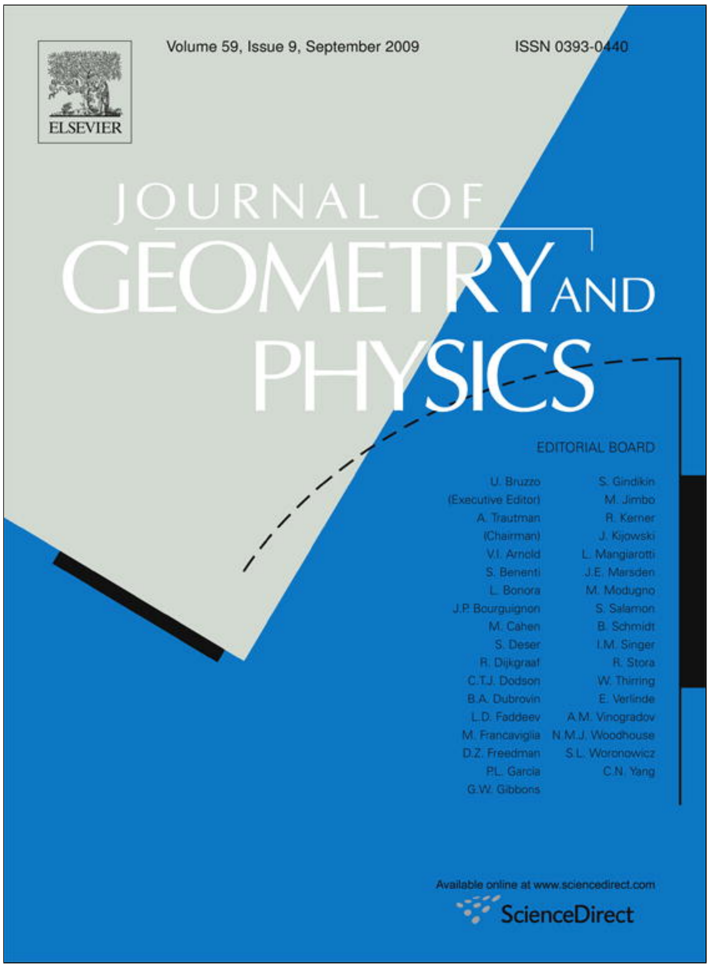
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Free field realizations of the elliptic affine Lie algebra $\mathfrak{sl}(2, R) \oplus (\Omega_R/\mathrm{d}R)$

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ABSTRACT

In this paper we construct two free field realizations of the elliptic affine Lie algebra $\mathfrak{sl}(2, R) \oplus \Omega_R/\mathrm{d}R$ where $R = \mathbb{C}[t, t^{-1}, u \mid u^2 = t^3 - 2bt^2 + t]$. The first realization provides an analogue of Wakimoto's construction for Affine Kac–Moody algebras, but in the setting of the elliptic affine Lie algebra. The second realization gives new types of representations analogous to Imaginary Verma modules in the Affine setting.

1. Introduction

Elliptic affine Lie algebras are a particular family of Krichever–Novikov Lie algebras related to Riemann surfaces [1,2]. These algebras were introduced by Krichever and Novikov in their study of string theory in Minkowski space.

The theory of highest weight modules for these algebras was developed by Sheinman in [3,4], see also [5]. Since Krichever–Novikov algebras are quasi-graded, their representation theory is quite different from the standard representation theory of Kac–Moody algebras. For instance, irreducible highest weight modules can have a two-dimensional subspace of highest weight vectors.

The goal of the present paper is to obtain free field realizations for the elliptic affine Lie algebra associated with $\mathfrak{sl}(2, \mathbb{C})$.

It is known from the work of Kassel and Loday (see [6,7]) that if R is a commutative algebra and $\mathfrak{g}$ is a simple Lie algebra, both defined over the complex numbers, then the universal central extension $\hat{\mathfrak{g}}$ of $L(\mathfrak{g}) = \mathfrak{g} \otimes R$ is the vector space $L(\mathfrak{g}) \oplus \Omega^1_R/\mathrm{d}R$ where $\Omega^1_R/\mathrm{d}R$ is the space of Kähler differentials modulo exact forms (see [7]). The vector space $\hat{\mathfrak{g}}$ is made into a Lie algebra by defining

$$[x \otimes f, y \otimes g] := [xy] \otimes fg + (x, y)\overline{f dg}, \quad [x \otimes f, \omega] = 0$$

for $x, y \in \mathfrak{g}, f, g \in R$, here $\omega \in \Omega^1_R/\mathrm{d}R$ and $(-, -)$ denotes the Killing form on $\mathfrak{g}$ and $\bar{a}$ denotes the image of $a \in \Omega^1_R$ in the quotient $\Omega^1_R/\mathrm{d}R$. A somewhat vague (due to the imprecision in the choice of R and hence imprecision in the description of

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the basis of $\Omega_R/\mathrm{d}R$) but natural question is whether there exist free field or Wakimoto type realizations of these algebras. The answer is well known from the work of M. Wakimoto if R is the ring of Laurent polynomials in one variable (see [8]). We solve this problem in the setting where $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $R = \mathbb{C}[t, t^{-1}, u|u^2 = t^3 - 2bt^2 + t]$ is the elliptic algebra. Related work on realizations of the universal central extension of $\mathfrak{sl}(2, \mathbb{C}) \otimes R$ can be found in the following papers; [9–16,3,8].

Before we begin we would like to mention a little genesis of elliptic affine algebras. In Kazhdan and Lusztig's explicit study of the tensor structure of modules for affine Lie algebras (see [17,18]) the ring of functions regular everywhere except at a finite number of points appears naturally. This algebra M. Bremner gave the name *n-point algebra*. On the other hand an elliptic algebra is an algebra of functions on an elliptic curve of genus one which may have poles at two points. This article deals with the particular example of a fixed nonsingular compact complex algebraic curve of genus 1 which Bremner denotes by Σ in [9]. This curve Σ can be represented as a quotient of the complex plane $\mathbb{C}$ by the lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\lambda$ with positive imaginary part of λ. Let R denote ring of meromorphic functions on Σ which are holomorphic outside of the set $\{0, \frac{1}{2}(1 + \lambda)\}$. Then Bremner has shown that $R \cong S_b := \mathbb{C}[t, t^{-1}, u|u^2 = t^3 - 2bt^2 + t]$ where $b \in \mathbb{C}$ is given below. As the latter, being $\mathbb{Z}_2$-graded, is more immediately amendable to the theatrics of conformal field theory, we choose to work with S_b instead. Moreover, Bremner has given an explicit description of the universal central extension of $\mathfrak{g} \otimes R$, in terms of Pollaczek polynomials. His description is recapitulated in what follows.

Our main result, Theorem 5.1, provides a natural free field realization in terms of a β–γ-system and the oscillator algebra of the elliptic affine Lie algebra when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Just as in the case of intermediate Wakimoto modules defined in [19], there two different realizations depending on two different normal orderings. The first realization is analogous to the construction of Wakimoto modules for Affine Lie algebras while the second one provides new modules our algebra analogous to Imaginary Verma modules [20].

2. The ring of elliptic functions $\mathbb{C}[t, t^{-1}, u|u^2 = t^3 - 2bt^2 + t]$

Fix a nonsingular compact complex algebraic curve of genus 1 which we denote by Σ. This curve Σ can be represented as a quotient of the complex plane $\mathbb{C}$ by the lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\lambda$ with positive imaginary part of λ. Let R denote ring of meromorphic functions on Σ which are holomorphic outside of the set $\{0, \frac{1}{2}(1 + \lambda)\}$.

Set $m = \wp\left(\frac{1}{2}(1 + \lambda)\right)$ where $\wp$ is the Weierstrass $\wp$ function:

$$\wp(z) := z^{-2} + \sum_{0\neq\omega\in\Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Then

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \quad g_2 = 60 \sum_{0\neq\xi\in\Lambda} \xi^{-4}, g_3 = 140 \sum_{0\neq\xi\in\Lambda} \xi^{-6}.$$

Proposition 2.1 (*[9], Prop. 4.1.*). *We have*

$$R \cong \mathbb{C}[t, t^{-1}, u|u^2 = t^3 - 2bt^2 + t]$$

where b is some constant determined by m.

3. The universal central extension of $\mathfrak{g} \otimes R$

We recall the realization of the universal central extension of $L(\mathfrak{g}) = \mathfrak{g} \otimes R$.

