dots, n_k} A(n_1, \dots, n_k)x^-(n_1) \dots x^-(n_k).$$

Using notation from the lemma above we have

$$\psi(s)x^-(n) = -tf_n(s)Kx^-(n+s) + q^{-2}x^-(n)\psi(s)$$

and thus

$$\begin{aligned}
 x^+(s)(x^-(n_1) \cdots x^-(n_k)\tilde{v}_\lambda) &= \frac{1}{q-q^{-1}}\psi(n_1+s)x^-(n_2) \cdots x^-(n_k)\tilde{v}_\lambda \\
 &\quad + x^-(n_1)x^+(s)x^-(n_2) \cdots x^-(n_k)\tilde{v}_\lambda \\
 &= -q^{2s+2n_1}f_{n_1}(s+n_1)Kx^-(n_1+n_2+s)x^-(n_3) \cdots x^-(n_k)\tilde{v}_\lambda \\
 &\quad + \frac{q^{-2}}{q-q^{-1}}x^-(n_2)\psi(n_1+s)x^-(n_3) \cdots x^-(n_k)\tilde{v}_\lambda \\
 &\quad + x(n_1)x^+(s)x^-(n_2) \cdots x^-(n_k)\tilde{v}_\lambda \\
 \\
 &= -q^{2s+2n_1}f_{n_1}(s)Kx^-(n_1+n_2+s)x^-(n_3) \cdots x^-(n_k)\tilde{v}_\lambda \\
 &\quad - q^{-2}q^{2s+2n_1}f_{n_1}(n_1+s)x^-(n_2)Kx^-(n_1+n_3+s)x^-(n_4) \cdots x^-(n_k)\tilde{v}_\lambda \\
 &\quad + \frac{q^{-4}}{q-q^{-1}}x^-(n_2)x^-(n_3)\psi(n_1+s)x^-(n_4) \cdots x^-(n_k)\tilde{v}_\lambda \\
 &\quad + \frac{1}{q-q^{-1}}x^-(n_1)\psi(n_2+s)x^-(n_3) \cdots x^-(n_k) \\
 &\quad + x^-(n_1)x^-(n_2)x^+(s)x^-(n_3) \cdots x^-(n_k)\tilde{v}_\lambda \\
 &\quad - q^{2s+2n_{k-1}}f_{n_{k-1}}(s+n_{k-1})x^-(n_1) \cdots x^-(n_{k-2})Kx^-(n_{k-1}+n_k+s)\tilde{v}_\lambda.
 \end{aligned}$$

We may order monomials in u in such a way that $n_1 \leq n_2 \leq \dots \leq n_k$. We also introduce lexicographical ordering among the monomials.

The smallest monomial in the image $x^+(s)(x^-(n_1) \dots x^-(n_k)\tilde{v}_\lambda)$ is

$$x^-(n_1) \dots x^-(n_{k-2})Kx^-(n_{k-1}+n_k+s)$$

up to a constant. It determines uniquely the first $k-2$ elements in the monomial and leaves a freedom in the choice of last two elements (remembering that u is homogeneous). Hence, we may assume that

$$u = x^-(n_1)x^-(n_2) \dots x^-(n_{k-2}) \sum_l B_l x^-(m-l)x^-(l),$$

for some fixed m , $n_1 \leq \dots \leq n_{k-2} \leq m-l \leq l$. Then

$$\begin{aligned}
 x^+(s)v &= x^+(s)u\tilde{v}_\lambda \\
 &= [x^+(s), x^-(n_1)x^-(n_2) \dots x^-(n_{k-2})] \sum_l B_l x^-(m-l)x^-(l)\tilde{v}_\lambda \\
 &\quad + x^-(n_1)x^-(n_2) \dots x^-(n_{k-2})[x^+(s), \sum_l B_l x^-(m-l)x^-(l)]\tilde{v}_\lambda.
 \end{aligned}$$

