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A Wakimoto Type Realization of Toroidal $\mathfrak{sl}_{n+1}$

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Abstract. The authors construct a Wakimoto type realization of toroidal $\mathfrak{sl}_{n+1}$. The representation constructed in this paper utilizes non-commuting differential operators acting on the tensor product of two polynomial rings in infinitely many commuting variables.

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1 Introduction

Toroidal Lie algebras were first introduced in [24] as a natural generalization of affine algebras. Given a finite-dimensional simple Lie algebra $\mathfrak{a}$, a toroidal algebra is a central extension of $\mathfrak{a}\otimes\mathbb{C}[t_1,\ldots,t_n,t_1^{-1},\ldots,t_n^{-1}]$, where the t_i are commuting variables. Toroidal algebras can be thought of as iterated loop algebras in many commuting variables. Such algebras can also be defined using generators and relations as we do here.

One motivation for the study of toroidal Lie algebras is for potential applications to mathematics and physics. For instance, one of the cocycles used in the construction of the toroidal extended affine Lie algebra is also used in Billig's study of a magnetic hydrodynamics equation with asymmetric stress tensor (see [7] and [6]). In addition, Billig [5] and independently Iohara, Saito and Wakimoto [17] derive Hirota bilinear equations arising from both homogeneous and principal realizations of the vertex operator representations of 2-toroidal Lie algebras of type A_l, D_l, E_l. They derive the hierarchy of Hirota equations and present their soliton-type solutions. In [20], Kakei, Ikeda and Takasaki constructed the hierarchy associated to the $(2+1)$-dimensional nonlinear Schrödinger (NLS) equation and show how the representation theory of toroidal $\mathfrak{sl}_2$ can be used to derive the Hirota-type equations for τ-functions. On the somewhat more mathematical side, in interesting work of Ginzburg, Kapranov and Vasserot [16] on Langland's reciprocity for algebraic surfaces, Hecke operators are constructed for vector bundles on an algebraic surface. The main point of their paper is that under certain conditions, the corresponding

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algebra of Hecke operators is the homomorphic image of a quantum toroidal algebra. One should also note some of the recent work of Slodowy [27], Berman and Moody [3], Benkart and Zelmanov [1] on generalized intersection matrix algebras involve their relationship to toroidal Lie algebras. In addition, Wakimoto's free field realization of affine $\widehat{\mathfrak{sl}_2}$ and Feigin and Frenkel's generalization to non-twisted affine algebras $\hat{\mathfrak{g}}$ play a fundamental role in describing integral solutions to the Knizhnik-Zamolodchikov equations (see for example [28], [14], [21], [12], [25] and [26]).

The representation constructed here is similar to what is often called a "free field" representation, that is, our Lie algebra elements will be realized as formal power series of non-commuting differentiable operators a_n ($n \in \mathbb{Z}$) acting on a given vector space V, where the formal power series associated with the Lie algebra become finite when applied to an element $v \in V$. Our representation is constructed by first finding a representation of an infinite-dimensional Heisenberg like algebra, and then "inducing" to the full toroidal algebra. The free field representation in this paper is a generalization of the second author [9, 10] which were in turn motivated by the work of Feigin and Frenkel [13, 15] constructing free field realizations of affine Kac-Moody and W-algebras, as well as [4]. A completely different representation of a class of toroidal algebras given by free bosonic fields appears in [18]. Interestingly, some free field representations of toroidal lie algebras can be used to construct vertex algebras (see [2]) of a certain type, where all simple graded modules can be classified (see [23]).

Part of our motivation for studying Wakimoto type realizations of toroidal $\mathfrak{sl}_{n+1}$ is to gain insight into the role of 2-cocycles in a more general construction of free field realizations for universal central extensions of Lie algebras of the form $\mathfrak{g} \otimes R$, where R is an algebra over the complex numbers. Another motivation is that they can often provide, in the generic setting, realizations in terms of partial differential operators of imaginary type Verma modules for toroidal Lie algebras. We plan to see how the realizations in this paper are related to these modules in future work.

2 Notation and Preliminary Setting

All vector spaces are over the field of complex numbers $\mathbb{C}$.

Let $A_n = (A_{ij})_{i,j=0}^n$ be the indecomposable Cartan matrix of affine type A_n^1 with $n \geq 2$. Let $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ denote the simple roots, a basis for the set of roots denoted by Δ. Let Q be the root lattice, i.e., the free $\mathbb{Z}$-module with generators $\alpha_0, \alpha_1, \ldots, \alpha_n$. The matrix A_n induces a symmetric bilinear form $(\cdot\,|\,\cdot)$ on Q satisfying $(\alpha_i|\alpha_j) = A_{ij}$. For $0 \leq i \leq n$, we set $\check{\alpha}_i := \alpha_i$.

We review some of the calculus of formal series following [22], and we introduce a slightly modified form of the $\boldsymbol{\lambda}$-bracket notation and Fourier transform of [19] which provides a very compressed notation, however many of the calculations are actually done in the more expanded form of [22]. As pointed out in [22], the formal calculus generalizes to several commuting variables, the case used here. Throughout this paper, $z_i, w_i, x_i, y_i, \lambda_i$ will denote mutually commuting formal variables with i ranging over some index set. We use multi-index notation, for a positive integer k, given an element $(m_0, m_1, \ldots, m_k) \in \mathbb{Z}^{k+1}$, we write $\mathbf{m} = (m_0, m_1, \ldots, m_k)$, and define $\mathbf{z}^{\mathbf{m}} = z_0^{m_0} z_1^{m_1} \cdots z_k^{m_k}$. Denote by $\mathbf{0}$ the k-tuple of all zeros, and by $\mathbf{1}$ the k-tuple of all ones. Fix a decomposition of $\mathbb{Z}^{k+1} = \mathbb{Z}_-^{k+1} \cup \{\mathbf{0}\} \cup \mathbb{Z}_+^{k+1}$ into three disjoint subsets such that $\mathbb{Z}_{\pm}^{k+1}$ are sets closed under vector addition, i.e., for example, if $\mathbf{j}, \mathbf{k} \in \mathbb{Z}_{+}^{k+1}$, then $\mathbf{j}+\mathbf{k} \in \mathbb{Z}_{+}^{k+1}$. Define $\mathbf{m} > \mathbf{0}$ if $\mathbf{m} \in \mathbb{Z}_{+}^{k+1}$ and $\mathbf{m} < \mathbf{0}$ if $\mathbf{m} \in \mathbb{Z}_{-}^{k+1}$. Define the function θ by $\theta(\mathbf{m}) = 1$ if $\mathbf{m} > 0$, and 0 otherwise.

