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# STRUCTURE OF THE NONSTANDARD CATEGORY OF HIGHEST WEIGHT MODULES

BEN COX*

*Dedicated to A. J. Coleman on his seventy-fifth Birthday*

ABSTRACT. In previous work of Futorny and Saifi, and independently the author, the structure of Verma modules induced from "nonstandard" Borel subalgebras of an affine Kac-Moody algebra $\hat{\mathfrak{g}}$ was analyzed. In this paper we define, for each highest weight $\lambda$, a category $\mathcal{O}_\lambda^X$ of representations of $\hat{\mathfrak{g}}$ that contain these "nonstandard" Verma modules and we show that this category is equivalent to the category $\mathcal{O}_\lambda(\hat{\mathfrak{k}})$ for a suitable infinite dimensional Lie subalgebra $\hat{\mathfrak{k}} \subset \hat{\mathfrak{g}}$. When $\hat{\mathfrak{k}}$ is an affine Kac-Moody algebra, we also obtain a BGG type resolution and BGG duality theorem in the setting of $\mathcal{O}_\lambda^X$.

## §0. Introduction:

In previous work of Futorny and Saifi, and independently the author, the structure of Verma modules $M^X(\lambda)$ induced from nonstandard Borel subalgebras was analyzed. More precisely the results can be described as follows: Let $\dot{\mathfrak{g}}$ denote a finite dimensional simple Lie algebra over $\mathbb{C}$, $\hat{\mathfrak{g}}$ the nontwisted affine Kac-Moody algebra associated to $\dot{\mathfrak{g}}$, $\mathfrak{H}$ the Cartan subalgebra of $\hat{\mathfrak{g}}$ and $\lambda \in \mathfrak{H}^*$ (Futorny and Saifi in addition analyzed the setting of twisted affine Kac-Moody algebras). Let $c$ denote a nonzero element in the center of $\hat{\mathfrak{g}}$ and suppose $\lambda(c) \neq 0$. If $\Pi$ is a set of simple roots of $\dot{\mathfrak{g}}$, and $X \subsetneq \Pi$ then one can construct from the data $X$, a nonstandard Borel subalgebra $\mathfrak{b}_+^X$ and a Verma module $M^X(\lambda)$ induced from $\mathfrak{b}_+^X$ and $\lambda$. It was shown in [C] and [FS] that the multiplicity of irreducible subquotients in a local composition series for $M^X(\lambda)$ is equal to the multiplicity of irreducible subquotients

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in a local composition series for a Verma module $M_{\hat{\mathfrak{k}}}(\lambda)$ for some infinite dimensional Lie subalgebra $\hat{\mathfrak{k}}$ (it has a triangular decomposition given in 1.4). This suggests that there might be an equivalence of categories lurking in the background (see [ES] Lemma 3.5). The main purpose of this article is to define the appropriate categories $\mathcal{O}_\lambda^X$ and $\mathcal{O}_\lambda(\hat{\mathfrak{k}})$ and show that they are equivalent. As a result of this equivalence, we obtain an analogue of a BGG type resolution and a BGG duality theorem when $\hat{\mathfrak{k}}$ (modulo a central ideal $\mathfrak{h}^X$) is an affine Kac-Moody algebra and $\lambda(c) \neq 0$.

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## §1. Notation:

**1.1** Let $\dot{\mathfrak{g}}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra of $\dot{\mathfrak{g}}$, $\dot{\Delta}$ its root system with respect to $\mathfrak{h}$, and $\Pi$ a set of simple roots and $\dot{\Delta}_+$ (resp. $\dot{\Delta}_-$) the set of positive (resp. negative) roots determined by $\Pi$. Let $\dot{\mathfrak{n}}_\pm := \oplus_{\alpha\in\dot{\Delta}_\pm}\dot{\mathfrak{g}}_\alpha$ so that $\dot{\mathfrak{g}} = \dot{\mathfrak{n}}_-\oplus\mathfrak{h}\oplus\dot{\mathfrak{n}}_+$ is the triangular decomposition of $\dot{\mathfrak{g}}$. For any Lie algebra $\mathfrak{a}$, let $L(\mathfrak{a}) = \mathfrak{a}\bigotimes\mathbb{C}[t,t^{-1}]$ be the loop algebra of $\mathfrak{a}$ and let $\hat{\mathfrak{g}} = L(\dot{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the associated nontwisted affine Kac-Moody algebra of $\dot{\mathfrak{g}}$ (see [K] and [MP] for more information about these algebras). We let $\mathfrak{H} = \mathfrak{h} + \mathbb{C}c + \mathbb{C}d$ denote the Cartan subalgebra of $\hat{\mathfrak{g}}$.

Let $\delta$ be the indivisible positive imaginary root for $\hat{\mathfrak{g}}$ and let $\Delta = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}, n \in \mathbb{Z}\}$ be the set of roots of $\hat{\mathfrak{g}}$. A subset $\Delta_+$ of $\Delta$ is called a *set of positive roots* if

(1) If $\alpha,\beta \in \Delta_+$ and $\alpha+\beta \in \Delta$, then $\alpha+\beta \in \Delta_+$.

(2) If $\alpha \in \Delta$, then either $\alpha \in \Delta_+$ or $-\alpha \in \Delta_+$.

(3) If $\alpha \in \Delta_+$, then $-\alpha \notin \Delta_+$.

A subalgebra $\mathfrak{b}$ of $\hat{\mathfrak{g}}$ is a *Borel subalgebra* if $\mathfrak{b} = \mathfrak{H} \oplus (\oplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha)$ for some set of positive roots $\Delta_+$ (see [JK] and [F]).

