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Intermediate Wakimoto modules for affine $\mathfrak{sl}(n+1,\mathbb{C})$

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Abstract

We construct certain boson-type realizations of affine $\mathfrak{sl}(n+1,\mathbb{C})$ that depend on a parameter $0 \leqslant r \leqslant n$ such that when $r = 0$ we get a Fock space realization appearing in [6] and when $r = n$ they are the Wakimoto modules described in the work of Feigin and Frenkel [7].

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1. Introduction

Wakimoto modules for affine Lie algebras were introduced by Feĭgin and Frenkel in [7] by a homological characterization. These modules admit a remarkable boson realization on a Fock space [17] (for $\hat{\mathfrak{sl}}(2)$), [9] (for $\hat{\mathfrak{sl}}(n)$) which plays an important role in the conformal field theory providing a new bosonization rule for the Wess–Zumino–Witten models. Wakimoto modules have a geometric interpretation as certain sheaves on a semi-infinite flag manifold [8]. They belong to the category $\mathcal{O}$ and generically are isomorphic to corresponding Verma modules.

Affine Lie algebras admit Verma-type modules associated with non-standard Borel subalgebras, see [5, 11, 13]. Modules associated with the *natural Borel subalgebra* were first introduced by Jakobsen and Kac in [13]. They were studied in [10] under the name of *imaginary Verma modules*.

A Fock space realization of the imaginary Verma modules for $\hat{\mathfrak{sl}}(2)$ was constructed by Bernard and Felder in [1] and then extended in [6] to the case of $\hat{\mathfrak{sl}}(n)$. These realizations are given generically by certain Wakimoto-type modules.

The main motivation for our work was a problem of finding suitable boson-type realizations for all Verma-type modules over $\hat{\mathfrak{sl}}(n+1)$. In theorem 3.1 we construct such realizations, *intermediate Wakimoto modules*, for a series of generic Verma-type modules depending on the parameter $0 \leqslant r \leqslant n$. If $r = n$ this construction coincides with the boson realization of Wakimoto modules in [7]. On the other hand, when $r = 0$ the obtained representation gives a Fock space realization described in [6]. One difficulty that arises in the study of Verma-type modules that are not induced from a standard Borel subalgebra is that certain of their weight spaces are infinite dimensional. On the other hand, the structure of representations that have infinite-dimensional weight spaces is an important problem that appears naturally in other contexts. Besides appearing in the representation theory of infinite-dimensional Heisenberg Lie algebras such representations also arise in the work of [3, 4]. Intermediate Wakimoto modules are another family of representations with certain weight spaces being infinite dimensional. We plan to discuss their detailed structure in a subsequent paper using the construction given in this paper. See the concluding remarks for the potential usefulness of our result.

2. Preliminaries

Fix a positive integer n, $0 \leqslant r \leqslant n$, $\gamma \in \mathbb{C}^*$. Set $k = \gamma^2 - (r+1)$. Let $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ and let E^{ij}, $i, j = 1, \dots, n+1$, be the standard basis for $\mathfrak{gl}(n+1, \mathbb{C})$. Set $H_i := E^{ii} - E^{i+1,i+1}$, $E_i := E^{i,i+1}$, $F_i := E^{i+1,i}$ which is a basis for $\mathfrak{sl}(n+1, \mathbb{C})$. Furthermore we denote the Killing form by $(X|Y) = \mathrm{tr}(XY)$ and $X_m = t^m \otimes X$ for $X, Y \in \mathfrak{g}$ and $m \in \mathbb{Z}$. Let $\{\alpha_1, \dots, \alpha_n\}$ be a base for Δ^+, the positive set of roots for $\mathfrak{g}$, such that $H_i = \check{\alpha}_i$ and let Δ_r be the root system with base $\{\alpha_1, \dots, \alpha_r\}$ ($\Delta_r = \emptyset$, if $r = 0$) of the Lie subalgebra $\mathfrak{g}_r = \mathfrak{sl}(r+1, \mathbb{C})$. A Cartan subalgebra $\mathfrak{H}$ (respectively $\mathfrak{H}_r$) of $\mathfrak{g}$ (respectively $\mathfrak{g}_r$) is spanned by H_i, $i = 1, \dots, n$ (respectively $i = 1, \dots, r$), and set $\mathfrak{H}_0 = 0$.

For any Lie algebra $\mathfrak{a}$, let $L(\mathfrak{a}) = \mathfrak{a} \bigotimes \mathbb{C}[t, t^{-1}]$ be the loop algebra of $\mathfrak{a}$. Then $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}(n+1, \mathbb{C}) = L(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{g}}_r = L(\mathfrak{g}_r) \oplus \mathbb{C}c \oplus \mathbb{C}d$ are the associated affine Kac–Moody algebras with $\hat{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $\hat{\mathfrak{H}}_r = \mathfrak{H}_r \oplus \mathbb{C}c \oplus \mathbb{C}d$ respectively.

The algebra $\hat{\mathfrak{g}}$ has generators E_{im}, F_{im}, H_{im}, $i = 1, \dots, n$, $m \in \mathbb{Z}$, and central element c with the product

$$[X_m, Y_n] = t^{m+n}[X, Y] + \delta_{m+n,0}\, m(X|Y)c.$$

2.1. Oscillator algebras

Let $\hat{\mathfrak{a}}$ be the infinite-dimensional Heisenberg algebra with generators $a_{ij,m}, a^*_{ij,m}$ and $\mathbf{1}$, $1 \leqslant i \leqslant j \leqslant n$ and $m \in \mathbb{Z}$, subject to the relations

$$[a_{ij,m}, a_{kl,n}] = [a^*_{ij,m}, a^*_{kl,n}] = 0 \qquad [a_{ij,m}, a^*_{kl,n}] = \delta_{ik}\delta_{jl}\delta_{m+n,0}\mathbf{1}$$
$$[a_{ij,m}, \mathbf{1}] = [a^*_{ij,m}, \mathbf{1}] = 0.$$

Such an algebra has a representation $\tilde{\rho} : \hat{\mathfrak{a}} \to \mathfrak{gl}(\mathbb{C}[\mathbf{x}])$ where

$$\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_{ij,m} | i, j, m \in \mathbb{Z}, 1 \leqslant i \leqslant j \leqslant n]$$

denotes the algebra over $\mathbb{C}$ generated by the indeterminates $x_{ij,m}$ and $\tilde{\rho}$ is defined by

$$\tilde{\rho}(a_{ij,m}) := \begin{cases} \partial/\partial x_{ij,m} & \text{if} \quad m \geqslant 0 \text{ and } j \leqslant r \\ x_{ij,m} & \text{otherwise} \end{cases}$$

$$\tilde{\rho}(a^*_{ij,m}) := \begin{cases} x_{ij,-m} & \text{if} \quad m \leqslant 0 \text{ and } j \leqslant r \\ -\partial/\partial x_{ij,-m} & \text{otherwise} \end{cases}$$

and $\tilde{\rho}(\mathbf{1}) = 1$. In this case $\mathbb{C}[\mathbf{x}]$ is an $\hat{\mathfrak{a}}$-module generated by $1 =: |0\rangle$, where

$$a_{ij,m}|0\rangle = 0 \quad m \geqslant 0 \text{ and } j \leqslant r \qquad a^*_{ij,m}|0\rangle = 0 \quad m > 0 \text{ or } j > r.$$

Let $\hat{\mathfrak{a}}_r$ denote the subalgebra generated by $a_{ij,m}$ and $a^*_{ij,m}$ and $\mathbf{1}$, where $1 \leqslant i \leqslant j \leqslant r$ and $m \in \mathbb{Z}$. If $r = 0$, we set $\hat{\mathfrak{a}}_r = 0$.