We work with formal series $a(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^{k+1}} a_{\mathbf{n}} \mathbf{z}^{-\mathbf{n}}$ with $a_{\mathbf{n}} \in \mathrm{End}(V)$ for a vector space V described below. The series in this paper are summable in the sense of [22], i.e., the coefficient of any monomial in the formal sum acts as a finite sum of operators when applied to any vector $v \in \mathrm{End}(V)$. To simplify notation, we denote $\mathbb{C}[[z_0, z_0^{-1}, z_1, z_1^{-1}, \dots, z_k, z_k^{-1}]]$ as $\mathbb{C}[[\mathbf{z}, \mathbf{z}^{-1}]]$. Define $\delta(\mathbf{z}) := \sum_{\mathbf{n}\in\mathbb{Z}^{k+1}} \mathbf{z}^{\mathbf{n}} \in \mathbb{C}[[\mathbf{z}, \mathbf{z}^{-1}]$. Similarly, $\delta(\mathbf{z}/\mathbf{w}) := \sum_{\mathbf{m}\in\mathbb{Z}^{k+1}} \mathbf{z}^{\mathbf{m}}\mathbf{w}^{-\mathbf{m}} \in \mathbb{C}[[\mathbf{z}, \mathbf{z}^{-1}, \mathbf{w}, \mathbf{w}^{-1}]]$ so that $\delta(\mathbf{z}/\mathbf{w}) = \prod_{i=0}^{k} \delta(z_i/w_i)$, where $\delta(z_i/w_i) = \sum_{k\in\mathbb{Z}} z_i^k w_i^{-k}$.

The following properties of δ hold (see [22, Proposition 2.1.8]) which we reproduce here in the multivariable setting.

Proposition 2.1.

(i) *Let* $f(\mathbf{z}) \in V[\mathbf{z}, \mathbf{z}^{-1}]$. *Then* $f(\mathbf{z})\delta(\mathbf{z}) = f(\mathbf{1})\delta(\mathbf{z})$.

(ii) *Let* $f(\mathbf{z}, \mathbf{w}) \in \mathrm{End}\, V[[\mathbf{z}, \mathbf{z}^{-1}, \mathbf{w}, \mathbf{w}^{-1}]]$ *such that* $\lim_{\mathbf{z}\to\mathbf{w}} f(\mathbf{z}, \mathbf{w})$ *exists. Then in* $\mathrm{End}(V)[\mathbf{z}, \mathbf{z}^{-1}, \mathbf{w}, \mathbf{w}^{-1}]]$, $f(\mathbf{z}, \mathbf{w})\delta(\mathbf{z}/\mathbf{w}) = f(\mathbf{z}, \mathbf{z})\delta(\mathbf{z}/\mathbf{w}) = f(\mathbf{w}, \mathbf{w})\delta(\mathbf{z}/\mathbf{w})$.

The formal residue for an element $f(\mathbf{z}) \in V[[\mathbf{z}, \mathbf{z}^{-1}]]$ is $\mathrm{Res}_{z_i} \sum_{\mathbf{n}\in\mathbb{Z}^{k+1}} a_{\mathbf{n}}\mathbf{z}^{\mathbf{n}} = \sum_{\substack{\mathbf{n} \\ n_i=-1}} a_{\mathbf{n}} z_i \mathbf{z}^{\mathbf{n}}$. Alternatively, we can define $\mathrm{Res}_{\mathbf{z}}$ as $\mathrm{Res}_{\mathbf{z}} \sum_{\mathbf{n}\in\mathbb{Z}^{k+1}} a_{\mathbf{n}}\mathbf{z}^{\mathbf{n}} = a_{-\mathbf{1}}$, the coefficient of $\mathbf{z}^{-1}$. We introduce a slightly modified form of Kac's $\boldsymbol{\lambda}$-bracket notation and Fourier transform (see [19]). For any $a(\mathbf{z}, \mathbf{w}) = \sum_{\mathbf{m},\mathbf{n}} a_{\mathbf{m},\mathbf{n}}\mathbf{z}^{\mathbf{m}}\mathbf{w}^{\mathbf{n}}$, we define the Fourier transform $F_{\mathbf{z},\mathbf{w}}^{\boldsymbol{\lambda}} a(\mathbf{z}, \mathbf{w}) = \mathrm{Res}_{z_0} \cdots \mathrm{Res}_{z_N} e^{\sum_{i=0}^{k} \lambda_i(z_i - w_i)} a(\mathbf{z}, \mathbf{w})$.