We now introduce some subalgebras of $\hat{\mathfrak{g}}$. For the rest of this article we assume $X \subsetneq \Pi$, $\dot{\Delta}^X$ the subroot system generated by $X$ and $\dot{\Delta}^X_\pm = \dot{\Delta}^X \cap \dot{\Delta}_\pm$. $X$ determines a reductive subalgebra $\dot{\mathfrak{m}}$ of $\dot{\mathfrak{g}}$: $\dot{\mathfrak{m}} = \dot{\mathfrak{m}}_-\oplus\mathfrak{h}\oplus\dot{\mathfrak{m}}_+$ where $\dot{\mathfrak{m}}_\pm = \oplus_{\alpha\in\dot{\Delta}^X_\pm}\dot{\mathfrak{g}}_\alpha$. $X$ also determines subalgebras

$\dot{\mathfrak{u}}_\pm = \oplus_{\alpha\in\dot{\Delta}_\pm\setminus\dot{\Delta}^X}\dot{\mathfrak{g}}_\alpha$ such that $[\dot{\mathfrak{u}}_\pm, \dot{\mathfrak{m}}] \subset \dot{\mathfrak{u}}_\pm$. Consequently the decomposition of $\dot{\mathfrak{g}} = \dot{\mathfrak{u}}_- \oplus \dot{\mathfrak{m}} \oplus \dot{\mathfrak{u}}_+$ induces a decomposition of $\hat{\mathfrak{g}}$: $\hat{\mathfrak{g}} = \mathfrak{u}_- \oplus \hat{\mathfrak{m}} \oplus \mathfrak{u}_+$ where $\hat{\mathfrak{m}} = L(\dot{\mathfrak{m}}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\mathfrak{u}_\pm = L(\dot{\mathfrak{u}}_\pm)$. We also set $\mathfrak{m}_\pm = (\dot{\mathfrak{m}} \bigotimes \mathbb{C}[t^{\pm1}]t^{\pm1}) \oplus \dot{\mathfrak{m}}_\pm$. Since $\dot{\mathfrak{m}}$ is reductive we have $\dot{\mathfrak{m}} = \mathfrak{h}^X \oplus \dot{\mathfrak{k}}$ where $\dot{\mathfrak{k}}$ is semisimple and $\mathfrak{h}^X := \{h \in \mathfrak{h} \,|\, \alpha(h) = 0 \quad \text{for all}\, \alpha \in \dot{\Delta}^X\}$ is the center of $\dot{\mathfrak{m}}$. Moreover $\dot{\mathfrak{k}} = \dot{\mathfrak{m}}_- \oplus \mathfrak{h}_X \oplus \dot{\mathfrak{m}}_+$ where $\mathfrak{h}_X = \sum_{\alpha\in\dot{\Delta}^X_+}[\dot{\mathfrak{g}}_\alpha, \dot{\mathfrak{g}}_{-\alpha}]$. From $\dot{\mathfrak{k}}$ we can also construct the affine Kac-Moody algebra $\hat{\mathfrak{k}} = L(\dot{\mathfrak{k}}) + \mathbb{C}c + \mathbb{C}d + \mathfrak{h}^X$ and define $\mathfrak{k}_\pm = (\dot{\mathfrak{k}} \otimes \mathbb{C}[t^{\pm1}]t^{\pm1}) \oplus \dot{\mathfrak{m}}_\pm$. Set $\mathfrak{n}_\pm = \mathfrak{m}_\pm \oplus \mathfrak{u}_\pm, \mathfrak{b}_\pm = \mathfrak{H} \oplus \mathfrak{n}_\pm$, and $\mathfrak{p}_\pm = \mathfrak{p}_\pm = \hat{\mathfrak{m}} \oplus \mathfrak{u}_\pm$. $\mathfrak{b}_\pm$ is a Borel subalgebra of $\hat{\mathfrak{g}}$.

**1.2.** The subalgebra

$$L = L^X := (\mathfrak{h}^X \otimes \mathbb{C}[t]t) \oplus \mathbb{C}c \oplus (\mathfrak{h}^X \otimes \mathbb{C}[t^{-1}]t^{-1}).$$

of $\hat{\mathfrak{m}}$ is a Heisenberg algebra with a $\mathbb{Z}$-grading determined by the degree of $t$. $L$ determines two subalgebras

$$L_+ = \mathfrak{h}^X \otimes \mathbb{C}[t]t$$

and

$$L_- = \mathfrak{h}^X \otimes \mathbb{C}[t^{-1}]t^{-1}.$$

Set $\mathfrak{b}_L = L_+ \oplus \mathbb{C}c$ and for $k \in \mathbb{C}$ let $\mathbb{C}_k = \mathbb{C}w$ denote the one dimensional $\mathfrak{b}_L$-module defined by $L_+w = 0$ and $cw = kw$. One says that a $\mathbb{Z}$-graded $L$-module $V$ satisfies *property* $\mathfrak{C}_k$ if (i) $c.v = kv$ for all $v \in V$ and (ii) there exists $N \in \mathbb{Z}$ such that $V_n = 0$ for $n > N$. Let $V^{L_+} = \{v \in V \,|\, \mathfrak{l}_+.v = 0\}$ denote the *L-invariant subspace* of $V$.

Let $U(\mathfrak{a})$ denote the universal enveloping algebra of $\mathfrak{a}$. Suppose $k \neq 0$. Then by [FLM], Theorem 1.7.3 or [K] section 9.13, every $L$-module satisfying property $\mathfrak{C}_k$ is isomorphic to a direct sum of copies of $U(L) \otimes_{U(\mathfrak{b}_L)} \mathbb{C}_k$. In addition if $M$ satisfies property $\mathfrak{C}_k$ then the map

$$\phi_M : U(L) \otimes_{U(\mathfrak{b}_L)} M^{L_+} \to M$$

induced by $u \otimes v \to u \cdot v$ is an isomorphism of $L$-modules.