Let $A_n = ((\alpha_i|\alpha_j))$ be the Cartan matrix for $\mathfrak{sl}(n+1, \mathbb{C})$ and let $\mathfrak{B}$ be the matrix whose entries are

$$\mathfrak{B}_{ij} := (\alpha_i|\alpha_j)\left(\gamma^2 - \delta_{i>r}\delta_{j>r}(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\right)$$

where

$$\delta_{i>r} = \begin{cases} 1 & \text{if} \quad i > r \\ 0 & \text{otherwise.} \end{cases}$$

In other words

$$\mathfrak{B} := \gamma^2 A_n - (r+1)\begin{pmatrix} 0 & 0 \\ 0 & A_{n-r} \end{pmatrix} + rE_{r+1,r+1}.$$

We also have the Heisenberg Lie algebra $\hat{\mathfrak{b}}$ with generators b_{im}, $1 \leqslant i \leqslant n$, $m \in \mathbb{Z}$, $\mathbf{1}$, and relations $[b_{im}, b_{jp}] = m\mathfrak{B}_{ij}\delta_{m+p,0}\mathbf{1}$ and $[b_{im}, \mathbf{1}] = 0$.

For each $1 \leqslant i \leqslant n$ fix $\lambda_i \in \mathbb{C}$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then the algebra $\hat{\mathfrak{b}}$ has a representation $\rho_\lambda : \hat{\mathfrak{b}} \to \mathfrak{g}l(\mathbb{C}[\mathbf{y}])$ where

$$\mathbb{C}[\mathbf{y}] := \mathbb{C}[y_{i,m}|i, m \in \mathbb{N}^*, 1 \leqslant i \leqslant n]$$

and ρ_λ is defined on $\mathbb{C}[\mathbf{y}]$ by

$$\rho_\lambda(b_{i0}) = \lambda_i \qquad \rho_\lambda(b_{i,-m}) = \mathbf{e}_i \cdot \mathbf{y}_m \qquad \rho_\lambda(b_{im}) = m\mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m} \qquad \text{for} \quad m > 0$$

and $\rho_\lambda(\mathbf{1}) = 1$. Here

$$\mathbf{y}_m = (y_{1m}, \ldots, y_{nm}) \qquad \frac{\partial}{\partial \mathbf{y}_m} = \left(\frac{\partial}{\partial y_{1m}}, \ldots, \frac{\partial}{\partial y_{nm}}\right)$$

and $\mathbf{e}_i$ are vectors in $\mathbb{C}^n$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$, where $\cdot$ means the usual dot product.

Note that since $\mathfrak{B}_{ij}$ is symmetric, it is orthogonally diagonalizable (i.e. there exists an orthogonal matrix P such that $P^t\mathfrak{B}P$ is a diagonal matrix), and hence we can find vectors $\mathbf{e}_i$ in $\mathbb{C}^n$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$. In fact for $m > 0$ and $n < 0$ we get

$$\begin{aligned}
[b_{im}, b_{jn}] &= \left[m\mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m}, \mathbf{e}_j \cdot \mathbf{y}_{-n}\right] \\
&= m\sum_{k,l}\left[e_{ik}\frac{\partial}{\partial y_{km}}, e_{jl}y_{l,-n}\right] \\
&= m\delta_{m+n,0}\sum_k e_{ik}e_{jk} = m\delta_{m+n,0}\mathfrak{B}_{ij}.
\end{aligned}$$

(See also [9].)

2.2. Formal distributions

We need some more notation that will simplify some of the arguments later. This notation follows roughly [14] and [15]: a *formal distribution* is an expression of the form

$$a(z, w, \ldots) = \sum_{m,n,\ldots\in\mathbb{Z}} a_{m,n,\ldots} z^m w^n$$

where the $a_{m,n,\ldots}$ lie in some fixed vector space V and $z, w, \ldots$ are formal variables. We define $\partial a(z) = \partial_z a(z) = \sum_n n a_n z^{n-1}$. We also have expansion about zero: there are two canonical embeddings of vector spaces $\iota_{z,w} : \mathbb{C}(z-w) \to \mathbb{C}[[z, w]]$ and $\iota_{w,z} : \mathbb{C}(z-w) \to \mathbb{C}[[z, w]]$ where $\iota_{z,w}(a(z, w))$ is a formal Laurent series expansion in z^{-1} and $-\iota_{w,z}(a(z, w))$ is a formal Laurent series expansion in z. The *formal delta function* $\delta(z-w)$ is the formal distribution

$$\delta(z-w) = z^{-1}\sum_{n\in\mathbb{Z}}\left(\frac{z}{w}\right)^n = \iota_{z,w}\left(\frac{1}{z-w}\right) - \iota_{w,z}\left(\frac{1}{z-w}\right).$$

For any sequence of elements $\{a_m\}_{m\in\mathbb{Z}}$ in the ring $\mathrm{End}(V)$, V being a vector space, the formal distribution

$$a(z) := \sum_{m\in\mathbb{Z}} a_m z^{-m-1}$$

is called a *field*, if for any $v \in V$, $a_m v = 0$ for $m \gg 0$. If $a(z)$ is a field, then we set

$$a(z)_- := \sum_{m\geqslant 0} a_m z^{-m-1} \qquad \text{and} \qquad a(z)_+ := \sum_{m<0} a_m z^{-m-1}.$$

In particular

$$\delta(z-w)_- = \iota_{z,w}\left(\frac{1}{z-w}\right) \qquad \delta(z-w)_+ = -\iota_{w,z}\left(\frac{1}{z-w}\right).$$

Note that

$$-\partial_z\delta(z-w) = \partial_w\delta(z-w) = \iota_{z,w}\left(\frac{1}{(z-w)^2}\right) - \iota_{w,z}\left(\frac{1}{(z-w)^2}\right).$$

The *normal ordered product* of two distributions $a(z)$ and $b(w)$ (and their coefficients) is defined by

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} : a_m b_n : z^{-m-1} w^{-n-1} =: a(z)b(w) := a(z)_+ b(w) + b(w)a(z)_-. \tag{2.1}$$