For $\mathbf{j} = (j_0, \dots, j_k) \in \mathbb{N}^{k+1}$, set $\mathbf{j}\,! = j_0! j_1! \cdots j_k!$, $\partial_{w_i}^{(j_i)} = \frac{1}{j_i!}\partial_{w_i}^{j_i}$, and $\partial^{(\mathbf{j})} = \prod_{j=0}^{k} \partial_{w_i}^{(j_i)}$. We also write $\boldsymbol{\lambda}^{(\mathbf{j})} := \boldsymbol{\lambda}^{\mathbf{j}}/\mathbf{j}\,! = \prod_{i=0}^{k} \frac{\lambda_i^{j_i}}{j_i!}$. Using this notation allows us to compress many of the formal series we will encounter, due to the following identity

$$F_{\mathbf{z},\mathbf{w}}^{\boldsymbol{\lambda}} \partial^{(\mathbf{j})} \delta(\mathbf{z}/\mathbf{w}) = \boldsymbol{\lambda}^{(\mathbf{j})}. \tag{1}$$

To prove identity (1), we recall a few properties shown in [19, Proposition 2.1]: For $j > 0$,

$$\mathrm{Res}_z \partial_w^{(j)} \delta(z/w) = 0, \;\; (z-w)\partial_w^{(j+1)}\delta(z/w) = \partial_w^{(j)}\delta(z/w), \;\; (z-w)^{j+1}\partial_w^{(j)}\delta(z/w) = 0.$$

Thus,

$$\begin{aligned} F_{\mathbf{z},\mathbf{w}}^{\boldsymbol{\lambda}} \partial^{(\mathbf{j})} \delta(\mathbf{z}/\mathbf{w}) &= \mathrm{Res}_{z_0} \cdots \mathrm{Res}_{z_n} e^{\sum_{i=0}^{k} \lambda_i(z_i - w_i)} \partial^{(\mathbf{j})} \delta(\mathbf{z}/\mathbf{w}) \\ &= \mathrm{Res}_{z_0} \cdots \mathrm{Res}_{z_n} \Big(\prod_{i=0}^{k} \Big(\sum_{k_i=0}^{\infty} \tfrac{1}{k_i!} \lambda_i^{k_i} (z_i - w_i)^{k_i} \Big) \prod_{i=0}^{k} \partial_{w_i}^{(j_i)} \delta(z_i/w_i) \Big) \end{aligned}$$

$$= \prod_{i=0}^{k} \mathrm{Res}_{z_i} \Big(\sum_{k_i=0}^{\infty} \tfrac{1}{k_i!} \lambda_i^{k_i} (z_i - w_i)^{k_i} \Big) \partial_{w_i}^{(j_i)} \delta(z_i/w_i) = \prod_{i=0}^{k} \tfrac{\lambda_i^{j_i}}{j_i!} = \boldsymbol{\lambda}^{(\mathbf{j})}.$$

If $a(\mathbf{z})$, $b(\mathbf{w})$ and $c^{\mathbf{j}}(\mathbf{w})$ are formal distributions satisfying

$$[a(\mathbf{z}), b(\mathbf{w})] = \sum_{\mathbf{j} \in \mathbb{N}^{k+1}} c^{\mathbf{j}}(\mathbf{w}) \partial^{(\mathbf{j})} \delta(\mathbf{z}/\mathbf{w}),$$

we have $F^{\boldsymbol{\lambda}}_{\mathbf{z},\mathbf{w}}[a(\mathbf{z}), b(\mathbf{w})] = \sum_{\mathbf{j} \in \mathbb{N}^{k+1}} c^{\mathbf{j}}(\mathbf{w}) \boldsymbol{\lambda}^{(\mathbf{j})}$. The $\boldsymbol{\lambda}$-*bracket* is defined as

$$[a(\mathbf{w})_{\boldsymbol{\lambda}} b(\mathbf{w})] = \sum_{\mathbf{j} \in \mathbb{N}^{k+1}} c^{\mathbf{j}}(\mathbf{w}) \boldsymbol{\lambda}^{(\mathbf{j})},$$

achieving the compressed notation. When the variables are clear from the context, we sometimes omit the formal multivariables $\mathbf{z}, \mathbf{w}$. Properties of the $\boldsymbol{\lambda}$-bracket that we use frequently include

$$[a_{\boldsymbol{\lambda}} [b_{\boldsymbol{\eta}} c]] = [[a_{\boldsymbol{\lambda}} b]_{\boldsymbol{\lambda}+\boldsymbol{\eta}} c] + [b_{\boldsymbol{\eta}} [a_{\boldsymbol{\lambda}} c]], \quad [a_{\boldsymbol{\lambda}} (bc)] = [a_{\boldsymbol{\lambda}} b] c + b [a_{\boldsymbol{\lambda}} c]. \tag{2}$$

2.1 The toroidal Lie algebra. Fix a positive integer N. We define the toroidal Lie algebra $\tau(A_n)$ by generators and relations. The generators are

$$K_{\mathbf{m},j}, \quad H_i(\mathbf{m}), \quad E_i(\mathbf{m}), \quad F_i(\mathbf{m}) \quad (0 \le i \le n,\ 0 \le j \le N,\ \mathbf{m} \in \mathbb{Z}^{N+1}).$$

We will use generating functions to write the relations of the algebra.

The generators of $\tau(A_n)$ $(1 \le i \le n,\ 0 \le s \le N)$ have generating functions:

$$\begin{aligned} K_s(\mathbf{z}) &= \sum_{\mathbf{m} \in \mathbb{Z}^{N+1}} K_{\mathbf{m},s} \mathbf{z}^{-\mathbf{m}}, \qquad & H_i(\mathbf{z}) &= \sum_{\mathbf{m} \in \mathbb{Z}^{N+1}} H_i(\mathbf{m}) \mathbf{z}^{-\mathbf{m}}, \\ E_i(\mathbf{z}) &= \sum_{\mathbf{m} \in \mathbb{Z}^{N+1}} E_i(\mathbf{m}) \mathbf{z}^{-\mathbf{m}}, & F_i(\mathbf{z}) &= \sum_{\mathbf{m} \in \mathbb{Z}^{N+1}} E_i(\mathbf{m}) \mathbf{z}^{-\mathbf{m}}. \end{aligned}$$