Theorem 3.1 (*[9], cf. Theorem 3.4*). *The space $\Omega^1_R/\mathrm{d}R$ has a basis*

$$\omega_0 := \overline{t^{-1}\mathrm{d}t}, \qquad \omega_- := \overline{t^{-2}u\,\mathrm{d}t}, \qquad \omega_+ := \overline{t^{-1}u\,\mathrm{d}t}.$$

There exists an automorphism τ of R given by

$$\tau(t) = t^{-1}, \qquad \tau(u) = t^{-2}u \tag{3.1}$$

which induces an automorphism on $\Omega^1_R/\mathrm{d}R$. This induced automorphism is simply the negative of the identity map.

We will now give Bremner's Fourier mode description of the relations satisfied by the basis elements $x \otimes t^n$, $x \otimes t^n u$, ω_0, $\omega_\pm$ of $\hat{\mathfrak{g}}$. First recall the *Pollaczek polynomials* $P_k(b) = P^\lambda_k(b; \alpha, \beta, \gamma)$, $\alpha, \beta, \gamma \in \mathbb{C}$ (see [21]), which are defined by the recursion

$$(k + \gamma)P_k(b) = 2[(k + \lambda + \alpha + \gamma - 1)b + \beta]P_{k-1}(b) - (k + 2\lambda + \gamma - 2)P_{k-2}(b). \tag{3.2}$$

Lemma 3.2 ([9], Lemma 4.4). *Consider the sequence of polynomials $p_k(b)$, and $q_k(b)$, defined by*

$$\overline{t^{k-2}u\,\mathrm{d}t} = p_k(b)\overline{t^{-1}u\,\mathrm{d}t} + q_k(b)\overline{t^{-2}u\,\mathrm{d}t}.$$

These polynomials are Pollaczek polynomials for the parameters $\lambda = -1/2$, $\alpha = 0$, $\beta = -1$, $\gamma = 1/2$ together with the initial conditions

$$p_0(b) = 0, \qquad p_1(b) = 1, \qquad q_0(b) = 1, \qquad q_1(b) = 0.$$

If we set

$$Q(x, b) := \sum_{n=0}^{\infty} q_n(b)x^n, \qquad P(x, b) := \sum_{n=0}^{\infty} p_n(b)x^n,$$

then it is straightforward to show that

$$x(x^2 - 2bx - 1)\frac{\mathrm{d}Q}{\mathrm{d}x} + \left[(2\lambda + \gamma)x^2 - 2x((\lambda + \alpha + \gamma)b + \beta) + \gamma\right] Q = \gamma,$$

and

$$x(x^2 - 2bx - 1)\frac{\mathrm{d}P}{\mathrm{d}x} + \left[(2\lambda + \gamma)x^2 - 2x((\lambda + \alpha + \gamma)b + \beta) + \gamma\right] P = (1 + \gamma)x.$$

Set $\alpha_\pm := b \pm \sqrt{b^2 - 1}$, $A_\pm := \lambda \mp (\alpha b + \beta)/\sqrt{b^2 - 1}$. Solving these differential equations above we get

$$Q(x, b) = \int_0^x \frac{\gamma\xi^{\gamma-1}(\xi - \alpha_1)^{A_+-1}(\xi - \alpha_2)^{A_--1}}{x^\gamma(x - \alpha_1)^{A_+}(x - \alpha_2)^{A_-}}\,\mathrm{d}\xi$$

and

$$P(x, b) = \int_0^x \frac{(1 + \gamma)\xi^\gamma(\xi - \alpha_1)^{A_+-1}(\xi - \alpha_2)^{A_--1}}{x^\gamma(x - \alpha_1)^{A_+}(x - \alpha_2)^{A_-}}\,\mathrm{d}\xi.$$

Theorem 3.3 ([9], Theorem 4.6). *The elliptic Lie algebra $\hat{\mathfrak{g}}$ has a $\mathbb{Z}/2\mathbb{Z}$-grading where*

$$\hat{\mathfrak{g}}^0 = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\omega_0, \qquad \hat{\mathfrak{g}}^1 = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]u \oplus \mathbb{C}\omega_- \oplus \mathbb{C}\omega_+.$$

For $x, y \in \mathfrak{g}$, the commutation relations defining $\hat{\mathfrak{g}}$ are

$$[x \otimes t^i, y \otimes t^j] = [xy] \otimes t^{i+j} + \delta_{i+j,0}j\omega_0$$

$$[x \otimes t^{i-1}u, y \otimes t^{j-1}u] = [xy] \otimes \left(t^{i+j-1} - 2bt^{i+j} + t^{i+j+1}\right) + (x, y)\omega_0 \left(-2jb\delta_{i+j,0} + \frac{1}{2}(j - i)(\delta_{i+j,-1} + \delta_{i+j,1})\right)$$

$$[x \otimes t^{i-1}u, y \otimes t^j] = [xy] \otimes t^{i+j-1}u + (x, y)j\Big(p_{|i+j|}(b)\omega_+ + q_{|i+j|}(b)\omega_-\Big),$$

for $i, j \in \mathbb{Z}$. In addition the elements ω_0, $\omega_\pm$ are central.

3.1. Formal distributions

We introduce some notation that will simplify later arguments. This notation follows roughly [22,23]: The *formal delta function* $\delta(z/w)$ is the formal distribution

$$\delta(z/w) = z^{-1} \sum_{n\in\mathbb{Z}} \left(\frac{z}{w}\right)^n.$$

Given a vector space V, for any sequence of elements $\{a_m\}_{m\in\mathbb{Z}}$ in the ring $\mathrm{End}(V)$, the formal distribution

$$a(z) := \sum_{m\in\mathbb{Z}} a_m z^{-m-1}$$

is called a *field*, if for any $v \in V$, $a_m v = 0$ for $m \gg 0$. If $a(z)$ is a field, then we set

$$a(z)_- := \sum_{m\geq 0} a_m z^{-m-1}, \quad \text{and} \quad a(z)_+ := \sum_{m<0} a_m z^{-m-1}. \tag{3.3}$$

The *normal ordered product* of two distributions $a(z)$ and $b(w)$ (and their coefficients) is defined by

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} :a_m b_n: z^{-m-1}w^{-n-1} =: {:}a(z)b(w){:} := a(z)_+ b(w) + b(w)a(z)_-. \tag{3.4}$$

Note that while $: a^1(z_1)\cdots a^m(z_m) :$ is always defined as a formal series, we will only define $: a(z)b(z) := \lim_{w\to z} : a(z)b(w) :$ for certain pairs $(a(z), b(w))$.

Then one defines recursively

$$: a^1(z_1)\cdots a^k(z_k) :=: a^1(z_1)\left(: a^2(z_2)\left(: \cdots : a^{k-1}(z_{k-1})a^k(z_k) :\right)\cdots :\right) :,$$

while normal ordered product

$$: a^1(z)\cdots a^k(z) := \lim_{z_1,z_2,\ldots,z_k\to z} : a^1(z_1)\left(: a^2(z_2)\left(: \cdots : a^{k-1}(z_{k-1})a^k(z_k) :\right)\cdots\right) :$$

will only be defined for certain k-tuples $(a^1,\ldots,a^k)$.

Let

$$\lfloor a(z)b(w)\rfloor = \lfloor ab\rfloor = a(z)b(w) - : a(z)b(w) := [a(z)_-, b(w)], \tag{3.5}$$

where $\lfloor a(z)b(w)\rfloor$ denotes the *contraction* of any two formal distributions $a(z)$ and $b(w)$.