**1.3** For any relation $R$ on the set of integers and $c$ an integer we set $\mathbb{Z}_{Rc} = \{a \in \mathbb{Z} | \, aRc\}$. Define

$$\Delta_+^X = \{\alpha + n\delta | \alpha \in \dot{\Delta}_+ \backslash \dot{\Delta}_+^X, n \in \mathbb{Z}\} \cup \{\alpha + n\delta | \alpha \in \dot{\Delta}^X \cup \{0\}, n \in \mathbb{Z}_{>0}\} \cup \dot{\Delta}_+^X \qquad \text{and}$$

$$\Delta_\pm(\hat{\mathfrak{m}}) = \dot{\Delta}_\pm^X \cup \{\alpha + n\delta | \, \alpha \in \dot{\Delta}^X \cup \{0\}, \pm n \in \mathbb{Z}_{>0}\}.$$

Let $Q_+^X$ (resp. $Q_+(\hat{\mathfrak{m}})$) denote the monoid in $\mathfrak{H}^*$ generated by $\Delta_+^X$ (resp. $\Delta_+(\hat{\mathfrak{m}})$). Define $\lambda <^X \mu$ if $\mu - \lambda \in Q_+^X$ and $\lambda <_{\hat{\mathfrak{m}}} \mu$ if $\mu - \lambda \in Q_+(\hat{\mathfrak{m}})$.

If the Coxeter-Dynkin diagram for $X$ is connected then we let $\Pi_{\hat{\mathfrak{m}}} = X \cup \{\alpha_{\hat{\mathfrak{m}}}\}$ denote the set of simple roots in $\Delta_+^X$ for $\hat{\mathfrak{m}}$ where $\alpha_{\hat{\mathfrak{m}}} = -\theta + \delta$ and $\theta$ is the maximal root of $\dot{\Delta}^X$ with respect to $\Pi_{\hat{\mathfrak{m}}}$. (In this case $\hat{\mathfrak{k}}$ is an affine Kac-Moody algebra.) We also let $\check{\beta}$ denote the coroot of $\beta \in \Pi_{\hat{\mathfrak{m}}}$.

**1.4** For $V$ a $\hat{\mathfrak{g}}$-module and $\lambda \in \mathfrak{H}^*$, let $V_\lambda := \{v | \, hv = \lambda(h)v \text{ for all } h \in \mathfrak{H}\}$ be the $\lambda$-*th weight space* of $V$. Let $P(V) = \{\lambda \in \mathfrak{H}^* | \, V_\lambda \neq 0\}$ be the set of *weights* of $V$ and define $\lambda \downarrow Q_+^X = \{\mu \in \mathfrak{H}^* | \, \mu \leq^X \lambda\}$ and $\lambda \downarrow Q_+(\hat{\mathfrak{m}}) = \{\mu \in \mathfrak{H}^* | \, \mu \leq_{\hat{\mathfrak{m}}} \lambda\}$.

For any Lie algebra $\mathfrak{a}$ let $\mathfrak{a} -$ mod denote the category of all left $\mathfrak{a}$-modules. For each $\lambda \in \mathfrak{H}$ we let $\mathcal{O}_\lambda(\hat{\mathfrak{m}})$ (resp. $\mathcal{O}_\lambda(\hat{\mathfrak{k}})$) denote full the subcategory of $\hat{\mathfrak{m}}$-mod (resp. $\hat{\mathfrak{k}}$-mod) whose objects $M$ satisfy

(1) $$M = \bigoplus_{\xi \in \lambda \downarrow Q_+(\hat{\mathfrak{m}})} M_\xi$$

and

(2) $$\dim M_\xi < \infty \quad \text{for all} \quad \xi \in \mathfrak{H}^*.$$

**Remark:** If the Coxeter-Dynkin diagram for $X$ is not connected then

$$S = (\mathfrak{H}, \hat{\mathfrak{k}}_+, Q_+(\hat{\mathfrak{m}}), \sigma|_{\hat{\mathfrak{k}}})$$

is not a triangular decomposition of $\hat{\mathfrak{k}}$. On the other hand if we let $Q_+$ denote the submonoid generated by the standard set of positive roots $\Delta^\Pi$, then $T = (\mathfrak{H}, \hat{\mathfrak{k}}_+, Q_+, \sigma|_{\hat{\mathfrak{k}}})$ is a regular triangular decomposition (see [C] §2 or [MP] Chapter 2). In this case $\mathcal{O}_\lambda(\hat{\mathfrak{k}})$ is a subcategory of the category $\mathcal{O}(\hat{\mathfrak{k}}, T)$ and all of the results that we will use below for $\mathcal{O}_\lambda(\hat{\mathfrak{k}})$ follow from corresponding results in [MP] and [RW].

For $\lambda \in \mathfrak{H}^*$ with $\lambda(c) \neq 0$, let $\mathcal{O}_\lambda^X$ denote the full subcategory of $\hat{\mathfrak{g}}$-mod whose objects $M$ satisfy

(1) $M = \bigoplus_{\xi \in \lambda \downarrow Q_+^X} M_\xi$,

(2) $M$ is generated as a $\hat{\mathfrak{g}}$-module by $\bigoplus_{\xi \in \lambda \downarrow Q_+(\hat{\mathfrak{m}})} M_\xi$, and

(3) $\bigoplus_{\xi \in \lambda \downarrow Q_+(\hat{\mathfrak{m}})} M_\xi$ is a module in $\mathcal{O}_\lambda(\hat{\mathfrak{m}})$.