Let $\partial_{z_i} = \frac{\partial}{\partial z_i}$ denote the formal differentiation. Define the operator $D_{z_i} = \frac{\partial}{\partial z_i}$, and $D := \sum_{s=0}^{N} D_{z_s}$ (the indeterminate in use is understood in the context of the formula). Then

$$K(\mathbf{z}) \cdot D = \sum_{i=0}^{N} K_i(\mathbf{z}) \tfrac{\partial}{\partial z_i}, \quad K(\mathbf{z}) = \sum_{i=0}^{N} K_i(\mathbf{z}) = \sum_{i=0}^{N} \sum_{\mathbf{m}} K_{\mathbf{m},i} \mathbf{z}^{-\mathbf{m}} z_i,$$
$$D \cdot K(\mathbf{z}) = \sum_{s=0}^{N} \sum_{\mathbf{m} \in \mathbb{Z}^{N+1}} m_p K_{\mathbf{m},s} z^{\mathbf{m}}.$$

The relations of the toroidal algebra $\tau(A_n)$ are given by the generating functions:

(R0) $K_i(\mathbf{z})$ is central, $D \cdot K(\mathbf{z}) = 0$;

(R1) $[H_i(\mathbf{z}), H_j(\mathbf{w})] = A_{ij} K(\mathbf{w}) \cdot D \delta(\mathbf{z}/\mathbf{w})$;

(R2) $[H_i(\mathbf{z}), E_j(\mathbf{w})] = A_{ij} E_j(\mathbf{w}) \delta(\mathbf{z}/\mathbf{w})$, $[H_i(\mathbf{z}), F_j(\mathbf{w})] = -A_{ij} F_j(\mathbf{w}) \delta(\mathbf{z}/\mathbf{w})$;

(R3) $[E_i(\mathbf{z}), F_j(\mathbf{w})] = -\delta_{i,j} \big(H_i(\mathbf{w}) + \frac{2}{A_{ij}} K(\mathbf{w}) \cdot D \big) \delta(\mathbf{w}/\mathbf{z})$;

(R4) $[E_i(\mathbf{z}), E_i(\mathbf{w})] = 0 = [F_i(\mathbf{z}), F_i(\mathbf{w})]$, $\operatorname{ad} E_i^{-A_{ij}+1}(\mathbf{z}) E_j(\mathbf{w}) = 0$ for $i \neq j$, $\operatorname{ad} F_i^{-A_{ij}+1}(\mathbf{z}) F_j(\mathbf{w}) = 0$ for $i \neq j$.

3 The Toroidal Heisenberg Algebra

Define the *toroidal Heisenberg algebra* $\mathfrak{B}$ as the Lie algebra with generators $b_i(\mathbf{r})$ ($1\leq i \leq n$) and $K_{\mathbf{r},p}$ ($0 \leq p \leq N$, $\mathbf{r} \in \mathbb{Z}^{n-1}$) which satisfy the following relations:

$$[b_i(\mathbf{r}), b_j(\mathbf{s})] = A_{ij} \sum_{p=0}^{N} \mathbf{r}_p K_{\mathbf{r}+\mathbf{s},p} \quad \text{and} \quad \sum_{p=0}^{N} \mathbf{r}_p K_{\mathbf{r},p} = 0 \quad \forall\, \mathbf{r} \in \mathbb{Z}^{N+1}. \tag{3}$$

Here A_{ij} denotes the (i,j)-th entry of the Cartan matrix A_n, where we have deleted the first row and column. If we set

$$b_0(\mathbf{m}) := -\sum_{i=1}^{n} b_i(\mathbf{m}), \tag{4}$$

then one can check that the first equality in (3) is also satisfied for $i = 0$ or $j = 0$.

3.1 Representation of the Heisenberg algebra. We define a polynomial ring over indeterminates indexed by $0 < i \leq n+1$ and $\mathbf{k} \in \mathbb{Z}^{N+1}$:

$$\mathbb{C}[\mathbf{y}] := \mathbb{C}\big[y_i(\mathbf{k}) \mid 0 < \mathbf{k} \in \mathbb{Z}^{N+1},\ 1 \leq i \leq n\big].$$

For fixed $\kappa_{\mathbf{m},p} \in \mathbb{C}$, $0 \leq p \leq N$ and $\lambda_i \in \mathbb{C}$, we define a map $\Phi : \mathfrak{B} \to \operatorname{End}\mathbb{C}[\mathbf{y}]$ below by an action on the generators. The construction of the map is similar to that appearing in [10]. The motivation for the definition of Φ uses heuristic ideas about how the toroidal Lie algebra "should" act on sections of certain (not well defined) line bundles. For readers who are interested in this heuristic type of construction, one could consult [13], [4] and [11]. The resulting map Φ is twisted as in [10] so that $\Phi(b_i(\mathbf{m}))$ is a well defined element of $\operatorname{End}\mathbb{C}[\mathbf{y}]$. The definition of $\Phi(b_0(\mathbf{m}))$ follows from the definition (4).