Note that the first part in the sum above will contribute smaller monomials than the second part. Hence, if $x^+(s)v = 0$ for any $s \in \mathbb{Z}$ then

$$[x^+(s), \sum_l B_l x^-(l)x^-(m-l)]\tilde{v}_\lambda = 0,$$

for all sufficiently large integers s . Define $A_l \in \mathbb{C}(q)$ such that

$$\sum_l A_l x^-(l)x^-(m-l) = \sum_l B_l x^-(m-l)x^-(l)$$

Applying Lemma 6.0.9 we obtain that all A_l are zero (and hence so are the B_l) and thus $v = 0$. This completes the proof. \square

Set $R_q(\lambda) = \sum_{\xi, |\xi| > 0} \tilde{M}_q(\lambda)_\xi$. Then $R_q(\lambda)$ is the unique maximal submodule of $\tilde{M}_q(\lambda)$ and $\dim \tilde{M}_q(\lambda)/R_q(\lambda) = 1$.

REMARK 6.0.11. *It was shown in [CFKM97], Theorem 5.4 that imaginary Verma module $M(\lambda)$ over affine $\hat{\mathfrak{sl}}(2)$ admits a quantum deformation to the imaginary Verma module $M_q(\lambda)$ over U_q in such a way that the dimensions of the weight spaces are invariant under the deformation, generalizing the Lusztig's deformation functor constructed originally for classical Verma modules [Lus88], see also [FGM98]. Theorem 6.0.10 shows that Lusztig's deformation functor preserves module structure in the case of imaginary Verma modules (see [Fut94]).*

7. Simplicity of \mathcal{N}_q^- as a \mathcal{K}_q -module

We will show that \mathcal{N}_q^- is simple as a module over \mathcal{K}_q .

LEMMA 7.0.12. *Let $P \in \mathcal{N}_q^-$. If $\Omega_\psi(s)P = 0$ for any $s \in \mathbb{Z}$, then P is a constant multiple of 1.*

PROOF. We may assume without loss of generality that P is a homogeneous element, say $P \in (\mathcal{N}_q^-)_{\lambda-\xi}$. We assume that $\xi \neq 0$. Then $\xi = n\alpha + m\delta$, $n > 0$, $m \in \mathbb{Z}$. Set $|\xi| = n$. We shall prove the lemma by induction on $|\xi|$.

Suppose $|\xi| = 1$. Then $P = x^-(m)$ and

$$\begin{aligned} \Omega_\psi(s)(P) &= \delta_{s,-m}\gamma^s + \sum_{r \in \mathbb{Z}} g(s-r)x^-(m-r+s)\gamma^r\Omega_\psi(r)1 \\ &= \delta_{s,-m}. \end{aligned}$$

Hence $\Omega_\psi(-m)(P) \neq 0$ unless $P = 0$.

Suppose $|\xi| > 1$. We assume $\Omega_\psi(l)(P) = 0$ for any $l \in \mathbb{Z}$ and then we use (4.18). For all k and m we get

$$(7.1) \quad \Omega_\psi(k)\Omega_\phi(m)(P) = \sum_{r \geq 0} g(r)\gamma^{2r}\Omega_\phi(r+m)\Omega_\psi(k-r)(P) = 0,$$

Hence by the induction hypothesis $\Omega_\phi(m)(P) = 0$ as $\Omega_\phi(m)(P) \in (\mathcal{N}_q^-)_{\lambda-\xi+1}$. Then $[x^+(m), P] = 0$ by (4.4).

Consider the imaginary Verma module $M_q(\lambda)$ with $\lambda(c) = 0$ and choose λ such that $\lambda(h) \neq 0$. Then $\tilde{M}_q(\lambda)$ is the unique irreducible quotient of $M_q(\lambda)$ and $v = Pv_\lambda$ is a nonzero element of the module $\tilde{M}_q(\lambda)$.

Thus

$$x^+(s)v = [x^+, P]\tilde{v}_\lambda + Px^+(s)\tilde{v}_\lambda = 0$$

for all $s \in \mathbb{Z}$.