For any $1 \leqslant i \leqslant j \leqslant n$, we define

$$a^*_{ij}(z) = \sum_{n\in\mathbb{Z}} a^*_{ij,n} z^{-n} \qquad a_{ij}(z) = \sum_{n\in\mathbb{Z}} a_{ij,n} z^{-n-1}$$

and

$$b_i(z) = \sum_{n\in\mathbb{Z}} b_{in} z^{-n-1}.$$

In this case

$$[b_i(z), b_j(w)] = \mathfrak{B}_{ij}\partial_w\delta(z-w) \qquad [a_{ij}(z), a^*_{kl}(w)] = \delta_{ik}\delta_{jl}\mathbf{1}\delta(z-w).$$

Observe that $a_{ij}(z)$ for $j > r$ is not a field whereas $a^*_{ij}(z)$ is always a field. We will call $a_{ij}(z)$ (resp. $a^*_{ij}(z)$) a *pure creation* (resp. *annihilation*) *operator* if $j > r$. Set

$$a_{ij}(z)_+ = a_{ij}(z) \qquad a_{ij}(z)_- = 0 \qquad a^*_{ij}(z)_+ = 0 \qquad a^*_{ij}(z)_- = a^*_{ij}(z)$$

if $j > r$.

Now we should point out that while : $a^1(z_1)\cdots a^m(z_m)$: is always defined as a formal series, we will only define : $a(z)b(z)$ ::= $\lim_{w\to z}$: $a(z)b(w)$: for certain pairs $(a(z), b(w))$. For example

$$: a_{ij}(z)a^*_{kl}(z) := \sum_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}} : a_{ij,n}a^*_{kl,m-n} :\right) z^{-m-1}$$

is well defined as an element in $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])[[z,z^{-1}]]$ for all $l > r$ (as $\tilde{\rho}(a^*_{kl,m}) := -\partial/\partial x_{kl,-m}$ for $l > r$) or if both $l \leqslant r$ and $j \leqslant r$ (see also the remarks after theorem 3.1).

Then one defines recursively

$$: a^1(z_1)\cdots a^k(z_k) :=: a^1(z_1)(: a^2(z_2)(: \cdots : a^{k-1}(z_{k-1})a^k(z_k) :)\cdots :) :$$

while the normal ordered product

$$: a^1(z)\cdots a^k(z) := \lim_{z_1,z_2,\ldots,z_k\to z} : a^1(z_1)(: a^2(z_2)(: \cdots : a^{k-1}(z_{k-1})a^k(z_k) :)\cdots) :$$

will only be defined for certain k-tuples $(a^1,\ldots,a^k)$.

Let

$$\lfloor ab\rfloor = a(z)b(w) - : a(z)b(w) := [a(z)_-, b(w)] \tag{2.2}$$

(half of $[a(z), b(w)]$) denote the *contraction* of any two formal distributions $a(z)$ and $b(w)$ where $a(z), b(z)$ are free fields or pure creation or annihilation operators. For example if $j, l \leqslant r$, then

$$\lfloor a_{ij}a^*_{kl}\rfloor = \sum_{m\geqslant 0}\delta_{ik}\delta_{jl}z^{-m-1}w^m = \delta_{i,k}\delta_{j,l}\delta_-(z-w) = \delta_{ik}\delta_{jl}\iota_{z,w}\left(\frac{1}{z-w}\right) \tag{2.3}$$

$$\lfloor a^*_{kl}a_{ij}\rfloor = -\sum_{n<0}\delta_{ik}\delta_{jl}z^n w^{-n-1} = -\delta_{ik}\delta_{jl}\delta_+(w-z) = \delta_{ik}\delta_{jl}\iota_{z,w}\left(\frac{1}{w-z}\right). \tag{2.4}$$

If $l > r$, then

$$\lfloor a_{ij}a^*_{kl}\rfloor = [a_{ij}(z)_-, a^*_{kl}(w)] = 0 \tag{2.5}$$

$$\lfloor a^*_{kl}a_{ij}\rfloor = [a^*_{kl}(z)_-, a_{ij}(w)] = -\delta_{ik}\delta_{jl}\delta(w-z). \tag{2.6}$$

Theorem 2.1 (Wick's theorem, [2], [12] or [14])**.** *Let $a^i(z)$ and $b^j(z)$ be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$*, satisfying*

(1) $[\lfloor a^i(z)b^j(w)\rfloor, c^k(x)_\pm] = [\lfloor a^ib^j\rfloor, c^k(x)_\pm] = 0$*, for all i, j, k and $c^k(x) = a^k(z)$ or $c^k(x) = b^k(w)$.*

(2) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ *for all i and j.*

(3) The products

$$\lfloor a^{i_1}b^{j_1}\rfloor\cdots\lfloor a^{i_s}b^{i_s}\rfloor : a^1(z)\cdots a^M(z)b^1(w)\cdots b^N(w) :_{(i_1,\ldots,i_s;j_1,\ldots,j_s)}$$

have coefficients in $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$ *for all subsets $\{i_1,\ldots,i_s\}\subset\{1,\ldots,M\}$, $\{j_1,\ldots,j_s\}\subset\{1,\ldots,N\}$. Here the subscript $(i_1,\ldots,i_s; j_1,\ldots,j_s)$ means that those factors $a^i(z)$, $b^j(w)$ with indices $i\in\{i_1,\ldots,i_s\}$, $j\in\{j_1,\ldots,j_s\}$ are to be omitted from the product : $a^1\cdots a^Mb^1\cdots b^N$: and when $s = 0$ we do not omit any factors.*

Then

$$: a^1(z)\cdots a^M(z) :: b^1(w)\cdots b^N(w) := \sum_{s=0}^{\min(M,N)} \sum_{i_1<\cdots<i_s,\, j_1\neq\cdots\neq j_s} \times \lfloor a^{i_1}b^{j_1}\rfloor \cdots \lfloor a^{i_s}b^{i_s}\rfloor : a^1(z)\cdots a^M(z)b^1(w)\cdots b^N(w) :_{(i_1,\ldots,i_s;\, j_1,\ldots,j_s)}.$$

Proof. Although it is essentially the same proof as in [12] or [14], we will repeat the argument here for the convenience of the reader.

To simplify notation we will write $a^i = a^i(z_i)$ and $b^j = b^j(w_j)$ below, hoping that it will not cause confusion. Moreover we define $\lfloor a^k b^k\rfloor = 0$ when $k > \min\{M, N\}$.