We call $\mathcal{O}_\lambda^X$ the *nonstandard* category $\mathcal{O}$ when $X \neq \Pi$. (The condition $\lambda(c) \neq 0$ is required to ensure that subquotients of modules in $\mathcal{O}_\lambda^X$ are still in $\mathcal{O}_\lambda^X$, see [F2] for examples where this does not occur.)

We define $R : \mathcal{O}_\lambda^X \to \mathcal{O}_\lambda(\hat{\mathfrak{m}})$ by

$$R(M) = \bigoplus_{\xi \in \lambda \downarrow Q_+(\hat{\mathfrak{m}})} M_\xi \quad \text{for} \quad M \in \mathcal{O}_\lambda^X$$

and

$$R(f) = f|_{R(M)}$$

for any $f \in \operatorname{Hom}_{\hat{\mathfrak{g}}}(M, N)$ in the category $\mathcal{O}_\lambda^X$. $R$ is exact since modules in $\mathcal{O}_\lambda(\hat{\mathfrak{m}})$ are $\mathfrak{H}$-semisimple.

A key result that we will need later about the functor $R$ is

**Lemma.** *([C], [FS]) Every nonzero submodule $N$ of $M(\lambda)$ is generated by $R(N)$.*

**1.5.** For $\lambda \in \mathfrak{H}^*$ define $M^X(\lambda) = U(\hat{\mathfrak{g}}) \bigotimes_{U(\mathfrak{b}_+^X)} \mathbb{C}_\lambda$ where $\mathbb{C}_\lambda$ is the usual one dimensional $\mathfrak{b}_+$-module. This is the (*nonstandard*) *Verma module* for the nonstandard Borel subalgebra $\mathfrak{b}_+$. Let $L(\lambda) = L^X(\lambda)$ denote the unique irreducible quotient of $M(\lambda) = M^X(\lambda)$. We can view $\mathbb{C}_\lambda$ as an $\mathfrak{m}_+ \oplus \mathfrak{H}$-module or $\mathfrak{k}_+ \oplus \mathfrak{H}$-module by restriction and then define $M_{\mathfrak{m}}(\lambda) = U(\hat{\mathfrak{m}}) \bigotimes_{U(\mathfrak{m}_+ \oplus \mathfrak{H})} \mathbb{C}_\lambda$ and $M_{\mathfrak{k}}(\lambda) = U(\hat{\mathfrak{k}}) \bigotimes_{U(\mathfrak{k}_+ \oplus \mathfrak{H})} \mathbb{C}_\lambda$. Let $L_{\mathfrak{m}}(\lambda)$ and $L_{\mathfrak{k}}(\lambda)$ denote the irreducible quotients of $M_{\mathfrak{m}}(\lambda)$ and $M_{\mathfrak{k}}(\lambda)$ respectively. Finally if $N$ is an $\hat{\mathfrak{m}}$-module then we can make it into a $\mathfrak{p}_+$-module by letting $\mathfrak{u}_+$ act by zero. Inducing up to $\hat{\mathfrak{g}}$ we obtain $U(N) := U(\hat{\mathfrak{g}}) \bigotimes_{U(\mathfrak{p}_+)} N$. Observe that $U$ defines a functor $\mathcal{O}_\lambda(\hat{\mathfrak{m}}) \to \mathcal{O}_\lambda^X$. The main goal of this paper is to show that $U \circ R$ and $R \circ U$ are both equivalent to identity functors and hence define an equivalence of categories.

## §2. Highest Weight Series and an Equivalence of Categories:

2.1 Let $\mathfrak{g}$ be a complex Lie algebra, $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$ and $\sigma : \mathfrak{g} \to \mathfrak{g}$ a linear involutive anti-automorphism (i.e. $\sigma^2 = 1_{\mathfrak{g}}$, and $\sigma([x,y]) = [\sigma(y), \sigma(x)]$ for all $x, y \in \mathfrak{g}$) such that

$$\mathfrak{a} + \sigma(\mathfrak{a}) = \mathfrak{g}.$$

Let $\lambda : \mathfrak{a} \to \mathbb{C}$ be a 1-dimensional representation of $\mathfrak{a}$. Following [JK] we say a representation $\pi : \mathfrak{g} \to gl(V)$ is a *highest weight representation of highest weight* $\lambda$ (with respect to $\mathfrak{a}$) if there exists a vector $v_\lambda \in V$ such that

$$\pi(U(\mathfrak{g}))v_\lambda = V \quad \text{and}$$
$$\pi(x)v_\lambda = \lambda(x)v_\lambda \quad \text{for} \quad x \in \mathfrak{a}.$$

Let $M$ be a $\mathfrak{g}$-module. A $\mathfrak{g}$*-highest weight series (with respect to* $\mathfrak{a}$*)* for $M$ is an increasing chain

$$(0) = M_0 \subset M_1 \subset M_2 \subset \cdots$$

of submodules of $M$ such that

(i) $\cup_{i=0}^{\infty} M_i = M$ and

(ii) $M_i/M_{i-1}$ is a highest weight module (with respect to $\mathfrak{a}$) for all $i$.

The following is a key result.