Theorem 3.4 (*Wick's Theorem, See [24,25] or [22]*). *Let $a^i(z)$ and $b^j(z)$ be formal distributions with coefficients in the associative algebra* $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$, *satisfying*

(1) $[\lfloor a^i(z)b^j(w)\rfloor, c^k(x)_\pm] = [\lfloor a^ib^j\rfloor, c^k(x)_\pm] = 0$, *for all i, j, k and $c^k(x) = a^k(z)$ or $c^k(x) = b^k(w)$.*
(2) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ *for all i and j.*
(3) *The products*

$$\lfloor a^{i_1}b^{j_1}\rfloor\cdots\lfloor a^{i_s}b^{i_s}\rfloor : a^1(z)\cdots a^M(z)b^1(w)\cdots b^N(w) :_{(i_1,\ldots,i_s;j_1,\ldots,j_s)}$$

have coefficients in $\mathrm{End}(\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])$ *for all subsets $\{i_1,\ldots,i_s\}\subset\{1,\ldots,M\}$, $\{j_1,\ldots,j_s\}\subset\{1,\ldots,N\}$. Here the subscript $(i_1,\ldots,i_s;j_1,\ldots,j_s)$ means that those factors $a^i(z)$, $b^j(w)$ with indices $i\in\{i_1,\ldots,i_s\}$, $j\in\{j_1,\ldots,j_s\}$ are to be omitted from the product $: a^1\cdots a^Mb^1\cdots b^N :$ and when $s=0$ we do not omit any factors.*

Then

$$\begin{aligned} : a^1(z)\cdots a^M(z) :: b^1(w)\cdots b^N(w) : &= \sum_{s=0}^{\min(M,N)} \sum_{i_1<\cdots<i_s, j_1\neq\cdots\neq j_s} \lfloor a^{i_1}b^{j_1}\rfloor\cdots\lfloor a^{i_s}b^{i_s}\rfloor : a^1(z)\cdots a^M(z) \\ &\quad\times b^1(w)\cdots b^N(w) :_{(i_1,\ldots,i_s;j_1,\ldots,j_s)}. \end{aligned}$$

For $m = i-\frac{1}{2}$, $i\in\mathbb{Z}+\frac{1}{2}$ and $x\in\mathfrak{g}$, define $x_{m+\frac{1}{2}} := x\otimes t^{i-\frac{1}{2}}u$ and $x_m := x\otimes t^m$. We set

$$x^1(z) := \sum_{m\in\mathbb{Z}} x_{m+\frac{1}{2}} z^{-m-1}, \qquad x(z) := \sum_{m\in\mathbb{Z}} x_m z^{-m-1}.$$

Then the relations in Theorem 3.3 can be rewritten as

$$\begin{aligned} &[x(z), y(w)] = [xy](w)\delta(z/w) - (x,y)\omega_0\partial_w\delta(z/w) \\ &[x^1(z), y^1(w)] = w(w^2-2bw+1)\left([x,y](w)\delta(z/w) - (x,y)\omega_0\partial_w\delta(z/w)\right) \\ &\qquad - (x,y)\frac{\omega_0}{2}\partial_w(w(w^2-2bw+1))\delta(z/w) \\ &[x^1(z), y(w)] = [x,y]^1(w)\delta(z/w) - (x,y)\left((P(w^{-1},b)+P(w,b))\omega_+ + (Q(w^{-1},b)+Q(w,b)-2)\omega_-\right)\delta(z/w) \\ &\qquad - (x,y)w\partial_w\left(\left((P(w^{-1},b)+P(w,b))\omega_+ + (Q(w^{-1},b)+Q(w,b)-2)\omega_-\right)\delta(z/w)\right) \\ &[x(z), y^1(w)] = [x,y]^1(w)\delta(w/z) - (x,y)\left((P(w^{-1},b)+P(w,b))\omega_+ + (Q(w^{-1},b)+Q(w,b)-2)\omega_-\right)w\partial_w\delta(z/w). \end{aligned}$$

4. Oscillator algebras

4.1. The β–γ system

Let $\hat{\mathfrak{a}}$ be the infinite-dimensional oscillator algebra with generators $a_n, a_n^*, a_n^1, a_n^{1*}$, $n\in\mathbb{Z}$ together with $\mathbf{1}$ satisfying the relations

$$\begin{aligned} &[a_n, a_m] = [a_m, a_n^1] = [a_m, a_n^{1*}] = [a_n^*, a_m^*] = [a_n^*, a_m^1] = [a_n^*, a_m^{1*}] = 0, \\ &[a_n^1, a_m^1] = [a_n^{1*}, a_m^{1*}] = 0 = [\mathfrak{a}, \mathbf{1}], \\ &[a_n, a_m^*] = \delta_{m+n,0}\mathbf{1} = [a_n^1, a_m^{1*}]. \end{aligned}$$

For $c = a, a^1$ and respectively $X = x, x^1$ with $r = 0$ or $r = 1$, we define a representation $\rho_r : \hat{\mathfrak{a}} \to \mathfrak{gl}(\mathbb{C}[x_n, x_n^1 \mid n \in \mathbb{Z}])$ of $\hat{\mathfrak{a}}$ in the Fock space $\mathbb{C}[x_n, x_n^1 \mid n \in \mathbb{Z}]$ by

$$\rho_r(c_m) := \begin{cases} \partial/\partial X_m & \text{if } m \geq 0, \text{ and } r = 0 \\ X_m & \text{otherwise,} \end{cases}$$

$$\rho_r(c_m^*) := \begin{cases} X_{-m} & \text{if } m \leq 0, \text{ and } r = 0 \\ -\partial/\partial X_{-m} & \text{otherwise} \end{cases}$$

and $\rho_r(\mathbf{1}) = 1$. These two representations can be constructed using induction: For $r = 0$ the representation ρ_0 is the $\hat{\mathfrak{a}}$-module generated by $1 =: |0\rangle$, where

$$a_m|0\rangle = a_m^1|0\rangle = 0, \quad m \geq 0, \qquad a_m^*|0\rangle = a_m^{1*}|0\rangle = 0, \quad m > 0.$$

For $r = 1$ the representation ρ_1 is the $\hat{\mathfrak{a}}$-module generated by $1 =: |0\rangle$, where

$$a_m^*|0\rangle = a_m^{1*}|0\rangle = 0, \quad m \in \mathbb{Z}.$$

If we write

$$\alpha(z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad \alpha^*(z) := \sum_{n\in\mathbb{Z}} a_n^* z^{-n}$$
$$\alpha^1(z) := \sum_{n\in\mathbb{Z}} a_n^1 z^{-n-1}, \qquad \alpha^{1*}(z) := \sum_{n\in\mathbb{Z}} a_n^{1*} z^{-n}$$

then

$$[\alpha(z), \alpha(w)] = [\alpha^*(z), \alpha^*(w)] = [\alpha^1(z), \alpha^1(w)] = [\alpha^{1*}(z), \alpha^{1*}(w)] = 0$$
$$[\alpha(z), \alpha^*(w)] = [\alpha^1(z), \alpha^{1*}(w)] = \mathbf{1}\delta(z/w).$$

Observe that $\rho_1(a(z))$ and $\rho_1(a^1(z))$ are not fields whereas $\rho_r(a^*(z))\rho_r(a^{1*}(z))$ are always the fields. Corresponding to these two representations there are two possible normal orderings: For $r = 0$ we use the usual normal ordering given by (3.3) and for $r = 1$ we define the *natural normal ordering* to be

$$\alpha(z)_+ = \alpha(z), \qquad \alpha(z)_- = 0$$
$$\alpha^1(z)_+ = \alpha^1(z), \qquad \alpha^1(z)_- = 0$$
$$\alpha^*(z)_+ = 0, \qquad \alpha^*(z)_- = \alpha^*(z),$$
$$\alpha^{1*}(z)_+ = 0, \qquad \alpha^{1*}(z)_- = \alpha^{1*}(z).$$

This means in particular that for $r = 0$ we get

$$\lfloor \alpha\alpha^* \rfloor = \delta_-(z/w) = \iota_{z,w}\left(\frac{1}{z-w}\right) \tag{4.1}$$

$$\lfloor \alpha^*\alpha \rfloor = -\delta_+(w/z) = \iota_{z,w}\left(\frac{1}{w-z}\right), \tag{4.2}$$

and for $r = 1$

$$\lfloor \alpha\alpha^* \rfloor = [\alpha(z)_-, \alpha^*(w)] = 0 \tag{4.3}$$
$$\lfloor \alpha^*\alpha \rfloor = [\alpha^*(z)_-, \alpha(w)] = -\delta(w/z), \tag{4.4}$$

where similar results hold for α^1. Notice that in both cases we have

$$\lfloor \alpha(z)\alpha^*(w) \rfloor - \lfloor \alpha^*(w)\alpha(z) \rfloor = \delta(z/w).$$

We will also need the following two results.

Theorem 4.1 (*Taylor's Theorem, [22], 2.4.3*). *Let $a(z)$ be a formal distribution. Then in the region $|z - w| < |w|$,*

$$a(z) = \sum_{j=0}^{\infty} \partial_w^{(j)} a(w)(z-w)^j. \tag{4.5}$$

Theorem 4.2 (*[22], Theorem 2.3.2*). *Set $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_n, x_n^1 | n \in \mathbb{Z}]$ and $\mathbb{C}[\mathbf{y}] = C[y_m, y_m^1 | m \in \mathbb{N}^*]$. Let $a(z)$ and $b(z)$ be formal distributions with coefficients in the associative algebra $\mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])$ where we are using the usual normal ordering. The following are equivalent*

(i) $[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z-w) c^j(w)$, *where* $c^j(w) \in \mathrm{End}(\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}])[[w, w^{-1}]]$.