The conclusion is true for $M = N = 1$ as follows from the definition of the contraction, (2.2). Suppose now $N > 1$ and $M = 1$. Then by hypotheses 1 and 2, and induction

$$\begin{aligned}
a : b^1\cdots b^N &:= ab^1_+ : b^2\cdots b^N : + a : b^2\cdots b^N : b^1_- \\
&= {}_+b^1_+ : b^2\cdots b^N : + \lfloor ab^1\rfloor : b^2\cdots b^N : + b^1_+ a_- : b^2\cdots b^N : + a : b^2\cdots b^N : b^1_- \\
&= a_+b^1_+ : b^2\cdots b^N : + \lfloor ab^1\rfloor : b^2\cdots b^N : + \sum_{j=2}^N \lfloor ab^j\rfloor b^1_+ : b^2\cdots \widehat{b^j}\cdots b^N : \\
&\quad + b^1_+ : b^2\cdots b^N : a_- + a_+ : b^2\cdots b^N : b^1_- \\
&\quad + \sum_{j=2}^N \lfloor ab^j\rfloor : b^2\cdots \widehat{b^j}\cdots b^N : b^1_- + : b^2\cdots b^N : b^1_- a_- \\
&= a_+ : b^1\cdots b^N : + \lfloor ab^1\rfloor : b^2\cdots b^N : \\
&\quad + \sum_{j=2}^N \lfloor ab^j\rfloor : b^1\cdots \widehat{b^j}\cdots b^N : + : b^1\cdots b^N : a_- \\
&= \; : ab^1\cdots b^N : + \sum_{j=1}^N \lfloor ab^j\rfloor : b^1\cdots \widehat{b^j}\cdots b^N : .
\end{aligned}$$

The hat above a factor means omit that factor. Note that at each step we are combining summations where all of the coefficients in each summand are in $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$ when $z_1, \ldots, z_M$ get replaced by z.

A similar argument proves the result for $M > 1$ and $N = 1$.

Now let us assume that M and N are greater than 1. Then using hypotheses 1, 2 and induction we get

$$\begin{aligned}
&\sum_{s\geqslant 0} \sum_{\substack{1\leqslant i_1<\cdots<i_s,\\ j_1\neq\cdots\neq j_s}} \lfloor a^{i_1}b^{j_1}\rfloor \cdots \lfloor a^{i_s}b^{i_s}\rfloor : a^1\cdots a^M b^1\cdots b^N :_{(i_1,\ldots,i_s;\, j_1,\ldots,j_s)} \\
&\quad = a^1_+ \sum_{s\geqslant 0} \sum_{\substack{2\leqslant i_1<\cdots<i_s\\ j_1\neq\cdots\neq j_s}} \lfloor a^{i_1}b^{j_1}\rfloor \cdots \lfloor a^{i_s}b^{i_s}\rfloor : a^2\cdots a^M b^1\cdots b^N :_{(i_1,\ldots,i_s;\, j_1,\ldots,j_s)} \\
&\qquad + \sum_{s\geqslant 0} \sum_{\substack{2\leqslant i_1<\cdots<i_s\\ j_1\neq\cdots\neq j_s}} \lfloor a^{i_1}b^{j_1}\rfloor \cdots \lfloor a^{i_s}b^{i_s}\rfloor : a^2\cdots a^M b^1\cdots b^N :_{(i_1,\ldots,i_s;\, j_1,\ldots,j_s)} a^1_- \\
&\qquad + \sum_{j=1}^N \lfloor a^1 b^j\rfloor : a^2\cdots a^M :: b^1\cdots \widehat{b^j}\cdots b^N :
\end{aligned}$$

$$= a^1_+ \sum_{s\geqslant 0} \sum_{\substack{2\leqslant i_1<\cdots<i_s \\ j_1\neq\cdots\neq j_s}} \lfloor a^{i_1} b^{j_1} \rfloor \cdots \lfloor a^{i_s} b^{i_s} \rfloor : a^2 \cdots a^M b^1 \cdots b^N :_{(i_1,\ldots,i_s;\, j_1,\ldots,j_s)}$$

$$+ \; : a^2 \cdots a^M :: a^1_- b^1 \cdots b^N : + \sum_{j=1}^{N} \lfloor a^1 b^j \rfloor : a^2 \cdots a^M :: b^1 \cdots \widehat{b^j} \cdots b^N :$$

$$= a^1_+ : a^2 \cdots a^M :: b^1 \cdots b^N : + : a^2 \cdots a^M : a^1_- : b^1 \cdots b^N :$$

$$= \; : a^1 \cdots a^M :: b^1 \cdots b^N : .$$

Since every step involves combining sums that are (by induction assumed to be) well defined when $z_1, \ldots, z_M$ and $w_1, \ldots w_N$ are replaced by z and w respectively, so is the resulting product. This completes the proof of the theorem. □

We will also need the following two results.

Theorem 2.2 (Taylor's theorem, [14], 2.4.3)**.** *Let $a(z)$ be a formal distribution. Then in the region $|z - w| < |w|$*

$$a(z) = \sum_{j=0}^{\infty} \partial_w^{(j)} a(w)(z-w)^j. \tag{2.7}$$

Theorem 2.3 ([14], theorem 2.3.2)**.** *Let $a(z)$ and $b(z)$ be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$. *The following are equivalent:*

(i) $[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z-w) c^j(w)$, *where* $c^j(w) \in \mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])[[w, w^{-1}]]$.

(ii) $\lfloor ab \rfloor = \sum_{j=0}^{N-1} \iota_{z,w} \left(\frac{1}{(z-w)^{j+1}} \right) c^j(w)$.

In other words the singular part of the *operator product expansion*

$$\lfloor ab \rfloor = \sum_{j=0}^{N-1} \iota_{z,w} \left(\frac{1}{(z-w)^{j+1}} \right) c^j(w)$$

completely determines the bracket of mutually local formal distributions $a(z)$ and $b(w)$. One writes

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}.$$

For example

$$b_i(z) b_j(w) \sim \frac{\delta_{ij}}{(z-w)^2}.$$

2.3. Verma type modules

For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$.

Let $\mathfrak{g}_\alpha$ be a root subspace of $\mathfrak{g}$ corresponding to a root α, $\mathfrak{n}^\pm = \oplus_{\alpha\in\Delta^+}\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{H} \oplus \mathfrak{n}^+$ a Cartan decomposition of $\mathfrak{g}$. Denote also $\mathfrak{n}_r^\pm = \mathfrak{n}^\pm \cap \mathfrak{g}_r$, $\mathfrak{n}^+(r) = \mathfrak{n}^+ \backslash \mathfrak{n}_r^+$,

$$\bar{B}_r = L(\mathfrak{n}^+(r)) \oplus \left(\mathfrak{n}_r^+ \otimes \mathbb{C}[t]\right) \oplus ((\mathfrak{n}_r^-) \oplus \mathfrak{H}) \otimes \mathbb{C}[t]t).$$

Then $B_r = \bar{B}_r \oplus \hat{\mathfrak{H}}$ is a Borel subalgebra of $\hat{\mathfrak{g}}$ for any $0 \leqslant r \leqslant n$.

Fix $\tilde{\lambda} \in \hat{\mathfrak{H}}^*$ and consider a $\hat{\mathfrak{g}}$-module

$$M_r(\tilde{\lambda}) = U(\hat{\mathfrak{g}}) \otimes_{U(B_r)} \mathbb{C}v_{\tilde{\lambda}}$$

where $\bar{B}_r v_{\tilde{\lambda}} = 0$ and $hv_{\tilde{\lambda}} = \tilde{\lambda}(h)v_{\tilde{\lambda}}$ for all $h \in \hat{\mathfrak{H}}$.