**2.2 Proposition.** *(a). ([DGK], [GL], [MP] and [RW]). Let $M \in \mathcal{O}(\mathfrak{g}, T)$ where $\mathfrak{g}$ is a Lie algebra with triangular decomposition $T$. Then $M$ has a highest weight series $\{M_i\}$ such if $M_i/M_{i-1}$ has highest weight $\lambda_i$, then $\lambda_i > \lambda_j$ implies that $i < j$.*

*(b). Let $M$ be a module in $\mathcal{O}_\lambda^X$ and suppose that $\lambda(c) \neq 0$. Then $M$ has a highest weight series $\{M_i\}$ where $M_i/M_{i-1}$ is isomorphic to $M(\lambda_i)/U(N_i)$ for some $N_i$ in $\mathcal{O}_\lambda(\hat{\mathfrak{m}})$.*

*Proof.* $R(M)$ is a module in $\mathcal{O}_\lambda(\hat{\mathfrak{m}})$ so by (a) $R(M)$ has an $\hat{\mathfrak{m}}$-highest weight series $\{N_i\}_{i=0}^\infty$. Define $M_i = U(\hat{\mathfrak{g}})N_i$. Since $P(M) \subset \lambda \downarrow Q_+^X$, we have that

$$U(\mathfrak{u}_+)\mathfrak{u}_+N_i \subset U(\mathfrak{u}_+)\mathfrak{u}_+(\bigoplus_{\beta \in Q_+(\hat{\mathfrak{m}})} M_{\lambda-\beta}) = 0.$$

By the Poincaré-Birkhoff-Witt Theorem it now follows that $M_i = U(\mathfrak{u}_-)\mathfrak{u}_-N_i$. We thus have a surjective $\hat{\mathfrak{g}}$-module homomorphism $\phi_i : U(\hat{\mathfrak{g}}) \bigotimes_{U(\mathfrak{p}_+)} N_i \to M_i$ induced by $u \otimes n \mapsto un$ for $u \in U(\hat{\mathfrak{g}}), n \in N_i$. Since $N_i/N_{i-1}$ is an $\hat{\mathfrak{m}}$-highest weight module of highest weight $\lambda_i$ we have a surjective $\hat{\mathfrak{m}}$-module map

$$M_{\hat{\mathfrak{m}}}(\lambda_i) \to N_i/N_{i-1}. \tag{1}$$

If we let $\mathfrak{u}_+$ act by zero on $M_{\hat{\mathfrak{m}}}(\lambda_i)$ then (1) induces a surjective $\hat{\mathfrak{g}}$-module map

$$M(\lambda_i) \to U(\hat{\mathfrak{g}}) \bigotimes_{U(\mathfrak{p}_+)} (N_i/N_{i-1}) \cong (U(\hat{\mathfrak{g}}) \bigotimes_{U(\mathfrak{p}_+)} N_i)/(U(\hat{\mathfrak{g}}) \bigotimes_{U(\mathfrak{p}_+)} N_{i-1}).$$

Combining this with the canonical map induced by $\phi_i$ we obtain a surjective $\hat{\mathfrak{g}}$-module map

$$M(\lambda_i) \to M_i/M_{i-1}.$$

Thus $M_i/M_{i-1}$ is a highest weight $\hat{\mathfrak{g}}$-module of highest weight $\lambda_i$. Since $R(M)$ generates $M$ as a $\hat{\mathfrak{g}}$-module and $R(M) = \cup_{i=0}^\infty N_i$ we have $M = \cup_{i=0}^\infty M_i$. Now every submodule of $M(\lambda_i)$ is induced from a module in $\mathcal{O}_{\lambda_i}^X(\hat{\mathfrak{m}})$ i.e. it is of the form $U(N)$ for some $N \in \mathcal{O}_{\lambda_i}^X(\hat{\mathfrak{m}})$ (Lemma 1.4), part (b) now follows. □

**2.3 Theorem.** *For all $\lambda \in \mathfrak{h}^*$, $\lambda(c) \neq 0$, $U \circ R$ (resp. $R \circ U$) is natural equivalent to the identity functor $1_{\mathcal{O}_\lambda^X}$ (resp. $1_{\mathcal{O}_\lambda(\hat{\mathfrak{m}})}$).*

*Proof.* First consider $N \in \mathcal{O}_\lambda^X(\hat{\mathfrak{m}})$. Viewing $N \subset R \circ U(N)$ we immediately obtain $R \circ U(N) = N$.

If $M \in \mathcal{O}_\lambda^X$, then $M$ has a $\hat{\mathfrak{g}}$-highest weight series $\{M_i\}$ with

$$M_i/M_{i-1} \cong U(M_{\mathfrak{m}}(\lambda_i)/N_{i-1})$$

for some $\hat{\mathfrak{m}}$-module $N_{i-1}$ by Proposition 2.2. Thus for all $i \geq 1$ we have $R(M_i/M_{i-1}) \cong M_{\hat{\mathfrak{m}}}(\lambda_i)/N_{i-1}$ as $\hat{\mathfrak{m}}$-modules so that $M_i/M_{i-1} \cong U \circ R(M_i/M_{i-1})$. A diagram chase shows that we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & U \circ R(M_i) \arrow[r] \arrow[d] & U \circ R(M_{i+1}) \arrow[r] \arrow[d] & U \circ R(M_{i+1}/M_i) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_i \arrow[r] & M_{i+1} \arrow[r] & M_i/M_{i+1} \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are the canonical maps induced by the obvious inclusions and projections. Lemma 1.4 implies that the left most vertical map is an isomorphism for $i = 1$ and the right map is an isomorphism for all $i$. Thus the Five Lemma and induction on $i$ implies that

$$U \circ R(M_i) \cong M_i$$

for all $i$. Using the Five Lemma and induction one can argue as above to make the identification $R(M_i) = N_i$ for all $i$. Thus

$$\begin{aligned}
U \circ R(M) = U\Big(\bigoplus_{\beta \in Q_+(\hat{\mathfrak{m}})} M_{\lambda-\beta}\Big) &= U(\cup_{i=1}^{\infty} N_i) \\
&= U(\cup_{i=1}^{\infty} R(M_i)) \cong \cup_{i=1}^{\infty} U \circ R(M_i) \\
&\cong \cup_{i=1}^{\infty} M_i = M
\end{aligned}$$

since tensoring commutes with direct limits (see [R], Corollary 2.10). Let $\psi_M$ denote the composition of the maps above. Now we need to check that the isomorphisms $\{\psi_M \mid M \in \mathcal{O}_\lambda^X\}$ define a natural equivalence.