(ii) $\lfloor ab \rfloor = \sum_{j=0}^{N-1} \iota_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right) c^j(w)$.

In other words the singular part of the *operator product expansion*

$$\lfloor ab\rfloor = \sum_{j=0}^{N-1} \iota_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right) c^j(w)$$

completely determines the bracket of mutually local formal distributions $a(z)$ and $b(w)$. One writes

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}.$$

4.2. *The elliptic Heisenberg algebra*

The Cartan subalgebra $\mathfrak{h}$ tensored with R generates a subalgebra of $\hat{\mathfrak{g}}$ which is an extension of an oscillator algebra. This extension motivates the following definition: The Lie algebra with generators b_m, b^1_m, $m \in \mathbb{Z}$, $\mathbf{1}_0$, $\mathbf{1}_\pm$, and relations

$$[b_m, b_n] = 2n\,\delta_{m+n,0}\mathbf{1}_0 \tag{4.6}$$

$$[b^1_m, b^1_n] = (n-m)(\delta_{m+n+2,-1} - 2b\,\delta_{m+n+2,0} + \delta_{m+n+2,1})\mathbf{1}_0 \tag{4.7}$$

$$[b^1_m, b_n] = 2n\Big(p_{|m+n+1|}(b)\mathbf{1}_+ + q_{|m+n+1|}(b)\mathbf{1}_-\Big) \tag{4.8}$$

$$[b_m, \mathbf{1}_0] = [b^1_m, \mathbf{1}_0] = [b_m, \mathbf{1}_\pm] = [b^1_m, \mathbf{1}_\pm] = 0 \tag{4.9}$$

we will give the appellation the *elliptic Heisenberg algebra* and denote it by $\hat{\mathfrak{h}}$.

This algebra has an involutive anti-automorphism σ:

$$\sigma(b_n) = -b_{-n}, \qquad \sigma(b^1_n) = -b^1_{-n-2}, \qquad \sigma(\mathbf{1}_0) = \mathbf{1}_0, \qquad \sigma(\mathbf{1}_\pm) = \mathbf{1}_\pm.$$

If we introduce the formal distributions

$$\beta(z) := \sum_{n\in\mathbb{Z}} b_n z^{-n-1}, \qquad \beta^1(z) := \sum_{n\in\mathbb{Z}} b^1_n z^{-n-1} \tag{4.10}$$

then using calculations done earlier for the elliptic Lie algebra we can see that the relations above can be rewritten in the form

$$[\beta(z), \beta(w)] = 2\mathbf{1}_0\partial_z\delta(z/w) = -2\mathbf{1}_0\partial_w\delta(z/w)$$

$$[\beta^1(z), \beta^1(w)] = -2\left(\frac{1}{2}\partial_w(w(w^2-2bw+1))\delta(z/w) + w(w^2-2bw+1)\partial_w\delta(z/w)\right)\mathbf{1}_0$$

$$[\beta^1(z), \beta(w)] = -2\left((P(w^{-1},b)+P(w,b))\mathbf{1}_+ + (Q(w^{-1},b)+Q(w,b)-2)\mathbf{1}_-\right)w\partial_w\delta(z/w).$$

Set

$$\hat{\mathfrak{h}}^\pm := \sum_{n\geqslant 0}\left(\mathbb{C}b_n + \mathbb{C}b^1_{n-1}\right), \qquad \hat{\mathfrak{h}}^0 := \mathbb{C}\mathbf{1}_- \oplus \mathbb{C}\mathbf{1}_0 \oplus \mathbb{C}\mathbf{1}_+ \oplus \mathbb{C}b_0 \oplus \mathbb{C}b^1_{-1}.$$

We introduce Borel type subalgebras of $\hat{\mathfrak{h}}$:

$$\hat{\mathfrak{b}} = \hat{\mathfrak{h}}^+ + \hat{\mathfrak{h}}^0, \qquad \hat{\mathfrak{b}}^- = \hat{\mathfrak{h}}^- + \hat{\mathfrak{h}}^0.$$

The defining relations above ensure that $\hat{\mathfrak{b}}$ is a subalgebra. Note that $\sigma(b^1_{-1}) = -b^1_{-1}$, $\sigma(\hat{\mathfrak{h}}^0) = \hat{\mathfrak{h}}^0$, and $\sigma(\hat{\mathfrak{h}}^+) = \hat{\mathfrak{h}}^-$. Moreover for $m, n \geq 0$ or for $m, n < -1$, we get

$$[b^1_m, b^1_n] = (n-m)(\delta_{m+n+2,-1} - 2b\,\delta_{m+n+2,0} + \delta_{m+n+2,1})\mathbf{1}_0 = 0. \tag{4.11}$$

Lemma 4.3. *Let $\mathcal{V} = \mathbb{C}\mathbf{v}_0 \oplus \mathbb{C}\mathbf{v}_1$ be a two-dimensional representation of $\hat{\mathfrak{b}}$ where $\hat{\mathfrak{h}}^+\mathbf{v}_i = 0$ for $i = 0, 1$. Suppose $\lambda, \mu, \nu, \varkappa, \chi_\pm, \chi_0 \in \mathbb{C}$ are such that*

$$b_0\mathbf{v}_0 = \lambda\mathbf{v}_0, \qquad b_0\mathbf{v}_1 = \lambda\mathbf{v}_1$$
$$b^1_{-1}\mathbf{v}_0 = \mu\mathbf{v}_0 + \nu\mathbf{v}_1, \qquad b^1_{-1}\mathbf{v}_1 = \varkappa\mathbf{v}_0 + \mu\mathbf{v}_1$$
$$\mathbf{1}_\pm\mathbf{v}_i = \chi_\pm\mathbf{v}_i, \qquad \mathbf{1}_0\mathbf{v}_i = \chi_0\mathbf{v}_i, \quad i = 0, 1.$$

Then $\chi_+ = \chi_- = 0$.

Proof. Since b_m acts by scalar multiplication for $m, n \geq 0$, the first defining relation (4.6) is satisfied for $m, n \geq 0$. The second relation (4.7) is also satisfied as the right-hand side is zero if $m \geq 0$, $n \geq 0$. If $n = 0$, then since b_0 acts by a scalar, the relation (4.8) leads to no condition on $\lambda, \mu, \nu, \varkappa, \chi_\pm, \chi_0 \in \mathfrak{h}_4^0$. If $m \geq 0$ and $n > 0$, the third relation becomes

$$0 = b_m^1 b_n \mathbf{v} - b_n b_m^1 \mathbf{v} = [b_m^1, b_n]\mathbf{v} = 2n\Big(p_{m+n+1}(b)\mathbf{1}_+ + q_{m+n+1}(b)\mathbf{1}_-\Big)\mathbf{v}.$$

Now

$$p_2(b) = \frac{4}{5}(b-1), \qquad q_2(b) = \frac{1}{5}, \qquad p_3(b) = \frac{1}{35}\left(32b^2 - 48b + 11\right), \qquad q_3(b) = \frac{4}{35}(2b-1)$$

which implies that $\chi_+ = \chi_- = 0$. $\square$

Let B_{-1}^1 denote the linear transformation on $\mathcal{V}$ that agrees with the action of b_{-1}^1. If we define the notion of a $\hat{\mathfrak{b}}$-submodule as is done in [4], Definition 1.2, then $\mathcal{V}$ above is an irreducible $\hat{\mathfrak{b}}$-module when $\varkappa\nu \neq 0$ i.e. if $\det B_{-1}^1 \neq \mu^2$. Later we will form a module for the elliptic affine algebra by creating the induced module for V. The resulting representation cannot be irreducible, if $\mathcal{V}$ were not irreducible itself (in the sense of Sheinman, [6]).

We would like to give the heuristic construction of formulae for the representations given below. Since we do not claim that any of this construction gives us a mathematically rigorous proof that we have obtain a representation, we will still need to check the defining relations are satisfied by the given formulae.