Module $M_r(\tilde{\lambda})$ is a particular case of Verma-type modules studied in [5, 11]. When $r = n$ it gives a usual Verma module construction. If $r = 0$ we get an imaginary Verma module.

Let $\tilde{\lambda}_r = \tilde{\lambda}|_{\hat{\mathfrak{H}}_r}$. Verma-type module $M_r(\tilde{\lambda})$ contains a $\hat{\mathfrak{g}}_r$-submodule $M(\tilde{\lambda}_r) = U(\hat{\mathfrak{g}}_r)(1 \otimes v_{\tilde{\lambda}})$ which is isomorphic to a usual Verma module for $\hat{\mathfrak{g}}_r$.

Theorem 2.4 ([5, 11])**.** *Let $\tilde{\lambda}(c) \neq 0$. Then the submodule structure of $M_r(\tilde{\lambda})$ is completely determined by the submodule structure of $M(\tilde{\lambda}_r)$. In particular, $M_r(\tilde{\lambda})$ is irreducible if $M(\tilde{\lambda}_r)$ is irreducible.*

3. Intermediate Wakimoto modules

Define

$$E_i(z) = \sum_{n\in\mathbb{Z}} E_{in} z^{-n-1} \qquad F_i(z) = \sum_{n\in\mathbb{Z}} F_{in} z^{-n-1}$$
$$H_i(z) = \sum_{n\in\mathbb{Z}} H_{in} z^{-n-1} \qquad 1 \leqslant i \leqslant n.$$

The defining relations between the generators of $\hat{\mathfrak{g}}$ can be written as follows,

(R1) $[H_i(z), H_j(w)] = (\alpha_i|\alpha_j)c\partial_w\delta(w - z)$

(R2) $[H_i(z), E_j(w)] = (\alpha_i|\alpha_j)E_j(z)\delta(w - z)$

(R3) $[H_i(z), F_j(w)] = -(\alpha_i|\alpha_j)F_j(z)\delta(w - z)$

(R4) $[E_i(z), F_j(w)] = \delta_{i,j}(H_i(z)\delta(w - z) + c\partial_w\delta(w - z))$

(R5) $[F_i(z), F_j(w)] = [E_i(z), E_j(w)] = 0 \quad \text{if} \quad (\alpha_i|\alpha_j) \neq -1$

(R6) $[F_i(z_1), F_i(z_2), F_j(w)] = [E_i(z_1), E_i(z_2), E_j(w)] = 0 \quad \text{if} \quad (\alpha_i|\alpha_j) = -1$

where $[X, Y, Z] := [X, [Y, Z]]$ is the Engel bracket for any three operators X, Y, Z.

Recall that $\mathbb{C}[\mathbf{x}]$ is an $\hat{\mathfrak{a}}$-module with respect to the representation $\tilde{\rho}$ and $\mathbb{C}[\mathbf{y}]$ is a $\hat{\mathfrak{b}}$-module with respect to ρ_λ. The main result of the paper is the following theorem where we define a representation

$$\rho : \hat{\mathfrak{g}} \to \mathfrak{gl}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]).$$

We use the notation $\rho(X_m) := \rho(X)_m$, for $X \in \mathfrak{g}$.

Theorem 3.1. *Let $\lambda \in \mathfrak{H}^*$ and set $\lambda_i = \lambda(H_i)$. The generating functions*

$$\rho(F_i)(z) = a_{ii} + \sum_{j=i+1}^{n} a_{ij} a^*_{i+1,j}$$

$$\rho(H_i)(z) = 2 : a_{ii} a^*_{ii} : + \sum_{j=1}^{i-1} (: a_{ji} a^*_{ji} : - : a_{j,i-1} a^*_{j,i-1} :)$$
$$+ \sum_{j=i+1}^{n} (: a_{ij} a^*_{ij} : - : a_{i+1,j} a^*_{i+1,j} :) + b_i$$

$$\rho(E_i)(z) =: a^*_{ii} \left(\sum_{k=1}^{i-1} a_{k,i-1} a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki} a^*_{ki} \right) : + \sum_{k=i+1}^{n} a_{i+1,k} a^*_{ik} - \sum_{k=1}^{i-1} a_{k,i-1} a^*_{ki}$$
$$- a^*_{ii} b_i - (\delta_{i>r}(r+1) + \delta_{i\leqslant r}(i+1) - \gamma^2) \partial a^*_{ii}$$

$$\rho(c) = \gamma^2 - (r+1)$$

define an action of the generators $E_{im}, F_{im}, H_{im}, i = 1, \ldots, n, m \in \mathbb{Z}$ and c, on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$. In the above a_{ij}, a^*_{ij} and b_i denote $a_{ij}(z)$, $a^*_{ij}(z)$ and $b_i(z)$ respectively.

Remark 3.2. One can see that the normal ordered products $: a_{ij}(z) a^*_{ij}(z) :$ are all well defined and thus so is $\rho(H_i)(z)$. Moreover a careful analysis of the other formal distributions shows that they too have coefficients in $\mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$. For example

$$\sum_{k=i+1}^{n} a_{i+1,k} a^*_{ik}$$

has summand $a_{i+1,k} a^*_{ik}$ that is certainly well defined if $k \leqslant r$ and for $k > r$ we have $\tilde{\rho}(a^*_{ik,m}) := -\partial/\partial x_{ik,-m}$ for all $m \in \mathbb{Z}$. Moreover the summation

$$\sum_{k=1}^{i-1} a_{k,i-1} a^*_{ki}$$

is well defined also for $i \leqslant r$ and for $i > r$ we have $\tilde{\rho}(a^*_{ki,m}) := -\partial/\partial x_{ki,-m}$.

Proof. The rather tedious proof is obtained by a routine application of Wick's theorem, Taylor's theorem and theorem 2.3 above. We have left the details in the appendix. □

Theorem 3.1 defines a boson-type realization of $\hat{\mathfrak{sl}}(n+1, \mathbb{C})$ and a module structure on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$ that depends on the parameter $0 \leqslant r \leqslant n$. We will call such a module an *intermediate Wakimoto module* and denote it by $W_{n,r}(\lambda, \gamma)$. The intermediate Wakimoto modules $W_{n,r}(\lambda, \gamma)$ have the property that the subalgebra $\bar{B}_r$ annihilates the vector $1 \otimes 1 \in \mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$, $h(1 \otimes 1) = \lambda(h)(1 \otimes 1)$ for all $h \in \mathfrak{H}$ and $c(1 \otimes 1) = (\gamma^2 - (r+1))(1 \otimes 1)$. Consider the $\hat{\mathfrak{g}}_r$-submodule $W = U(\hat{\mathfrak{g}}_r)(1 \otimes 1) \simeq W_{r,r}(\lambda, \gamma)$ of $W_{n,r}(\lambda, \gamma)$. Then W is isomorphic to the Wakimoto module $W_{\lambda(r),\tilde{\gamma}}$ ([9]) where $\lambda(r) = \lambda|_{\mathfrak{H}_r}$, $\tilde{\gamma} = \gamma^2 - (r+1)$.