Suppose that $M, N$ are two modules in $\mathcal{O}_\lambda^X$ and $f \in \mathrm{Hom}_{\hat{\mathfrak{g}}}(M, N)$. Then $f(M_{\lambda-\beta}) \subset N_{\lambda-\beta}$ for all $\beta \in Q_+(\hat{\mathfrak{m}})$, so that $R(f) \in \mathrm{Hom}_{\hat{\mathfrak{m}}}(R(M), R(N))$. Since $M$ is generated as a

$\hat{\mathfrak{g}}$-module by $R(M)$, we have that $f$ is completely determined by $R(f)$. Consequently the diagram below is commutative:

$$\begin{array}{ccc} U\circ R(M) & \xrightarrow{\psi_M} & M \\ {\scriptstyle 1\otimes R(f)}\downarrow & & {\scriptstyle f}\downarrow \\ U\circ R(N) & \xrightarrow{\psi_N} & N. \end{array}$$

Hence $U\circ R \cong 1_{\mathcal{O}_\lambda^X}$.

Similarly if $M', N'$ are two modules in $\mathcal{O}_\lambda(\hat{\mathfrak{m}})$ and $g \in \operatorname{Hom}_{\hat{\mathfrak{m}}}(M', N')$, then

$$\begin{array}{ccc} M' & \xrightarrow{\sim} & R\circ U(M') \\ {\scriptstyle g}\downarrow & & {\scriptstyle R(1\otimes g)}\downarrow \\ N' & \xrightarrow{\sim} & R\circ U(N') \end{array}$$

is a commutative diagram where the horizontal maps are induced by inclusions and thus $R\circ U$ is naturally equivalent to $1_{\mathcal{O}_\lambda(\hat{\mathfrak{m}})}$. $\square$

We would now like to show $\mathcal{O}_\lambda(\hat{\mathfrak{m}})$ and $\mathcal{O}_\lambda(\hat{\mathfrak{k}})$ are equivalent categories. Since $\alpha(h)=0$ for $h\in\mathfrak{h}^X$ and $\alpha\in\Delta_+^X$ we have $[\hat{\mathfrak{k}}, L_+]=0$. If $M\in\mathcal{O}_\lambda(\hat{\mathfrak{m}})$, then this implies that $M^{L_+}$ is a $\hat{\mathfrak{k}}$-submodule of $M$. Consequently the functor $\operatorname{Inv}:\mathcal{O}_\lambda(\hat{\mathfrak{m}})\to\mathcal{O}_\lambda(\hat{\mathfrak{k}})$ given by

$$\operatorname{Inv}(M)=M^{L_+} \quad\text{and}\quad \operatorname{Inv}(f)=f|_{M^{L_+}}$$

for $M,N\in\mathcal{O}_\lambda(\hat{\mathfrak{m}})$ and $f\in\operatorname{Hom}_{\mathfrak{m}}(M,N)$ is well-defined. We have another canonical functor $\operatorname{Ind}:\mathcal{O}_\lambda(\hat{\mathfrak{k}})\to\mathcal{O}_\lambda(\hat{\mathfrak{m}})$ given by induction

$$\operatorname{Ind}(N)=U(\hat{\mathfrak{m}})\bigotimes_{U(\hat{\mathfrak{k}}\oplus L_+)} N \quad\text{and}\quad \operatorname{Ind}(f)=1\bigotimes f$$

where $N,N'\in\mathcal{O}_\lambda(\hat{\mathfrak{k}})$, $f\in\operatorname{Hom}_{\mathfrak{k}}(N,N')$ and $L_+$ acts by zero on $N$.

Suppose that $V\in\mathcal{O}_\lambda(\hat{\mathfrak{m}})$. Then the Poincaré-Birkhoff-Witt theorem gives us an isomorphism of $L_-$-modules

$$U(\hat{\mathfrak{m}})\bigotimes_{U(\hat{\mathfrak{k}}\oplus L_+)} V^{L_+}\cong U(L_-)\bigotimes_{U(\mathfrak{b}_L)} V^{L_+} \quad \text{(see 1.2 for notation)}$$

which we can compose with the isomorphism $\phi_V$ given in section 1.2 to obtain an $L_-$-module isomorphism

$$\psi_V : \mathrm{Ind} \circ \mathrm{Inv}\,(V) = U(\hat{\mathfrak{m}}) \bigotimes_{U(\hat{\mathfrak{k}}\oplus L_+)} V^{L+} \to V.$$

In fact a straightforward computation shows this is the canonical $\hat{\mathfrak{m}}$-module homomorphism induced from $u \otimes v \to uv$. This isomorphism also shows us that $V$ is generated by $V^{L+}$ as an $\hat{\mathfrak{m}}$-module and thus

$$\psi_W \circ (\mathrm{Ind} \circ \mathrm{Inv}\,)(f) = f \circ \psi_V$$

for any $f \in \mathrm{Hom}_{\hat{\mathfrak{k}}}(V, W)$ and $W \in \mathcal{O}_\lambda(\hat{\mathfrak{m}})$. Consequently $\mathrm{Ind} \circ \mathrm{Inv} \cong I_{\mathcal{O}_\lambda(\hat{\mathfrak{m}})}$.