Consider $V = \mathcal{N}_q^-v \subset \tilde{M}_q(\lambda)$. Then V is a nonzero proper submodule of $\tilde{M}_q(\lambda)$ which is a contradiction by Theorem 6.0.10. This completes the proof. \square

REMARK 7.0.13. *Suppose $|\xi| = 2$. We will give a direct proof of Lemma 7.0.12 in this case without the use of Theorem 6.0.10.*

Let

$$P = \sum_{n_1, n_2, n_1+n_2=m} A(n_1, n_2) x^-(n_1) x^-(n_2).$$

We can assume that $n_1 \leq n_2$ in all the monomials in P . Then

$$\begin{aligned} \Omega_\psi(s)(P) &= \sum_{n_1, n_2, n_1+n_2=m} A(n_1, n_2) (\delta_{n_1, -s} x^-(n_2) + g(n_2 + s) x^-(m + s)) \gamma^s \\ &= A(-s, m + s) \delta_{n_1, -s} x^-(m + s) \gamma^s \\ &\quad + \left(\sum_{n_1} A(n_1, m - n_1) g(m + s - n_1) \right) x^-(m + s) \gamma^s. \end{aligned}$$

If $\Omega_\psi(s)(P) = 0$ for all s then, in particular,

$$\sum_{n_1} A(n_1, m - n_1) g(m + s - n_1) = 0$$

for any s sufficiently large. Since $g(p) = (q^4 - 1)q^{-2p-2}$ for $p \geq 1$, we will get

$$(q^4 - 1)q^{-2(m+s+1)} \sum_{n_1} q^{2n_1} A(n_1, m - n_1) = 0$$

implying $\sum_{n_1} q^{2n_1} A(n_1, m - n_1) = 0$. Note that this relation does not depend on s .

Choose now the smallest among n_1 , say $n_1 = r$, with $A(n_1, m - n_1) \neq 0$ and apply $\Omega_\psi(-r)$:

$$\Omega_\psi(-r)P = A(r, m-r)x(m-2r)\gamma^s + \sum_{n_1} A(n_1, m-n_1)g(m-n_1-r)x^-(m-2r)\gamma^s = 0,$$

Now

$$\sum_{n_1} A(n_1, m - n_1) g(m - n_1 - r) = (q^4 - 1)q^{-2m+2r-2} \sum_{n_1} A(n_1, m - n_1) q^{2n_1} = 0.$$

Hence

$$0 = \Omega_\psi(-r)P = A(r, m - r)x(m - 2r)\gamma^s,$$

but then $A(r, m - r) = 0$ which is a contradiction.

We have Suppose $m - 2r > 0$. Then

$$\sum_{n_1} A(n_1, m - n_1) g(m - n_1 - r) = (q^4 - 1)q^{-2(m-r+1)} \sum_{n_1} q^{2n_1} A(n_1, m - n_1) = 0,$$

and

$$\sum_{n_1} A(n_1, m - n_1) g(m - n_1 - r) x^-(m - 2r) \gamma^s = 0.$$

Thus $A(r, m - r) = 0$, which is a contradiction.

If $m = 2r$ then we have a unique monomial $x^-(r)^2$ in P due to the chosen ordering. Hence,

$$\Omega_\psi(-r)P = A(r, r)(1 + g(0))x^-(0)\gamma^s = A(r, r)(1 + q^{-2})x^-(0)\gamma^s = 0$$

implies $A(r, r) = 0$. This is again a contradiction. Therefore, there exists s such that $\Omega_\psi(s)P \neq 0$. Note that in fact we proved that in the case $|\xi| = 2$, $\Omega_\psi(s)P \neq 0$ for all $s \in \mathbb{Z}$.

Lemma 7.0.12 implies immediately the following result.

THEOREM 7.0.14. *The algebra \mathcal{N}_q^- is simple as a \mathcal{K}_q -module.*

COROLLARY 7.0.15. *The form defined in Proposition 5.0.7 is non-degenerate.*

PROOF. By Proposition 5.0.7 the radical of the form $(\ , \)$ is a \mathcal{K}_q -submodule of \mathcal{N}_q^- and since $(1, 1) = 1$, the radical must be zero. \square

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