Proposition 3.1. (Realization of the Toroidal Heisenberg Algebra) *Fix $\kappa_{\mathbf{m},p} \in \mathbb{C}$, $0 \leq p \leq N$ and $\lambda_i \in \mathbb{C}$, where $0 \leq i \leq n$. Assume*

$$\sum_{p=0}^{N} m_p \kappa_{\mathbf{m},p} = 0 \ \text{ for all } \mathbf{m}, \tag{5}$$

$$\sum_{p=0}^{N} m_p \kappa_{-\mathbf{m}-\mathbf{n},p} = 0 \ \text{ for } \mathbf{m} > 0 \text{ and } \mathbf{n} > 0. \tag{6}$$

Then the map $\Phi : \mathfrak{B} \to \operatorname{End}\mathbb{C}[\mathbf{y}]$ given by

$$\begin{aligned}
\Phi\big(b_i(\mathbf{m})\big) =\ & \theta(-\mathbf{m}) \sum_{p=0}^{N} \sum_{\mathbf{s}>0} \Big(\partial_{y_{i-1}(\mathbf{s})} - \partial_{y_i(\mathbf{s})}\Big) m_p \kappa_{-\mathbf{m}+\mathbf{s},p} \\
& + \theta(\mathbf{m}) \sum_{p=0}^{N} \sum_{\mathbf{s}>0} \Big(\partial_{y_{i-1}(\mathbf{s})} - 2\partial_{y_i(\mathbf{s})} + \partial_{y_{i+1}(\mathbf{s})}\Big) m_p \kappa_{-\mathbf{m}+\mathbf{s},p} \\
& + \theta(-\mathbf{m}) y_i(-\mathbf{m}) - \delta_{\mathbf{m},0}\lambda_i, \\
\Phi(K_{\mathbf{m}+\mathbf{n},p}) =\ & -\kappa_{-\mathbf{m}-\mathbf{n},p}
\end{aligned}$$

for $1 \leq i \leq n$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^{N+1}$ defines a representation $\mathfrak{B}$ on $\mathbb{C}[\mathbf{y}]$.

Proof. For $1 < i, j \leq n$, we have

$$\begin{aligned}
&\big[\Phi\big(b_i(\mathbf{m})\big), \Phi\big(b_j(\mathbf{n})\big)\big] \\
&= \theta(-\mathbf{n})\theta(-\mathbf{m})\big(\delta_{j,i-1} - \delta_{j,i}\big) \sum_{p=0}^{N} m_p \kappa_{-\mathbf{m}-\mathbf{n},p} \\
&\quad + \theta(-\mathbf{n})\theta(\mathbf{m})\big(\delta_{j,i-1} - 2\delta_{i,j} + \delta_{j,i+1}\big) \sum_{p=0}^{N} m_p \kappa_{-\mathbf{m}-\mathbf{n},p} \\
&\quad - \theta(-\mathbf{m})\theta(-\mathbf{n})\big(\delta_{i,j-1} - \delta_{i,j}\big) \sum_{q=0}^{N} n_q \kappa_{-\mathbf{m}-\mathbf{n},q} \\
&\quad - \theta(-\mathbf{m})\theta(\mathbf{n})\big(\delta_{i,j-1} - 2\delta_{i,j} + \delta_{i,j+1}\big) \sum_{q=0}^{N} n_q \kappa_{-\mathbf{m}-\mathbf{n},q} \\
&= \big(\theta(\mathbf{n})\theta(-\mathbf{m}) + \theta(-\mathbf{n})\theta(\mathbf{m}) + \theta(-\mathbf{m})\theta(-\mathbf{n})\big)\big(\delta_{j,i-1} - 2\delta_{i,j} + \delta_{j,i+1}\big) \sum_{p=0}^{N} m_p \kappa_{-\mathbf{m}-\mathbf{n},p} \\
&= \big(\delta_{j,i-1} - 2\delta_{i,j} + \delta_{j,i+1}\big) \sum_{p=0}^{N} m_p \kappa_{-\mathbf{m}-\mathbf{n},p} = -A_{ij} \sum_{p=0}^{N} m_p \kappa_{-\mathbf{m}-\mathbf{n},p},
\end{aligned}$$

where in the last two equalities we used the hypotheses (5) and (6), respectively. The remaining relations are also straightforward. ☐

4 Main Result, the Representation of the Toroidal Algebra

Let $i, j \leq n+1$, $\mathbf{m} \in \mathbb{Z}^{N+1}$ and

$$\mathbb{C}[\mathbf{x}] := \mathbb{C}\big[x_{ij}(\mathbf{m}) \,|\, 0 < i < j \leq n+1,\, \mathbf{m} \in \mathbb{Z}^{N+1}\big].$$

The elements $x_{ij}(\mathbf{m})$ act via multiplication on the ring $\mathbb{C}[\mathbf{x}]$, and hence on the ring $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$ (as $x_{ij}(\mathbf{m}) \otimes 1$). Define the following differential operators acting on the polynomial ring $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$:

$$a_{ij,\mathbf{m}} := -x_{ij}(\mathbf{m}), \quad a^*_{ij,\mathbf{m}} := \tfrac{\partial}{\partial x_{ij}(-\mathbf{m})}\,. \tag{7}$$

With corresponding generating functions

$$a_{ij}(\mathbf{z}) = \sum_{\mathbf{m} \in \mathbb{Z}^{N+1}} a_{ij,\mathbf{m}} \mathbf{z}^{-\mathbf{m}}, \quad a^*_{ij}(\mathbf{z}) = \sum_{\mathbf{m} \in \mathbb{Z}^{N+1}} a^*_{ij,\mathbf{m}} \mathbf{z}^{-\mathbf{m}}, \quad \kappa_s(\mathbf{z}) = \sum_{\mathbf{m} \in \mathbb{Z}^{N+1}} \kappa_{\mathbf{m},s} \mathbf{z}^{\mathbf{m}},$$

define the operators

$$\kappa(\mathbf{z}) \cdot D = \sum_{i=0}^{N} \kappa_i(\mathbf{z}) \tfrac{\partial}{\partial z_i}, \quad \kappa(\mathbf{z}) = \sum_{i=0}^{N} \kappa_i(\mathbf{z}) = \sum_{i=0}^{N} \sum_{\mathbf{m}} \kappa_{\mathbf{m},i} \mathbf{z}^{\mathbf{m}} z_i.$$

Note that $\kappa(\mathbf{z}) \cdot D$ is a weighted version of Euler's differential operator. The operators $\Phi(b_i)$ commute with the $a_{ij,\mathbf{m}}$, $a^*_{ij,\mathbf{m}}$ and act on $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$ as $1 \otimes \Phi(b_i)$.