Conversely suppose that $N \in \mathcal{O}_\lambda(\hat{\mathfrak{k}})$. Then viewing $N$ as a trivial $L_+$-submodule of $U(L) \otimes_{U(\mathfrak{b}_L)} N$ we have

$$(U(L) \bigotimes_{U(\mathfrak{b}_L)} N)^{L+} = N$$

by 1.2. Since the $U(\hat{\mathfrak{m}}) \otimes_{U(\hat{\mathfrak{k}}\oplus L_+)} N \cong U(L) \otimes_{U(\mathfrak{b}_L)} N$ as $L$-modules we have that the canonical inclusion $N \subset U(\hat{\mathfrak{m}}) \otimes_{U(\hat{\mathfrak{k}}\oplus L_+)} N$ $(n \mapsto 1 \otimes n)$ induces an isomorphism

$$\mathrm{Inv} \circ \mathrm{Ind}\,(N) = (U(\hat{\mathfrak{m}}) \bigotimes_{U(\hat{\mathfrak{k}}\oplus L_+)} N)^{L+} \cong (U(L) \bigotimes_{U(\mathfrak{b}_L)} N)^{L+} = N.$$

Certainly $\mathrm{Inv} \circ \mathrm{Ind}\,(f) = f$ for all $f \in \mathrm{Hom}_{\hat{\mathfrak{k}}}(N, P)$ with $P \in \mathcal{O}_\lambda(\hat{\mathfrak{k}})$, and consequently we have

**2.4 Theorem.** *If $\lambda(c) \neq 0$ then the categories $\mathcal{O}_\lambda^X$, $\mathcal{O}_\lambda(\hat{\mathfrak{m}})$ and $\mathcal{O}_\lambda(\hat{\mathfrak{k}})$ are all equivalent.*

*Proof.* The above paragraphs prove that $\mathcal{O}_\lambda(\hat{\mathfrak{m}})$ and $\mathcal{O}_\lambda(\hat{\mathfrak{k}})$ are equivalent and by Theorem 2.3 we have that $\mathcal{O}_\lambda^X$ and $\mathcal{O}_\lambda(\hat{\mathfrak{m}})$ are equivalent. $\square$

## §3. Consequences of the Equivalence of Categories:

**3.1** Set $P_{\hat{\mathfrak{k}}} = \{\alpha \in Q_+(\hat{\mathfrak{m}}) \mid \mu + \alpha \notin \lambda \downarrow Q_+(\hat{\mathfrak{m}})\}$ and

$$V = U(\hat{\mathfrak{k}}_+)/ \bigoplus_{\alpha\in P_{\hat{\mathfrak{k}}}} U(\hat{\mathfrak{k}}_+)_\alpha, \quad P_{\hat{\mathfrak{k}}}(\mu) = U(\hat{\mathfrak{k}}) \bigotimes_{U(\hat{\mathfrak{k}}_+\oplus\mathfrak{h})} (V \otimes \mathbb{C}_\mu)$$

where $V \bigotimes \mathbb{C}_\mu$ has the tensor product structure as an $\mathfrak{k}_+ \oplus \mathfrak{H}$-module. Then $P_{\mathfrak{k}}(\mu)$ is a projective module in $\mathcal{O}_\lambda(\hat{\mathfrak{k}})$ and it has a unique finitely generated indecomposable summand $I_{\mathfrak{k}}(\mu)$ such that $L_{\mathfrak{k}}(\mu)$ is an irreducible quotient of $I_{\mathfrak{k}}(\mu)$ (see [MP] Chapter 2, or [RW] §5 and §6). We let $\mathfrak{u}_+ \oplus L_+$ act trivially on $V$, $\mathbb{C}_\mu$, and $I_{\mathfrak{k}}(\mu)$ so that $V \otimes \mathbb{C}_\mu$ and $I_{\mathfrak{k}}(\mu)$ become $\mathfrak{b}_+$-modules. Set

$$P^X(\mu) = U(\hat{\mathfrak{g}}) \bigotimes_{U(\mathfrak{b}_+)} (V \bigotimes \mathbb{C}_\mu), \qquad I^X(\mu) = U(\hat{\mathfrak{g}}) \bigotimes_{U(\mathfrak{b}_+)} I_{\mathfrak{k}}(\mu).$$

A module $M$ in $\mathcal{O}_\lambda^X$ is said to have a *Verma composition series* if there is a chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M \tag{3.2}$$

such that $M_{i+1}/M_i \cong M(\lambda_i)$ for some $\lambda_i \in \lambda \downarrow Q_+^X$. If $M$ has a Verma composition series then we let $[M : M(\lambda)]$ denote the number of occurences of $M(\lambda)$ as a subquotient in the composition series above. If $\eta \in \mathfrak{H}^*$ and $M$ has a local composition series (defined as in [K] or [MP]) at $\eta$ then we let $[M : L^X(\mu)]$ denote the number of occurences of $L^X(\mu)$ in this series for $\mu \geq \eta$. We will see below that these numbers are independent of the series defining them.