To this end we consider the following expression (which should lie in a Borel subgroup $\hat{B}_-$ of $\hat{\mathfrak{b}}^-$)

$$\exp\left(\sum_{m<0} y_m b_m\right)\exp\left(\sum_{m<-1} y_m^1 b_m^1\right),$$

where y_m, y_n^1 are coordinate functions. Consider the representation $\mathcal{V}$ defined above with $\mathbf{w} \in \mathcal{V}$. Since $\exp A \exp B = \exp(\exp(\operatorname{ad} A)B)\exp A$ we have for $k > 0$.

$$\begin{aligned}
&\exp(-tb_k)\exp\left(\sum_{m<0} y_m b_m\right)\exp\left(\sum_{m<-1} y_m^1 b_m^1\right)\mathbf{w} \\
&= \exp\left(\exp\operatorname{ad}(-tb_k)\sum_{m<0} y_m b_m\right)\exp\left(\exp\operatorname{ad}(-tb_k)\sum_{m<-1} y_m^1 b_m^1\right)\mathbf{w} \\
&= \exp\left(\sum_{m<0} y_m b_m - t\sum_{m<0} y_m[b_k, b_m]\right)\exp\left(\sum_{m<-1} y_m^1 b_m^1 - t\sum_{m<-1} y_m^1 [b_k, b_m^1]\right)\mathbf{w} + \mathcal{O}(t^2) \\
&= \exp\left(\sum_{m<0} y_m b_m - t\sum_{m<0} y_m 2m\delta_{k+m,0}\mathbf{1}_0\right) \\
&\quad\times \exp\left(\sum_{m<-1} y_m^1 b_m^1 + t\sum_{m<-1} y_m^1 2k\Big(p_{|m+k+1|}(b)\mathbf{1}_+ + q_{|m+k+1|}(b)\mathbf{1}_-\Big)\right)\mathbf{w} + \mathcal{O}(t^2).
\end{aligned}$$

Under the assumption $[A, A, B] = 0$ we can use Campbel–Baker–Hausdorff formula:

$$\log(\exp(A + tB)\exp(-tB)) = A - \frac{t}{2}[A, B] + O(t^2).$$

This leads to

$$\begin{aligned}
&\exp(-tb_k)\exp\left(\sum_{m<0} y_m b_m\right)\exp\Big(\sum_{m<-1} y_m^1 b_m^1\Big)\mathbf{w} \\
&= \exp\left(\sum_{m<0} y_m b_m\right)\exp\left(\sum_{m<-1} y_m^1 b_m^1\right)\exp\left(-t\sum_{m<0} y_m 2m\delta_{k+m,0}\mathbf{1}_0\right) \\
&\quad\times \exp\left(t\sum_{m<-1} y_m^1 2k\Big(p_{|m+k+1|}(b)\mathbf{1}_+ + q_{|m+k+1|}(b)\mathbf{1}_-\Big)\right)\mathbf{w} + \mathcal{O}(t^2) \\
&= \exp\left(\sum_{m<0} y_m b_m\right)\exp\left(\sum_{m<-1} y_m^1 b_m^1\right)\exp\left(t\, 2ky_{-k}\mathbf{1}_0\right)\mathbf{w} + \mathcal{O}(t^2).
\end{aligned}$$

Thus we set

$$\rho(b_k) = -2k\chi_0 y_{-k}.$$

After conjugating the linear maps ρ (defined at the moment only on the b_k with $k > 0$) with the anti-automorphism σ together with the "anti-Fourier transform"

$$\Phi : y_k \mapsto -\partial_{y_k}, \qquad \partial_{y_k} \mapsto -y_k,$$

we get a new linear map which we also denote by ρ:

$$\rho(b_{-k}) = -2k\chi_0\, \partial_{y_{-k}}, \quad k > 0.$$

The formula for ρ on the other basis elements of $\hat{\mathfrak{h}}$ are obtained in a similar fashion and are given in the following

Proposition 4.4. *Let $\mathcal{M} = \mathbb{C}[y_{-n}, y^1_{-m} \mid m, n \in \mathbb{N}^*] \otimes \mathcal{V}$. Then for $k > 0$*

$$\begin{aligned}
&\rho(b_k) = -y_{-k}, \qquad \rho(b^1_{-k-1}) = y^1_{-k-1}\\
&\rho(b_{-k}) = -2k\chi_0\, \partial_{y_{-k}},\\
&\rho(b^1_k) = -\left((2k+1)\partial_{y^1_{-k-1}} - 4(k+1)b\, \partial_{y^1_{-k-2}} + (2k+3)\partial_{y^1_{-k-3}}\right)\chi_0\\
&\rho(b^1_0) = -\left(\partial_{y^1_{-1}} - 4b\, \partial_{y^1_{-2}} + 3\partial_{y^1_{-3}}\right)\chi_0\\
&\rho(b^1_{-1}) = y^1_{-1} - \partial_{y^1_{-2}}\chi_0 + B^1_{-1}\\
&\rho(b_0) = \lambda \qquad \rho(\mathbf{1}_0) = \chi_0, \qquad \rho(\mathbf{1}_\pm) = 0
\end{aligned}$$

defines a representation of $\hat{\mathfrak{h}}$ on $\mathcal{M}$.

Proof. The defining relations on the generators of $\hat{\mathfrak{h}}$ can easily be checked. For example, for $m \geq 0$ and $n > 0$ we have

$$\begin{aligned}
[\rho(b_m), \rho(b_{-n})] &= [-y_{-m}, -2n\chi_0\partial_{y_{-n}}]\\
&= -2n\,\delta_{m-n,0}\rho(\mathbf{1}_0),
\end{aligned}$$

or

for $n > 1$ and $m \geq 0$,

$$\begin{aligned}
[\rho(b^1_{-n}), \rho(b^1_m)] &= \left[y^1_{-n}, -((2m+1)\partial_{y^1_{-m-1}} + 4(m+1)b\,\partial_{y^1_{-2-m}} - (2m+3)\partial_{y^1_{-m-3}})\chi_0\right]\\
&= -\left(-(2m+1)\delta_{-n+m,-1} + 4(m+1)b\,\delta_{-n+m,-2} - (2m+3)\delta_{-n+m,-3}\right)\chi_0\\
&= (m+n)\left(\delta_{-n+m,-1} - 2b\,\delta_{-n+m,-2} + \delta_{-n+m,-3}\right)\chi_0.
\end{aligned}$$

All other relations are straightforward. □

4.3. The imaginary Borel subgroup

Let e, h, f be the standard basis of $sl(2)$. Then $e_n, e^1_n, f_n, f^1_n, h_n, h^1_n, w_0, w_\pm$ is a basis of $\hat{\mathfrak{g}}$, where $x_n = x \otimes t^n$, $x^1_n = x \otimes ut^n$. Denote by $\mathfrak{N}_+$ (respectively $\mathfrak{N}_-$) a subalgebra spanned by e_n, e^1_n, $n \in \mathbb{Z}$, h_m, h^1_m, $m > 0$ (respectively f_n, f^1_n, $n \in \mathbb{Z}$, h_m, h^1_m, $m < 0$) and set $\mathfrak{H} = \mathbb{C}h \oplus \mathbb{C}w_0 \oplus \mathbb{C}w_+ \oplus \mathbb{C}w_-$. Then

$$\hat{\mathfrak{g}} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+.$$

We call the subalgebra $\mathfrak{B} = \mathfrak{N}_+ \oplus H$ the Imaginary Borel subalgebra of $\hat{\mathfrak{g}}$. We will make use of this Borel subalgebra to construct Wakimoto type representations of $\hat{\mathfrak{g}}$.