Consider $\tilde{\lambda} \in \hat{\mathfrak{H}}^*$ such that $\tilde{\lambda}|_{\mathfrak{H}} = \lambda$, $\tilde{\lambda}(c) = \gamma^2 - (r+1)$, a Verma-type module $M_r(\tilde{\lambda})$ and its $\hat{\mathfrak{g}}_r$-submodule $M(\tilde{\lambda}_r)$. Suppose that $M(\tilde{\lambda}_r)$ is irreducible. In this case the Wakimoto module $W_{\lambda(r),\tilde{\gamma}}$ is isomorphic to $M(\tilde{\lambda}_r)$. Let $\tilde{W} = U(\hat{\mathfrak{g}}) W_{\lambda(r),\tilde{\gamma}}$ and assume that $\lambda(c) \neq 0$. Then by theorem 2.4 the module $M_r(\tilde{\lambda})$ is irreducible and therefore isomorphic to $\tilde{W}$. Hence theorem 3.1 also provides a boson-type realization for generic Verma-type modules.

4. Concluding remarks

We gave a realization of the intermediate Wakimoto module for $\hat{\mathfrak{sl}}(n+1, \mathbb{C})$. We list below related problems and planned future work:

(1) We believe that generically Verma-type modules and intermediate Wakimoto modules are isomorphic (the same way as generic Verma modules are isomorphic to classical Wakimoto modules). Preliminary calculations using (A.3) below give a proof of this for $\hat{\mathfrak{sl}}(n+1, \mathbb{C})$ when $r = n-1$. This should be explored further.

(2) Verma-type modules have a complicated structure when the centre c acts as zero (see for example [10]). The realization above given in theorem 3.1 yields information about the structure of these modules at least in the case of $\hat{\mathfrak{sl}}(2, \mathbb{C})$. These calculations are inspired by work done in [9] when $c = -\check{h}$ is at the singular hyperplane.

(3) A similar realization must exist for all Verma-type modules over $\hat{\mathfrak{sl}}(n+1, \mathbb{C})$ and other affine Lie algebras. It would be of interest to us if one could give a characterization and proof of the existence of intermediate Wakimoto modules using semi-infinite flag manifolds and their cohomology (see [8, 16]).

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Appendix: Proof of theorem 3.1

Set

$$\mathcal{H}_i(z) := 2 : a_{ii}a^*_{ii} : + \sum_{j=1}^{i-1} (: a_{ji}a^*_{ji} : - : a_{j,i-1}a^*_{j,i-1} :) + \sum_{j=i+1}^{n} (: a_{ij}a^*_{ij} : - : a_{i+1,j}a^*_{i+1,j} :).$$

In the above a_{ij} and a^*_{ij} denote $a_{ij}(z)$, $a^*_{ij}(z)$ respectively.

For any $\alpha \in \Delta^+$ we can find unique $1 \leqslant k \leqslant l \leqslant n$ such that

$$\alpha = \alpha_k + \cdots + \alpha_l.$$

Set $a_\alpha := a_{kl}$ and $a^*_\alpha := a^*_{kl}$. Observe that

$$(\alpha_i|\alpha) = \sum_{j=k}^{l} (\alpha_i|\alpha_j) = (2\delta_{ik}\delta_{il} + \delta_{k<i}(\delta_{il} - \delta_{i-1,l}) + \delta_{l>i}(\delta_{ik} - \delta_{i+1,k})).$$

Moreover we have

$$\lfloor a_\alpha a^*_\beta \rfloor = \begin{cases} \delta_{\alpha,\beta}\iota_{z,w}\left(\frac{1}{z-w}\right) & \text{if} \quad \alpha, \beta \in \Delta^+_r \\ 0 & \text{otherwise} \end{cases} \tag{A.1}$$

$$\lfloor a^*_\alpha a_\beta \rfloor = \begin{cases} -\delta_{\alpha,\beta}\iota_{z,w}\left(\frac{1}{z-w}\right) & \text{if} \quad \alpha, \beta \in \Delta^+_r \\ -\delta_{\alpha,\beta}\delta(w-z) & \text{otherwise.} \end{cases} \tag{A.2}$$

Since this is the case we can rewrite

$$\mathcal{H}_i(z) := \sum_{\alpha \in \Delta^+} (\alpha_i|\alpha) : a_\alpha a^*_\alpha : .$$

Lemma 5.1. *For* $1 \leqslant i \leqslant j \leqslant n, \alpha, \beta \in \Delta^+$,

$$[\mathcal{H}_i(z), a_\alpha(w)] = -\delta_{1\leqslant i\leqslant r}\delta_{\alpha\in\Delta_r^+}(\alpha_i|\alpha)a_\alpha(z)\delta(z-w)$$
$$[\mathcal{H}_i(z), a_\alpha^*(w)] = \delta_{1\leqslant i\leqslant r}\delta_{\alpha\in\Delta_r^+}(\alpha_i|\alpha)a_\alpha^*(z)\delta(z-w)$$
$$[\mathcal{H}_i(z), \partial_w a_\alpha^*(w)] = \delta_{1\leqslant i\leqslant r}\delta_{\alpha\in\Delta_r^+}(\alpha_i|\alpha)a_\alpha^*(z)\partial_w(z-w)$$
$$[\mathcal{H}_i(z), \mathcal{H}_j(w)] = -(\alpha_i|\alpha_j)\left((1-\delta_{i>r}\delta_{j>r})(r+1)+\frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\right)\partial_w\delta(z-w)$$
$$\begin{aligned}[\mathcal{H}_i(z), :a_\alpha(w)a_\beta^*(w)a_\gamma^*(w):] &= (\alpha_i|\beta+\gamma-\alpha) :a_\alpha(w)a_\beta^*(w)a_\gamma^*(w): \delta(z-w)\\ &\quad - \delta_{\alpha\in\Delta_r^+}(\alpha_i|\alpha)(\delta_{\alpha,\beta}a_\gamma^*(w)+\delta_{\alpha,\gamma}a_\beta^*(w))\partial_w\delta(z-w)\end{aligned}$$
$$\begin{aligned}\left[\mathcal{H}_{\alpha_i}(z), :a_\alpha(w)a_\beta(w)a_\gamma^*(w):\right] &= (\alpha_i|\gamma-\alpha-\beta) :a_\alpha(w)a_\beta(w)a_\gamma^*(w): \delta(z-w)\\ &\quad - \delta_{\gamma\in\Delta_r^+}(\alpha_i|\gamma)(\delta_{\gamma,\beta}a_\gamma(w)+\delta_{\alpha,\gamma}a_\beta(w))\partial_w\delta(z-w).\end{aligned}$$