Theorem 4.1. (Realization) *Let $\kappa_{\mathbf{m},l}$ be fixed complex numbers satisfying conditions* (5) *and* (6) *and fix $\lambda_i \in \mathbb{C}$ for $0 \leq i \leq n$. Then the generating functions given below*

$$\rho(F_r)(\mathbf{z}) \,=\, a_{r,r+1}(\mathbf{z}) - \sum_{j=1}^{r-1} a_{j,r+1}(\mathbf{z}) a^*_{jr}(\mathbf{z}),$$

$$
\begin{aligned}
\rho(H_r)(\mathbf{z}) &= 2a_{r,r+1}(\mathbf{z})a^*_{r,r+1}(\mathbf{z}) + \sum_{i=1}^{r-1}\left(a_{i,r+1}(\mathbf{z})a^*_{i,r+1}(\mathbf{z}) - a_{ir}(\mathbf{z})a^*_{ir}(\mathbf{z})\right) \\
&\quad + \sum_{j=r+2}^{n+1}\left(a_{rj}(\mathbf{z})a^*_{rj}(\mathbf{z}) - a_{r+1,j}(\mathbf{z})a^*_{r+1,j}(\mathbf{z})\right) + \Phi(b_r)(\mathbf{z}), \\
\rho(E_r)(\mathbf{z}) &= a_{r,r+1}(\mathbf{z})a^*_{r,r+1}(\mathbf{z})a^*_{r,r+1}(\mathbf{z}) \\
&\quad - \sum_{j=r+2}^{n+1} a_{r+1,j}(\mathbf{z})a^*_{rj}(\mathbf{z}) + \sum_{j=1}^{r-1} a_{jr}(\mathbf{z})a^*_{j,r+1}(\mathbf{z}) \\
&\quad + \sum_{j=r+2}^{n+1}\left(a_{rj}(\mathbf{z})a^*_{rj}(\mathbf{z}) - a_{r+1,j}(\mathbf{z})a^*_{r+1,j}(\mathbf{z})\right) a^*_{r,r+1}(\mathbf{z}) \\
&\quad + a^*_{r,r+1}(\mathbf{z})\Phi(b_r)(\mathbf{z}) + \kappa\cdot Da^*_{r,r+1}(\mathbf{z})
\end{aligned}
$$

for $1 \le r \le n$*, together with*

$$
\begin{aligned}
\rho(E_0)(\mathbf{z}) &= -a_{1,n+1}(\mathbf{z}), \\
\rho(H_0)(\mathbf{z}) &= -\sum_{r=1}^{n}\rho(H_r)(\mathbf{z}) \\
&= -\sum_{r=1}^{n} a_{r,n+1}(\mathbf{z})a^*_{r,n+1}(\mathbf{z}) - \sum_{r=2}^{n+1}(\mathbf{z})a_{1r}(\mathbf{z})a^*_{1r}(\mathbf{z}) + \Phi(b_0)(\mathbf{z}), \\
\rho(F_0)(\mathbf{z}) &= \sum_{1\le r<j\le n+1} -a_{rj}(\mathbf{z}) \sum_{\mathbf{q};j=q_i;r\ge q_{i-1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}}(\mathbf{z})a^*_{r,n+1}(\mathbf{z}) \\
&\quad - \sum_{1\le r<n+1}\ \sum_{r\ge q_i,\mathbf{q}}\ \prod_{j=1}^{i} a^*_{q_j q_{j+1}}(\mathbf{z})\Phi(b_r)(\mathbf{z}) \\
&\quad - \sum_{1\le r<n+1}\ \sum_{r=q_i,\mathbf{q}}\ \prod_{j=1}^{i-1} a^*_{q_j q_{j+1}}(\mathbf{z})\kappa\cdot Da^*_{r,n+1}(\mathbf{z}),
\end{aligned}
$$

define an action of the generators $E_r(\mathbf{m})$*,* $F_r(\mathbf{m})$ *and* $H_r(\mathbf{m})$ *and a representation of* $\tau(A_n)$ *on the Fock space* $\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}]$*. In the partitions above,* $1 = q_1 < q_2 < \cdots < q_i$*,* $q_{i+1} = n+1$*. In addition,* $K_{\mathbf{m},l}$ *acts as left multiplication by* $-\kappa_{-\mathbf{m},l}$*.*

Note that one should also have $\rho(E_{lk}) = -a_{kl} + \sum_{j=1}^{k-1} a_{jl}a^*_{jk}$ for $k < l$, but we do not seem to need this general formula, so we do not determine whether it is always true.

5 Proof of the Main Result

We should point out that the proof requires very lengthy (at least to us) calculations. We have selected representative portions of the calculations to include here, from an original manuscript of over one hundred pages, a version of which is available (see [8]). Calculations similar to those omitted can be found in [9] and [10], students may also wish to specialize to the special cases of type A_2 and A_3 especially the latter which is a good guide for the general setting of $n \ge 2$.