5. Two realizations of the elliptic affine algebra $\hat{\mathfrak{g}}$

Our main result is the following

Theorem 5.1. *Let $r \in \{0, 1\}$, with the corresponding normal ordering defined above. Assume that $\chi_+ = \chi_- = 0$, $\chi_0 \in \mathbb{C}$ and $\mathcal{V}$ as in Proposition 4.4. Then the following defines a representation of the elliptic affine algebra $\hat{\mathfrak{g}}$ on $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}] \otimes \mathcal{V}$:*

$$\begin{aligned}
&\theta(\omega_\pm) = 0, \qquad \theta(\omega_0) = \chi_0 = \chi_0 - 4\delta_{r,0}\\
&\theta(f(z)) = -\alpha, \qquad \theta(f^1(z)) = -\alpha^1,\\
&\theta(h(z)) = 2\left(:\alpha\alpha^*: + :\alpha^1\alpha^{1*}:\right) + \beta\\
&\theta(h^1(z)) = 2\left(:\alpha^1\alpha^*: + z(1-2bz+z^2):\alpha\alpha^{1*}:\right) + \beta^1
\end{aligned}$$

Table 1

$[\cdot_\lambda \cdot]$	$f(w)$	$f^1(w)$	$h(w)$	$h^1(w)$	$e(w)$	$e^1(w)$
$f(z)$	0	0	$*$	$*$	$*$	$*$
$f^1(z)$		0	$*$	$*$	$*$	$*$
$h(z)$			$*$	$*$	$*$	$*$
$h^1(z)$				$*$	$*$	$*$
$e(z)$					0	0
$e^1(z)$						0

$$\begin{aligned}\theta(e^1(z)) =: &\ \alpha^1(z)\alpha^*(z)^2 : + z\left(z^2-2bz+1\right)\left(: \alpha^1(z)\alpha^{1*}(z)^2 : + 2 : \alpha(z)\alpha^*(z)\alpha^{1*}(z) :\right)\\ &+ \beta^1(z)\alpha^*(z) + z\left(z^2-2bz+1\right)\beta(z)\alpha^{1*}(z)\\ &+ \chi_0 z(1-2bz+z^2)\partial_z a^{1*}(z) + \frac{1}{2}\chi_0\partial_z[z(1-2bz+z^2)]a^{1*}(z)\end{aligned}$$

$$\theta(e(z)) =: \alpha(\alpha^*)^2 : + z(1-2bz+z^2) : \alpha(\alpha^{1*})^2 : + 2 : \alpha^1\alpha^*\alpha^{1*} : + \beta\alpha^* + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*.$$

In the formulas above we omitted the variables in the fields whenever it does not create any confusion. In the next section we explain in what sense $r = 0$ and $r = 1$ give two different realizations of the elliptic affine Lie algebra $\hat{\mathfrak{g}}$.

Before we go through the proof it will be fruitful to introduce V. Kac's λ-notation (see [22] Section 2.2 and [26] for some of its properties) used in operator product expansions. If $a(z)$ and $b(w)$ are formal distributions, then

$$[a(z), b(w)] = \sum_{j=0}^{\infty} \frac{(a_{(j)}b)(w)}{(z-w)^{j+1}}$$

is transformed under the *formal Fourier transform*

$$F^{\lambda}_{z,w} a(z,w) = \mathrm{Res}_z e^{\lambda(z-w)} a(z,w),$$

into the sum

$$[a_\lambda b] = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a_{(j)} b.$$

Proof. We need to check that Table 1 is preserved under θ. In Table 1 $*$ represents nonzero formal distributions that are obtained from the defining relations. The proof is carried out using Wick's Theorem together with Taylor's Theorem.

$$[\theta(f)_\lambda\theta(f)] = 0, \qquad [\theta(f)_\lambda\theta(f^1)] = 0,$$

$$[\theta(f)_\lambda\theta(h)] = -\Big[\alpha_\lambda\Big(2\left(\alpha\alpha^* + \alpha^1\alpha^{1*}\right) + \beta\Big)\Big] = -2\alpha = -2\theta(f),$$

$$[\theta(f)_\lambda\theta(h^1)] = -\left[\alpha_\lambda\left(2\left(\alpha^1\alpha^* + P^2\alpha\alpha^{1*}\right) + \beta^1\right)\right] = -2\alpha^1 = 2\theta(f^1),$$

$$\begin{aligned}[\theta(f)_\lambda\theta(e)] &= -\Big[\alpha_\lambda\Big(: \alpha(\alpha^*)^2 : + \beta\alpha^* + \chi_+\partial(P\alpha^{1*}) + P^2 : \alpha(\alpha^{1*})^2 : + 2 : \alpha^1\alpha^*\alpha^{1*} : + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\Big)\Big]\\ &= -2\left(: \alpha\alpha^* : - : \alpha^1\alpha^{1*} :\right) - \beta - \chi_0\partial = -\theta(h) - \chi_0\lambda\end{aligned}$$

$$\begin{aligned}[\theta(f)_\lambda\theta(e^1)] &= -\Big[\alpha_\lambda\Big(: \alpha^1(\alpha^*)^2 : + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^*\\ &\quad + P^2\Big(: \alpha^1(\alpha^{1*})^2 : + 2 : \alpha\alpha^*\alpha^{1*} : + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*}\Big)\Big)\Big]\\ &= -2\left(: \alpha^1\alpha^* : + P^2 : \alpha\alpha^{1*} :\right) - \beta^1 - \chi_+ P\partial = -\theta(h^1) - \chi_+ P\lambda\end{aligned}$$

$$[\theta(f^1)_\lambda\theta(f^1)] = 0$$

$$[\theta(f^1)_\lambda\theta(h)] = -\left[\alpha^1_\lambda\left(2\left(: \alpha\alpha^* : + : \alpha^1\alpha^{1*} :\right) + \beta\right)\right] = -2\alpha^1 = 2\theta(f^1),$$

$$[\theta(f^1)_\lambda\theta(h^1)] = -\left[\alpha^1_\lambda\left(2\left(: \alpha^1\alpha^* : + P^2 : \alpha\alpha^{1*} :\right) + \beta^1\right)\right] = -2P^2\alpha^1 = P^2\theta(f^1),$$

$$\begin{aligned}[\theta(f^1)_\lambda\theta(e)] &= -\Big[\alpha^1_\lambda\Big(: \alpha(\alpha^*)^2 : + \beta\alpha^* + \partial(P\alpha^{1*}) + P^2 : \alpha(\alpha^{1*})^2 : + 2 : \alpha^1\alpha^*\alpha^{1*} : + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\Big)\Big]\\ &= -\Big(\chi_+(\partial P + P\lambda) + 2P^2 : \alpha\alpha^{1*} : + 2 : \alpha^1\alpha^* : + \beta^1\Big) = -\theta(h^1) - \chi_+(\partial P + P\partial)\end{aligned}$$

$$\begin{aligned}[\theta(f^1)_\lambda\theta(e^1)] &= -\Big[\alpha^1_\lambda\Big(: \alpha^1(\alpha^*)^2 : + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^*\\ &\quad + P^2\Big(: \alpha^1(\alpha^{1*})^2 : + 2 : \alpha\alpha^*\alpha^{1*} : + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*}\Big)\Big)\Big]\\ &= -\Big(\chi_0 P(\partial P)\alpha^{1*} + P^2\left(2\left(: \alpha^1\alpha^{1*} : + : \alpha\alpha^* :\right) + \beta + \chi_0\lambda\right)\Big)\\ &= -\left(P^2\theta(h) + \chi_0 P\partial P + P^2\chi_0\lambda\right)\end{aligned}$$

$$\begin{aligned}[\theta(h)_\lambda\theta(h)] &= \left[\left(2\left(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:\right)+\beta\right)_\lambda\left(2\left(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:\right)+\beta(w)\right)\right]\\ &= 4(-:\alpha\alpha^*:+:\alpha^*\alpha:-:\alpha^1\alpha^{1*}:+:\alpha^{1*}\alpha^1:)-8\delta_{r,0}\lambda+[\beta_\lambda\beta]\\ &= -2(4\delta_{r,0}+\chi_0)\lambda .\end{aligned}$$

Thus

$$[\theta(h(z)),\theta(h(w))]=-2(4\delta_{r,0}+\chi_0)\partial_w\delta(z/w).$$

Next we calculate

$$\begin{aligned}[\theta(h)_\lambda\theta(h^1)] &= \left[\left(2\left(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:\right)+\beta\right)_\lambda\left(2\left(:\alpha^1\alpha^*:+P^2:\alpha\alpha^{1*}:\right)+\beta^1\right)\right]\\ &= \Big(\left(:\alpha^*\alpha^1:-:\alpha^1\alpha^*:\right)+P^2\left(-:\alpha\alpha^{1*}:+:\alpha^{1*}\alpha:\right)\Big)+[\beta_\lambda\beta^1].\end{aligned}$$

Since $[a_n,a_m^{1*}]=[a_n^1,a_m^*]=0$, we have

$$\left[\theta(h(z)),\theta(h^1(w))\right]=[\beta(z),\beta^1(w)]=-2\chi_+\left(1-2bw+w^2\right)^{1/2}\partial_w\delta(z/w).$$