Lemma 5.2 (R1)**.**

$$[\rho(H_i)(z), \rho(H_j)(w)] = (\alpha_i|\alpha_j)\rho(c)\partial_w\delta(z-w).$$

Proof. We use lemma 5.1 in the following calculations:

$$\begin{aligned}[\rho(H_i)(z), \rho(H_j)(w)] &= [\mathcal{H}_i(z)+b_i(z), \mathcal{H}_j(z)+b_j(z)]\\ &= \Bigg(-(\alpha_i|\alpha_j)\left((1-\delta_{i>r}\delta_{j>r})(r+1)+\frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}\right)\\ &\quad + (\alpha_i|\alpha_j)\left(\gamma^2-\delta_{i>r}\delta_{j>r}(r+1)+\frac{r}{2}\delta_{i,r+1}\delta_{j,r+1})\right)\Bigg)\partial_w\delta(z-w)\\ &= (\alpha_i|\alpha_j)\rho(c)\partial_w\delta(z-w).\end{aligned}$$
□

Lemma 5.3 (R2)**.**

$$[\rho(H_i)(z), \rho(E_j)(w)] = (\alpha_i|, \alpha_j)\rho(E_j)(z)\delta(z/w).$$

Proof. We will use lemma 5.1 repeatedly:

$$\begin{aligned}&[\mathcal{H}_i(z), \rho(E_j)(w)]\\ &\quad= [\mathcal{H}_i(z), :a_{jj}^*\left(\sum_{k=1}^{j-1}a_{k,j-1}a_{k,j-1}^*-\sum_{k=1}^{j}a_{kj}a_{kj}^*\right): + \sum_{k=j+1}^{n}a_{j+1,k}a_{jk}^*\\ &\qquad - \sum_{k=1}^{j-1}a_{k,j-1}a_{kj}^* - a_{jj}^*b_j - (\delta_{j>r}(r+1)+\delta_{j\leqslant r}(j+1)-\gamma^2)\partial a_{jj}^*]\\ &\quad= \sum_{k=1}^{j-1}((\alpha_i|\alpha_j) :a_{jj}^*a_{k,j-1}a_{k,j-1}^*: \delta(z-w) - \delta_{j-1\leqslant r}(\alpha_i|\alpha_k\\ &\qquad + \cdots + \alpha_{j-1})a_{jj}^*(w)\partial_w\delta(z-w)) - \sum_{k=1}^{j}((\alpha_i|\alpha_j) :a_{jj}^*a_{kj}a_{kj}^*: \delta(z-w)\\ &\qquad - \delta_{j\leqslant r}(\alpha_i|\alpha_k+\cdots+\alpha_j)(\delta_{jk}a_{kk}^*(w)+a_{jj}^*(w))\partial_w\delta(z-w))\\ &\qquad + (\alpha_i|\alpha_j)\sum_{k=j+1}^{n}a_{j+1,k}(z)a_{jk}^*(w)\delta(z-w) - (\alpha_i|\alpha_j)\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{k=1}^{j-1} a_{k,j-1}(z)a^*_{kj}(w)\delta(z-w) - (\alpha_i|\alpha_j)a^*_{jj}(z)b_j(w)\delta(z-w)\\
&- (\alpha_i|\alpha_j)(\delta_{j>r}(r+1) + \delta_{j\leqslant r}(j+1) - \gamma^2)a^*_{jj}(z)\partial_w\delta(z-w)\\
&= (\alpha_i|\alpha_j)\Bigg(: a^*_{jj}\left(\sum_{k=1}^{j-1} a_{k,j-1}a^*_{k,j-1} - \sum_{k=1}^{j} a_{kj}a^*_{kj}\right) :\\
&+ \sum_{k=j+1}^{n} a_{j+1,k}a^*_{jk} - \sum_{k=1}^{j-1} a_{k,j-1}a^*_{kj} - a^*_{jj}b_j\Bigg)\delta(z-w)\\
&+ \sum_{k=1}^{j} \delta_{1\leqslant j\leqslant r}(\alpha_i|\alpha_k + \cdots + \alpha_j)(\delta_{jk}a^*_{kk}(w) + a^*_{jj}(w))\partial_w\delta(z-w)\\
&- \sum_{k=1}^{j-1}(\delta_{1\leqslant j-1\leqslant r}(\alpha_i|\alpha_k + \cdots + \alpha_{j-1})a^*_{jj}(w)\partial_w\delta(z-w))\\
&- (\alpha_i|\alpha_j)(\delta_{j>r}(r+1) + \delta_{j\leqslant r}(j+1) - \gamma^2)a^*_{jj}(z)\partial_w\delta(z-w).
\end{aligned}
$$

The last term of $\rho(H_i)(z)$ gives us

$$[b_i(z), \rho(E_j)(w)] = -\mathfrak{B}_{ij}a_{jj}(w)\partial_w\delta(z-w).$$

Putting these computations together we get

$$[\rho(H_i)(z), \rho(E_j)(w)] = (\alpha_i|\alpha_j)\rho(E_j)(w)\delta(z-w).$$

□

Lemma 5.4 (R3)**.**

$$[\rho(H_i)(z), \rho(F_j)(w)] = -(\alpha_i|\alpha_j)\rho(F_j)(z)\delta(z-w).$$

Proof. The proof follows from lemma 5.1:

$$
\begin{aligned}
&[\rho(H_i)(z), \rho(F_j)(w)]\\
&= [\mathcal{H}_i(z), \rho(F_j)(w)] = [\mathcal{H}_i(z), a_{jj}(w) + \sum_{k=j+1}^{n} a_{jk}(w)a^*_{j+1,k}(w)]\\
&= -(\alpha_i|\alpha_j)a_{jj}(w)\delta(z-w) + \sum_{k=j+1}^{n} [\mathcal{H}_i(z), a_{jk}(w)]a^*_{j+1,k}(w)\\
&+ \sum_{k=j+1}^{n} a_{jk}(w)[\mathcal{H}_i(z), a^*_{j+1,k}(w)]\\
&= \Bigg(-(\alpha_i|\alpha_j)a_{jj}(w) - \sum_{k=j+1}^{n} (\alpha_i|\alpha_j + \cdots + \alpha_k)a_{jk}(z)a^*_{j+1,k}(w)\\
&+ \sum_{k=j+1}^{n} (\alpha_i|\alpha_{j+1} + \cdots + \alpha_l)a_{jk}(w)a^*_{j+1,k}(z)\Bigg)\delta(z-w)\\
&= -(\alpha_i|\alpha_j)\rho(F_j)(z)\delta(z-w).
\end{aligned}
$$

□

Lemma 5.5 (R4)**.**

$$[\rho(E_i)(z), \rho(F_j)(w)] = \delta_{i,j}(\rho(H_i)(z))\delta(z-w) + \rho(c)\partial_w\delta(z-w))$$