Let $\Phi(b_r) := \Phi(b_r)(\mathbf{z}) = \sum_{\mathbf{m}} \Phi(b_r)(\mathbf{m})\mathbf{z}^{-\mathbf{m}}$, then we can write the last calculation in the proof of Proposition 3.1 as $[\Phi(b_r)_{\boldsymbol{\lambda}}\Phi(b_s)] = A_{rs}\sum_{l=0}^{N}\rho(K_l)\lambda_l = -A_{rs}\kappa\cdot\boldsymbol{\lambda}$. Set

$K_l(\mathbf{w}) := \big(\sum_{\mathbf{r}} K_{\mathbf{r},l} w^{-\mathbf{r}}\big) w_l$. Relations (R0) are satisfied by the definition of $\kappa_{\mathbf{m},l}$. The relations (R1)–(R4) will follow if the following $\boldsymbol{\lambda}$-brackets are satisfied:

(T1) $[\rho(H_i)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(H_j)(\mathbf{w})] = A_{ij}\sum_{l=0}^{N}\rho(K_l)(\mathbf{w})\lambda_l$ $(0 \le i,j \le n)$;

(T2) $[\rho(H_i)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(E_j)(\mathbf{w})] = A_{ij}\rho(E_j)(\mathbf{w})$, $[\rho(H_i)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(F_j)(\mathbf{w})] = -A_{ij}\rho(F_j)(\mathbf{w})$;

(T3) $[\rho(E_i)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(F_j)(\mathbf{w})] = -\delta_{i,j}\Big(\rho(H_i)(\mathbf{w}) + \frac{2}{A_{ij}}\sum_{l=0}^{N}\rho(K_l)(\mathbf{w})\lambda_l\Big)$;

(T4) $[\rho(E_i)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(E_j)(\mathbf{w})] = 0 = [\rho(F_i)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(F_j)(\mathbf{w})]$ if $|i-j| \neq 1$.
$[\rho(E_i(\mathbf{w})_{\boldsymbol{\lambda}})[\rho(E_i)(\mathbf{w})_{\boldsymbol{\mu}}\rho(E_j)(\mathbf{w})]] = 0$ if $i = j \pm 1$.
$[\rho(F_i)(\mathbf{w})_{\boldsymbol{\lambda}}[\rho(F_i)(\mathbf{w})_{\boldsymbol{\mu}}\rho(F_j)(\mathbf{w})]] = 0$ if $i = j \pm 1$.

Proof. We demonstrate how to write the relation (R1) in $\boldsymbol{\lambda}$-bracket form:

$$\begin{aligned}&\sum_{\mathbf{m},\mathbf{n}}[H_i(\mathbf{m}), H_j(\mathbf{n})]\mathbf{z}^{-\mathbf{m}}\mathbf{w}^{-\mathbf{n}}\\ &= A_{ij}\sum_{l=0}^{N}K_l(\mathbf{w})\partial_{w_l}\delta(\mathbf{z}/\mathbf{w}) = A_{ij}\sum_{\mathbf{j}\in\mathbb{N}^{N+1}}c^{\mathbf{j}}(\mathbf{w})\partial^{(\mathbf{j})}\delta(\mathbf{z}/\mathbf{w}),\end{aligned}$$

where $c^{\mathbf{j}}(\mathbf{w})$ is defined as follows: we take $\mathbf{e}_l$ to be the N-tuple with 1 in the l-th position and zeros elsewhere, and define $c^{\mathbf{e}_l}(\mathbf{w}) = K_l(\mathbf{w})$ and $c^{\mathbf{j}}(\mathbf{w}) = 0$ if $\mathbf{j} \neq e_l$ for some $0 \le l \le N$. Applying $F^{\boldsymbol{\lambda}}$ gives the result. □

5.1 Preliminary lemmas. We have the following identities for $a_{i,j,\mathbf{m}}$ and $a^*_{i,j,\mathbf{m}}$ as in (7), whose proofs carry over from [10, Lemma 4.1]. The identities are written which we write in terms of the $\boldsymbol{\lambda}$-bracket. In the interest of compressing the notation, we will often suppress the variables $\mathbf{z}, \mathbf{w}$ in the computations, especially when using the $\boldsymbol{\lambda}$-notation, where the presence of the multivariable $\mathbf{w}$ is assumed.

Lemma 5.1. [10] *Let $i,j,k,l \in \mathbb{Z}$. Then for the generating functions $a_{ij}(\mathbf{w})$ and $a^*_{ij}(\mathbf{w})$, the following identities hold:*

(a) $[a_{ij}(\mathbf{w})\boldsymbol{\lambda} a^*_{kl}(\mathbf{w})] = \delta_{i,k}\delta_{j,l}$,

(b) $[a_{ij}(\mathbf{w})a^*_{ij}(\mathbf{w})\boldsymbol{\lambda} a_{ij}(\mathbf{w})a^*_{ij}(\mathbf{w})] = 0$,

(c) $[a_{ij}(\mathbf{w})_{\boldsymbol{\lambda}}\kappa\cdot Da^*_{kl}(\mathbf{w})] = \delta_{i,k}\delta_{j,l}\sum_{p=0}^{N}\kappa_p\lambda_p = [\kappa\cdot Da^*_{kl}(\mathbf{w})\boldsymbol{\lambda} a_{ij}(\mathbf{w})]$,

(d) $\sum_{j=r+2}^{n+1}\sum_{k=1}^{s-1}\Big[a_{ks}(\mathbf{w})a^*_{k,s+1}(\mathbf{w})\boldsymbol{\lambda} a_{rj}(\mathbf{w})a^*_{rj}(\mathbf{w})\Big] = -\delta_{s,r+1}a_{r,r+1}(\mathbf{w})a^*_{r,r+2}(\mathbf{w})$,

(e) $\sum_{j=1}^{r-1}\sum_{k=s+2}^{n+1}\Big[a_{s+1,k}(\mathbf{w})a^*_{sk}(\mathbf{w})\boldsymbol{\lambda} a_{jr}(\mathbf{w})a^*_{j,r+1}(\mathbf{w})\Big] = 0$,

(f) $\sum_{j=r+2}^{n+1}\sum_{k=1}^{s-1}\Big[a_{ks}(\mathbf{w})a^*_{k,s+1}(\mathbf{w})\boldsymbol{\lambda} a_{r+1,j}(\mathbf{w})a^*_{r+1,j}(\mathbf{w})\Big] = 0$,