If $\hat{\mathfrak{k}}$ (modulo the central ideal $\mathfrak{h}^X$) is an affine Kac-Moody algebra then we will let $W_{\hat{\mathfrak{k}}}$ denote the Weyl group for $\hat{\mathfrak{k}}$. Let $l$ denote the length function on $W_{\hat{\mathfrak{k}}}$ and $s_\beta$ the reflection with respect to a real root $\beta \in \Delta_+(\hat{\mathfrak{m}})$. Let $W_{\hat{\mathfrak{k}}}^{(j)}$ denote the elements in $W_{\hat{\mathfrak{k}}}$ of length $j$. We write $w \leftarrow w'$ if $w = s_\beta w'$ and $l(w) = l(w') + 1$. The usual *Bruhat order* on $W_{\hat{\mathfrak{k}}}$ is given by $w \leq w'$ if $w = w'$ or if there exists $w_1, \ldots, w_r \in W_{\hat{\mathfrak{k}}}$ such that

$$w = w_1 \overset{\gamma_1}{\leftarrow} w_2 \overset{\gamma_2}{\leftarrow} \cdots \overset{\gamma_{r-1}}{\leftarrow} w_r = w'$$

for some real roots $\gamma_i \in \Delta_+(\hat{\mathfrak{m}})$. We also define the dot action of $W_{\hat{\mathfrak{k}}}$ on $\mathfrak{H}^*$ by

$$w \cdot \mu = w(\mu + \rho_{\hat{\mathfrak{k}}}) - \rho_{\hat{\mathfrak{k}}}$$

where $\rho_{\hat{\mathfrak{k}}} \in \mathfrak{H}^*$ is any fixed element such that $\rho_{\hat{\mathfrak{k}}}(\check{\beta}) = 1$ for all $\beta \in \Pi_{\hat{\mathfrak{m}}}$. In addition let $P^+(\hat{\mathfrak{k}}) = \{\lambda \in \mathfrak{H}^* | \, \lambda(\check{\alpha}) \geq 0 \text{ for all } \alpha \in \Delta_+(\hat{\mathfrak{m}}_+)\}$ be the positive root lattice for $\hat{\mathfrak{k}}$.

**3.3 Proposition.** *Suppose $\lambda \in \mathfrak{H}^*$ with $\lambda(c) \neq 0$.*

(i) *For $\mu \in \lambda \downarrow Q_+(\hat{\mathfrak{m}})$ the modules $P^X(\mu)$ and $I^X(\mu)$ are projective modules in $\mathcal{O}_\lambda^X$. Moreover $I^X(\mu)$ is indecomposable and $L^X(\mu)$ is the unique subquotient of $I^X(\mu)$.*

(ii) *$P^X(\mu)$ has a Verma module composition series and for all $\nu \in \lambda \downarrow Q_+^X$*

$$[P^X(\mu) : M^X(\nu)] = \begin{cases} \dim V_{\lambda-\mu} = \dim \operatorname{Hom}_{\hat{\mathfrak{g}}}(P^X(\mu), M^X(\nu)) & \text{if } \nu \leq_{\hat{\mathfrak{m}}} \mu \leq_{\hat{\mathfrak{m}}} \lambda \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *(BGG duality). Let $M$ be an object in $\mathcal{O}_\lambda^X$ and $\mu \in \lambda \downarrow Q_+(\hat{\mathfrak{m}})$ then*

$$[M : L^X(\mu)] = \dim \operatorname{Hom}_{\hat{\mathfrak{g}}}(I^X(\mu), M), \quad \text{and} \quad P^X(\mu) = \bigoplus m_\mu(\nu) I^X(\nu)$$

*where $m_\mu(\nu) = \dim \operatorname{Hom}_{\hat{\mathfrak{g}}}(P^X(\mu), L^X(\nu))$. Moreover*

$$[M^X(\mu) : L^X(\nu)] = [I^X(\nu) : M^X(\mu)]$$

(iv) *Suppose $\hat{\mathfrak{k}}$ is an affine Kac-Moody algebra. Let $\mu \in P^+(\hat{\mathfrak{k}})$, $w, w' \in W_{\hat{\mathfrak{k}}}$. Then*

$$\begin{aligned} \dim \operatorname{Hom}_{\hat{\mathfrak{g}}}(M^X(w \cdot \mu), M^X(w' \cdot \mu)) \leq 1 \Leftrightarrow \quad & w' \leq w \\ & \Leftrightarrow (M^X(w \cdot \mu) : L^X(w' \cdot \mu)) \neq 0. \end{aligned}$$

(v) *(Strong BGG resolution). Suppose $\hat{\mathfrak{k}}$ (modulo the central ideal $\mathfrak{h}^X$) is an affine Kac-Moody algebra. If $w \leq w'$ and $\mu \in P^+(\hat{\mathfrak{k}}) \cap \lambda \downarrow Q_+^X$ then by (iv) we can fix inclusions*

$$i_{w,w'} : M^X(w \cdot \mu) \to M^X(w' \cdot \mu).$$

*Set $C_j = \bigoplus_{w \in W_{\hat{\mathfrak{k}}}^{(j)}} M^X(w \cdot \mu)$. Note that $C_0 = M^X(\mu)$ so that there exists a canonical projection $d_0 : M^X(\mu) \to L^X(\mu)$. For $(w_1, w_2) \in W^{(j)} \times W^{(j-1)}$ there exists $c(w_1, w_2) \in \{-1, 1\}$ such if we define*

$$b^j_{w_1,w_2} = \begin{cases} c(w_1, w_2) & \text{if } w_1 \leftarrow w_2 \\ 0 & \text{otherwise} \end{cases}$$

*and if $d_j : C_j \to C_{j-1}$ is given by $d_j = \oplus b^j_{w_1,w_2} i_{w_1,w_2}$ then the sequence*

$$\cdots \to C_j \xrightarrow{d_j} C_{j-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} L^X(\mu) \longrightarrow 0$$

*is exact (see [RW] 9.6 for a detailed description of $c(w_1, w_2)$).*

*Proof.* The proof is a consequence of the equivalence of categories 2.3, Remark 1.4 together with the following results from [MP] and [RW]: (i) from §3 [RW] or [MP] Chapter 2, (ii) and (iii) from §5 and §6 of [RW] or [MP] Chapter 2, (iv) from [RW] Theorem 8.15, and (v) from [RW] Theorem 9.7. □

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