Proof. First we take $i = j$. Now for the convenience of the reader we recall that $\rho(E_i)(z)$ is equal to

$$: a_{ii}^*\left(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^* - \sum_{k=1}^{i} a_{ki}a_{ki}^*\right) : + \sum_{k=i+1}^{n} a_{i+1,k}a_{ik}^* - \sum_{k=1}^{i-1} a_{k,i-1}a_{ki}^*$$
$$- a_{ii}^* b_i - (\delta_{i>r}(r+1) + \delta_{i\leqslant r}(i+1) - \gamma^2)\partial a_{ii}^*$$

and thus the first summand of $\rho(F_i)(w) = a_{ii} + \sum_{l=i+1}^{n} a_{il}a_{i+1,l}^*$ brackets with $\rho(E_i)(z)$ to give us

$$\left(2 : a_{ii}(z)a_{ii}^*(z) : - : \sum_{k=1}^{i-1}(a_{k,i-1}a_{k,i-1}^* - a_{ki}a_{ki}^*) : + b_i(z)\right)\delta(z-w)$$
$$+ (\delta_{i>r}(r+1) + \delta_{i\leqslant r}(i+1) - \gamma^2)\partial_z\delta(z-w).$$

The second summation in $\rho(F_i)(w)$ contributes

$$\sum_{l=i+1}^{n}[\rho(E_i)(z), a_{il}(w)a_{i+1,l}^*(w)]$$
$$= \sum_{l=i+1}^{n}\left[\sum_{k=i+1}^{n} a_{i+1,k}(z)a_{ik}^*(z), a_{il}(w)a_{i+1,l}^*(w)\right]$$
$$= \sum_{l=i+1}^{n}(a_{il}(z)a_{il}^*(z) - a_{i+1,l}(z)a_{i+1,l}^*(z))\delta(z-w) - \delta_{i+1\leqslant r}(r-i)\partial_w\delta(z-w).$$

Adding these two summations, we arrive at the desired result.

Now consider the case $|i-j| \geqslant 1$. Then $\rho(F_j)(w)$ acts as $a_{jj} + \sum_{l=j+1}^{n} a_{jl}a_{j+1,l}^*$. First we have

$$[E_i(z), a_{jj}(w)] = \left[: a_{ii}^*\left(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^* - \sum_{k=1}^{i} a_{ki}a_{ki}^*\right) : + \sum_{k=i+1}^{n} a_{i+1,k}a_{ik}^* - \sum_{k=1}^{i-1} a_{k,i-1}a_{ki}^*\right.$$
$$= \left[: a_{ii}^*\left(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^* - \sum_{k=1}^{i} a_{ki}a_{ki}^*\right) :, a_{jj}(w)\right]$$
$$= -\delta_{j,i-1}a_{i-1,i-1}(z)a_{i,i}^*(z)\delta(z-w)$$

by lemma 5.1(d). Next we have

$$\left[E_i(z), \sum_{l=j+1}^{n} a_{jl}(w)a_{j+1,l}^*(w)\right]$$
$$= \left[: a_{ii}^*\left(\sum_{k=1}^{i-1} a_{k,i-1}a_{k,i-1}^* - \sum_{k=1}^{i} a_{ki}a_{ki}^*\right) : + \sum_{k=i+1}^{n} a_{i+1,k}a_{ik}^* - \sum_{k=1}^{i-1} a_{k,i-1}a_{ki}^*\right.$$
$$\left. - a_{ii}^* b_i + (\gamma^2 - \delta_{i+1\leqslant r}(i+1))\partial a_{ii}^*, \sum_{l=j+1}^{n} a_{jl}(w)a_{j+1,l}^*(w)\right]$$

$$= \left[: a^*_{ii} \left(\sum_{k=1}^{i-1} a_{k,i-1} a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki} a^*_{ki} \right) : - \sum_{k=1}^{i-1} a_{k,i-1} a^*_{ki}, \sum_{l=j+1}^{n} a_{jl}(w) a^*_{j+1,l}(w) \right]$$

by lemma 5.1(a)

$$= \delta_{j,i-1} a_{i-1,i-1}(z) a^*_{ii}(z) \delta(z-w) \qquad \text{by lemma 5.1(b) and (c).}$$

Adding the last two calculations finishes the proof of this lemma. □

We are now left with the Serre relations:

Lemma 5.6 (R5/R6)**.**

$$[\rho(F_i)(z), \rho(F_j)(w)] = [\rho(E_i)(z), \rho(E_j)(w)] = 0 \qquad \text{if} \quad (\alpha_i|\alpha_j) \neq -1$$

$$[\rho(F_i)(z_1), \rho(F_i)(z_2), \rho(F_j)(w)] = [\rho(E_i)(z_1), \rho(E_i)(z_2), \rho(E_j)(w)] = 0$$

$$\text{if} \quad (\alpha_i|\alpha_j) = -1.$$

Proof. Let us check the relations for $\rho(F_i)$. (The Serre relations were already known to hold true for the F_i, see [9], but we provide a proof for the convenience of the reader.) A straightforward calculation shows

$$\left[a_{ii}(z), a_{jj} + \sum_{l=j+1}^{n} a_{jl} a^*_{j+1,l} \right] = \delta_{i,j+1} a_{j,j+1}(w) \delta(z-w).$$

Moreover

$$\left[\sum_{k=i+1}^{n} a_{ik}(z) a^*_{i+1,k}(z), a_{jj}(w) + \sum_{l=j+1}^{n} a_{jl}(w) a^*_{j+1,l}(w) \right]$$

$$= \left(\delta_{i,j+1} \sum_{k=j+2}^{n} a_{jk}(z) a^*_{j+2,k}(z) - \delta_{j,i+1} \sum_{k=i+2}^{n} a_{ik}(z) a^*_{i+2,k}(z) - \delta_{j,i+1} a_{i,i+1}(z) \right)$$

$$\times \delta(z-w)$$

by lemma 5.1(a). Thus

$$[\rho(F_i)(z), \rho(F_j)(w)] = (\delta_{i,j+1} a_{j,j+1}(w) - \delta_{j,i+1} a_{i,i+1}(z)) \delta(z-w)$$

$$+ \left(\delta_{i,j+1} \sum_{k=j+2}^{n} a_{jk}(z) a^*_{j+2,k}(z) - \delta_{j,i+1} \sum_{k=i+2}^{n} a_{ik}(z) a^*_{i+2,k}(z) \right) \delta(z-w). \qquad \text{(A.3)}$$

Note the above is zero if $|i-j| \neq 1$ which is precisely when $(\alpha_i|\alpha_j) \neq -1$. As the first index in a_{kl} (resp. a^*_{kl}) in $\rho(F_i)(z)$ is i (resp. $i+1$) we also get

$$[\rho(F_i)(z_1), \rho(F_i)(z_1), \rho(F_j)(w)] = 0.$$

This completes the proof of the relations R5 and R6 for $\rho(F_i)(z)$. The Serre relations, R5 and R6, for $\rho(E_i)(z)$ are quite tedious and can be found at the first author's Web page[3]. □

[3] http://math.cofc.edu/faculty/cox/papers/

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