We continue with

$$\begin{aligned}[\theta(h^1)_\lambda\theta(h^1)] &= \left[\left(2\left(:\alpha^1\alpha^*:+P^2:\alpha\alpha^{1*}:\right)+\beta^1\right)_\lambda\left(2\left(:\alpha^1\alpha^*:+P^2:\alpha\alpha^{1*}:\right)+\beta^1\right)\right]\\ &= 4P^2\left(-:\alpha\alpha^*:+:\alpha^{1*}\alpha^1:\right)+4P^2\left(-:\alpha^1\alpha^{1*}:+:\alpha^*\alpha:\right)+8\delta_{r,0}P^2\lambda+8\delta_{r,0}P(\partial P)+[\beta^1_\lambda\beta^1]\\ &= \delta_{r,0}P^2\lambda+8\delta_{r,0}P(\partial P)-2\chi_0(P^2\lambda+P(\partial P)).\end{aligned}$$

Note that $:a(z)b(z):$ and $:b(z)a(z):$ are usually not equal, but $:\alpha^1(w)\alpha^{*1}(w):=:\alpha^{1*}(w)\alpha^1(w):$ and $:\alpha(w)\alpha^*(w):=:\alpha^*(w)\alpha(w):$. Thus we have

$$\left[\theta(h^1(z)),\theta(h^1(w))\right]=(4\delta_{r,0}+\chi_0)\left(-2\left(w^2-2bw+1\right)\partial_w\delta(z/w)+2(b-w)\delta(z/w)\right).$$

Next we calculate the h's paired with the e's:

$$\begin{aligned}[\theta(h)_\lambda\theta(\mathrm{e}^1)] =\ & \bigg[\Big(2\left(:\alpha\alpha^*:+:\alpha^1\alpha^{1*}:\right)+\beta\Big)_\lambda\Big(:\alpha^1(\alpha^*)^2:+\beta^1\alpha^*+\chi_0P(\partial P)\alpha^{1*}+\chi_+P\partial\alpha^*\\ &+P^2\left(:\alpha^1(\alpha^{1*})^2:+2:\alpha\alpha^*\alpha^{1*}:+\beta\alpha^{1*}+\chi_0\partial\alpha^{1*}\right)\Big)\bigg]\\ =:\ & \alpha^1(\alpha^*)^2:+2\beta^1\alpha^*+2\chi_0P(\partial P)\alpha^{1*}+2\chi_+P\alpha^*\lambda+2\chi_+\partial P+4P^2\Big(:\alpha^*\alpha\alpha^{1*}:-2\delta_{r,0}\alpha^{1*}\lambda\Big)\\ &-2P^2\Big(:\alpha^1(\alpha^{1*})^2:+\beta\alpha^{1*}+\chi_0\lambda\Big)+\alpha^*[\beta_\lambda\beta^1]+P^2\alpha^{1*}[\beta_\lambda\beta].\end{aligned}$$

From this we can conclude

$$\begin{aligned}\left[\theta(h(z)),\theta(\mathrm{e}^1(w))\right] =\ & \left(:\alpha^1(w)\alpha^*(w)^2:+\beta^1(w)\alpha^*(w)+\chi_0(w-b)\alpha^{1*}(w)\right)\delta(z/w)\\ &+2\left(\chi_+(1-2bw+w^2)^{1/2}\partial_w\alpha^*(w)\right)\delta(z/w)\\ &+2\left(w^2-2bw+1\right)\Big(2:\alpha(w)\alpha^*(w)\alpha^{1*}(w):+\chi_0\partial_w\alpha^{1*}(w)\Big)\delta(z/w)\\ &+2\left(w^2-2bw+1\right)\Big(:\alpha^1(w)\alpha^{1*}(w)\alpha^{1*}(w)+\beta(w)\alpha^{1*}(w)\Big)\delta(z/w)\\ &-2\left(w^2-2bw+1\right)(4\delta_{r,0}+\chi_0-\chi_0)\alpha^{1*}(w)\partial_w\delta(z/w)\\ =\ & 2\theta(\mathrm{e}^1(w))\delta(z/w)-2\left(w^2-2bw+1\right)(4\delta_{r,0}+\chi_0-\chi_0)\alpha^{1*}(w)\partial_w\delta(z/w).\end{aligned}$$

Next we must calculate

$$\begin{aligned}[\theta(h)_\lambda\theta(e)] =\ & 2\Big[:\alpha\alpha^*:_\lambda\Big(:\alpha(\alpha^*)^2:+\beta\alpha^*+\chi_+\partial P\alpha^{1*}+\chi_+P\partial\alpha^{1*}\\ &+P^2:\alpha(\alpha^{1*})^2:+2:\alpha^1\alpha^*\alpha^{1*}:+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*\Big)\Big]\\ &+2\Big[:\alpha^1\alpha^{1*}:_\lambda\Big(:\alpha(\alpha^*)^2:+\beta\alpha^*+\chi_+(\partial P)\alpha^{1*}+\chi_+P\partial\alpha^{1*}\\ &+P^2:\alpha(\alpha^{1*})^2:+2:\alpha^1\alpha^*\alpha^{1*}:+\beta^1\alpha^{1*}+\chi_0\partial\alpha^*\Big)\Big]\\ &+\Big[\beta_\lambda\Big(:\alpha(\alpha^*)^2:+\beta\alpha^*+\chi_+(\partial P)\alpha^{1*}+\chi_+P\partial\alpha^{1*}\\ &+P^2:\alpha(\alpha^{1*})^2:+2:\alpha^1\alpha^*\alpha^{1*}:+\beta^1\alpha^{1*}+\chi_0\partial_w\alpha^*\Big)\Big]\end{aligned}$$

$$= 2\Big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+\partial(P\alpha^{1*})$$
$$+ P^2 :\alpha(\alpha^{1*})^2: + 2 :\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\Big)2\chi_0\alpha^*\lambda - 8\delta_{r,0}\alpha^*\lambda - 2\chi_0\alpha^*\lambda.$$

As a consequence we get

$$[\theta(h(z)), \theta(e(w))] = 2\theta(e(w))\delta(z/w) + 2(\chi_0 - \chi_0 - 4\delta_{r,0})\alpha^*(w)\partial_w\delta(z/w).$$

Next we calculate

$$\begin{aligned}[\theta(h^1)_\lambda\theta(\mathrm{e}^1)] &= \Big[\Big(2\left(:\alpha^1\alpha^*: + P^2 :\alpha\alpha^{1*}:\right) + \beta^1\Big)_\lambda \Big(:\alpha^1(\alpha^*)^2: + \beta^1\alpha^* + \chi_0 P(\partial P)\alpha^{1*} + \chi_+ P\partial\alpha^* \\ &\quad + P^2\left(:\alpha^1(\alpha^{1*})^2: + 2:\alpha\alpha^*\alpha^{1*}: + \beta\alpha^{1*} + \chi_0\partial\alpha^{1*}\right)\Big)\Big] \\ &= 2P^2 :\alpha(\alpha^*)^2: + 2P^2\beta\alpha^* + 2\chi_0P^2\partial\alpha^* + 2P^4 :\alpha(\alpha^{1*})^2: \\ &\quad + 4P^2 :\alpha^1\alpha^{1*}\alpha^*: + 2P^2\beta^1\alpha^{1*} + 2\chi_+P^3\partial\alpha^{1*} + 2\chi_+P^2(\partial P)\alpha^{1*} \\ &\quad + 2\chi_0\alpha^*(P(\partial P) + P^2\lambda) - 8\delta_{r,0}\alpha^*(P(\partial P) + P^2\lambda) - 2\chi_0\alpha^*(P(\partial P) + P^2\lambda) \\ &= 2P^2\theta(e).\end{aligned}$$

The final calculation for the Cartan generators is

$$\begin{aligned}[\theta(h^1)_\lambda\theta(e)] &= \Big[\Big(2\left(:\alpha^1\alpha^*: + P^2 :\alpha\alpha^{1*}:\right) + \beta^1\Big)_\lambda \Big(:\alpha(\alpha^*)^2: + \beta\alpha^* + \chi_+(\partial P)\alpha^{1*} + \chi_+P\partial\alpha^{1*} \\ &\quad + P^2 :\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta^1\alpha^{1*} + \chi_0\partial\alpha^*\Big)\Big] \\ &= -2 :\alpha^1(\alpha^*)^2 + 2\chi_+(\partial P)\alpha^* + 2\chi_+P\partial\alpha^* + 2\chi_+P\alpha^*\lambda \\ &\quad - 2P^2 :\alpha^1(\alpha^{1*})^2: + 4P^2 :\alpha\alpha^*\alpha^{1*}: - 4\delta_{r,0}P^2\alpha^{1*}\lambda + 4 :\alpha^1(\alpha^*)^2: + \beta^1\alpha^* \\ &\quad + 4P^2 :\alpha\alpha^*\alpha^{*1}: + 2P^2\beta\alpha^{*1} + 4P^2 :\alpha^1(\alpha^{1*})^2: - 4P^2 :\alpha\alpha^*\alpha^{1*}: \\ &\quad - 4\delta_{r,0}P^2\alpha^{1*}\lambda - 8\delta_{r,0}P(\partial P)\alpha^{1*} + 2\chi_0P^2\alpha^{*1}\lambda + 2\chi_0P^2\partial\alpha^{1*} + 4\chi_0P(\partial P)\alpha^{1*} \\ &\quad - 2\chi_+(P\lambda + \partial P)\alpha^* - 2\chi_0\alpha^{1*}(P(\partial P) + P^2\lambda) \\ &= 2 :\alpha^1(\alpha^*)^2 + 2\chi_+P\partial\alpha^* + 2P^2 :\alpha^1(\alpha^{1*})^2: + + \beta^1\alpha^* + 4P^2 :\alpha\alpha^*\alpha^{*1}: \\ &\quad + 2P^2\beta\alpha^{*1} + 2\chi_0P^2\partial\alpha^{1*} + 2\chi_0P(\partial P)\alpha^{1*} + 2(\chi_0 - 4\delta_{r,0} - \chi_0)\alpha^{*1}(P(\partial P) + P^2\lambda) = 2\theta(\mathrm{e}^1).\end{aligned}$$

Similar calculations show that

$$[\theta(e)_\lambda\theta(e)] = [\theta(e)_\lambda\theta(\mathrm{e}^1)] = [\theta(\mathrm{e}^1)_\lambda\theta(\mathrm{e}^1)] = 0.$$

We leave the details to the reader. This completes the proof of the theorem. □

6. Jakobsen–Kac realizations

Now we relate the representations constructed in Theorem 5.1 with the representations constructed by Jakobsen and Kac [16].

If R is an associative (but not necessarily commutative) algebra over $\mathbb{C}$, then a linear map $\phi : R \to \mathbb{C}$ is called a *trace* on R if

$$\phi(ab) = \phi(ba) \quad \forall a, b \in R.$$

If ϕ is a trace on R, then the space of matrices

$$\mathfrak{sl}_2(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle|\, a, b, c, d \in R,\ \text{and } \phi(a+d) = 0 \right\}$$

is a Lie algebra under the usual commutator.

Note that $\mathfrak{sl}_2(R)$ always contains a subalgebra isomorphic to $\mathfrak{sl}(2, R) = \mathfrak{sl}_2 \otimes R$.

The subspace

$$\mathfrak{s} := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle|\, a, b, d \in R,\ \text{and } \phi(a+d) = 0 \right\}$$

is also clearly a subalgebra of $\mathfrak{sl}_2(R)$. Let $\mathbb{C}_\phi = \mathbb{C}\mathbf{v}_\phi$ be the one-dimensional $\mathfrak{s}$-module defined by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mathbf{v}_\phi := \phi(a)\mathbf{v}_\phi.$$

Thus one can define an induced module

$$M(\phi) := U(\mathfrak{sl}_2(R)) \otimes_{U(\mathfrak{s})} C_\phi.$$

If R is a commutative ring with basis $\{a_\beta\}_{\beta\in I}$ with structure constants $c^\gamma_{\alpha\beta}$ so that

$$a_\alpha a_\beta = c^\gamma_{\alpha\beta} a_\gamma$$

then elements of the form

$$(a_{\alpha_1} f)\cdots(a_{\alpha_n} f).v_\phi$$

constitute a basis of $M(\phi)$. Here $af := \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$.

In [16] Jakobsen and Kac realized $M(\phi)$ of $\mathfrak{sl}_2(R)$ on the space of polynomials $\mathbb{C}[x_\beta | \beta \in I]$ as follows. Let $\tilde{\rho}$ be the corresponding representation. We have

$$\begin{aligned}
&\rho(a_{\alpha_0} f) = x_{\alpha_0},\\
&\rho(a_{\alpha_0} h) = -2c^\gamma_{\alpha_0\alpha} x_\gamma \frac{\partial}{\partial x_\alpha} + \phi(a_{\alpha_0})\\
&\rho(a_{\alpha_0} e) = -c^\gamma_{\alpha_0\alpha} c^\delta_{\gamma\beta} x_\delta \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x_\beta} + \phi(a_\gamma) c^\gamma_{\alpha_0\alpha}\frac{\partial}{\partial x_\alpha}.
\end{aligned}$$

Now consider $R = \mathbb{C}[t, t^{-1}, u | u^2 = t^3 - 2bt^2 + t]$ with a basis $\{a_n = t^n, a^1_n := t^n u \mid n \in \mathbb{Z}\}$. Then

$$a_m a_n = a_{m+n}, \qquad a_m a^1_n = a^1_{m+n}, \qquad a^1_m a^1_n = a_{m+n+3} - 2b a_{m+n+2} + a_{m+n+1}.$$

The representation $\tilde{\rho}$ of $\mathfrak{sl}_2(R)$ on $\mathbb{C}[x_n, x^1_m | m, n \in \mathbb{Z}]$ induces a representation of $\mathfrak{sl}(2, R)$ given by

$$\begin{aligned}
&\rho(t^m f) = x_m, \qquad \rho(t^m u f) = x^1_m\\
&\rho(t^m h) = -2\sum_p \left(x_{m+p}\partial_{x_p} + x^1_{m+p}\partial_{x^1_p}\right) + \phi(t^m)\\
&\rho(t^m u\, h) = -2\sum_p \left(x^1_{m+p}\partial_{x_p} + \left(x_{m+p+2} - 2x_{m+p+1} + x_{m+p}\right)\partial_{x^1_p}\right) + \phi(t^m u)\\
&\rho(t^m e) = -\sum_{n,q}\left(x_{m+n+q}\partial_{x_n}\partial_{x_q} + x^1_{m+n+q}\partial_{x_n}\partial_{x^1_q} + x^1_{m+n+q}\partial_{x^1_n}\partial_{x_q}\right)\\
&\qquad - \sum_{n,q}\left(x_{m+n+q+2}\partial_{x^1_n}\partial_{x^1_q} - 2bx_{m+n+q+1}\partial_{x^1_n}\partial_{x^1_q} + x_{m+n+q}\partial_{x^1_n}\partial_{x^1_q}\right) + \sum_p\left(\phi(t^{m+p})\partial_{x_p} + \phi(t^{m+p}u)\partial_{x^1_p}\right)\\
&\rho(t^m u\, e) = -\sum_{n,q} x^1_{m+n+q}\partial_{x_n}\partial_{x_q} - \sum_{n,q}\left(x_{m+n+q+2} - 2bx_{m+n+q+1} + x_{m+n+q}\right)\partial_{x_n}\partial_{x^1_q}\\
&\qquad - \sum_{n,q}\left(x_{m+n+q+2} - 2bx_{m+n+q+1} + x_{m+n+q}\right)\partial_{x^1_n}\partial_{x_q} - \sum_{n,q}\left(x^1_{m+n+q+2} - 2bx^1_{m+n+q+1} + x^1_{m+n+q}\right)\partial_{x^1_n}\partial_{x^1_q}\\
&\qquad + \sum_p\left(\phi(t^{m+p}u)\partial_{x_p} + \phi(t^{m+p}(t^2 - 2bt - 1))\partial_{x^1_p}\right).
\end{aligned}$$

One can see that, up to a change in sign, Jakobsen and Kac's representation is a quotient of the representation that we have constructed in Theorem 5.1 for the universal central extension of $\mathfrak{sl}(2, R)$ when $r = 1$ and $\phi = 0$. For example if one looks at the Fourier modes of

$$:\alpha(\alpha^*)^2: + z(1 - 2bz + z^2):\alpha(\alpha^{1*})^2: + 2:\alpha^1\alpha^*\alpha^{1*}: + \beta\alpha^*$$

in $\theta(e(z))$, one gets $\rho(t^m e)$ up to a difference in sign.

When $r = 0$ the Wakimoto type module constructed in Theorem 5.1 appears to be related to the Verma module for $\tilde{\mathfrak{g}}$ with highest weight subspace $\mathcal{V}$ given in Lemma 4.3. We conjecture that generically the Wakimoto type module constructed in Theorem 5.1 ($r = 0$) and the corresponding Verma module are isomorphic.

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