(g) $$\begin{aligned}&\sum_{j=r+2}^{n+1}\sum_{k=s+2}^{n+1}\Big[a_{s+1,k}(\mathbf{w})a^*_{sk}(\mathbf{w})_{\boldsymbol{\lambda}}\big(a_{rj}(\mathbf{w})a^*_{rj}(\mathbf{w}) - a_{r+1,j}(\mathbf{w})a^*_{r+1,j}(\mathbf{w})\big)\Big]\\ &= -2\delta_{r,s}\sum_{j=r+2}^{n+1}a_{r+1,j}(\mathbf{w})a^*_{rj}(\mathbf{w}) + \delta_{r,s+1}\sum_{j=r+2}^{n+1}a_{rj}(\mathbf{w})a^*_{r-1,j}(\mathbf{w})\\ &\quad + \delta_{s,r+1}\sum_{j=r+3}^{n+1}a_{r+2,k}(\mathbf{w})a^*_{r+1,j}(\mathbf{w}),\end{aligned}$$

(h) $\sum_{j=1}^{r-1}\left[a^*_{s,s+1}(\mathbf{w})_{\boldsymbol\lambda} a_{jr}a^*_{j,r+1}(\mathbf{w})\right] = -\delta_{r,s+1}a^*_{r-1,r+1}(\mathbf{w})$,

(i) $\sum_{j=r+2}^{n+1}\left[a^*_{s,s+1}(\mathbf{w})_{\boldsymbol\lambda} a_{r+1,j}(\mathbf{w})a^*_{rj}(\mathbf{w})\right] = -\delta_{r+1,s}a^*_{r,r+2}(\mathbf{w})$.

Proof. Only statement (c) is new,

$$[a_{ij}(\mathbf{z}),\kappa(\mathbf{w})\cdot Da^*_{kl}(\mathbf{w})] = \sum_{\mathbf{m}}\sum_{p=0}^{N}\sum_{\mathbf{n},\mathbf{q}}\kappa_{\mathbf{n},p}[a_{ij}(\mathbf{m}),a^*_{kl}(\mathbf{q})]\mathbf{w}^{\mathbf{n}}w_p\tfrac{\partial}{\partial w_p}\mathbf{z}^{-\mathbf{m}}\mathbf{w}^{-\mathbf{q}}$$
$$= \delta_{i,k}\delta_{j,l}\sum_{p=0}^{N}\sum_{\mathbf{n}}\kappa_{\mathbf{n},p}\mathbf{w}^{\mathbf{n}}\sum_{\mathbf{m}}m_p\mathbf{z}^{-\mathbf{m}}\mathbf{w}^{\mathbf{m}} = \delta_{i,k}\delta_{j,l}\sum_{p=0}^{N}\sum_{\mathbf{n}}\kappa_{\mathbf{n},p}\mathbf{w}^{\mathbf{n}}w_p\tfrac{\partial}{\partial w_p}\delta(\mathbf{z}/\mathbf{w})$$

and

$$[\kappa(\mathbf{z})\cdot Da^*_{kl}(\mathbf{z}),a_{ij}(\mathbf{w})]$$
$$= \sum_{\mathbf{m}}\sum_{p=0}^{N}\sum_{\mathbf{n},\mathbf{q}}\kappa_{\mathbf{n},p}[a^*_{kl}(\mathbf{q}),a_{ij}(\mathbf{m})]\,\mathbf{z}^{\mathbf{n}}\mathbf{w}^{-\mathbf{m}}z_p\tfrac{\partial}{\partial z_p}\mathbf{z}^{-\mathbf{q}}$$
$$= -\delta_{i,k}\delta_{j,l}\sum_{p=0}^{N}\sum_{\mathbf{n}}\kappa_{\mathbf{n},p}\mathbf{z}^{\mathbf{n}}\sum_{\mathbf{m}}m_p\mathbf{z}^{\mathbf{m}}\mathbf{w}^{-\mathbf{m}} = \delta_{i,k}\delta_{j,l}\sum_{p=0}^{N}\sum_{\mathbf{n}}\kappa_{\mathbf{n},p}\mathbf{z}^{\mathbf{n}}w_p\tfrac{\partial}{\partial w_p}\delta(\mathbf{z}/\mathbf{w})$$
$$= \delta_{i,k}\delta_{j,l}\sum_{p=0}^{N}\sum_{\mathbf{n}}\kappa_{\mathbf{n},p}\mathbf{w}^{\mathbf{n}}w_p\tfrac{\partial}{\partial w_p}\delta(\mathbf{z}/\mathbf{w}) + \delta_{i,k}\delta_{j,l}\sum_{p=0}^{N}w_p\tfrac{\partial}{\partial w_p}\Big(\sum_{\mathbf{n}}\kappa_{\mathbf{n},p}\mathbf{w}^{\mathbf{n}}\Big)\delta(\mathbf{z}/\mathbf{w})$$
$$= \delta_{i,k}\delta_{j,l}\sum_{p=0}^{N}\sum_{\mathbf{n}}\kappa_{\mathbf{n},p}\mathbf{w}^{\mathbf{n}}w_p\tfrac{\partial}{\partial w_p}\delta(\mathbf{z}/\mathbf{w}) + \delta_{i,k}\delta_{j,l}\sum_{\mathbf{n}}\Big(\sum_{p=0}^{N}n_p\kappa_{\mathbf{n},p}\Big)\mathbf{w}^{\mathbf{n}}\delta(\mathbf{z}/\mathbf{w})$$
$$= \delta_{i,k}\delta_{j,l}\sum_{p=0}^{N}\sum_{\mathbf{n}}\kappa_{\mathbf{n},p}\mathbf{w}^{\mathbf{n}}w_p\tfrac{\partial}{\partial w_p}\delta(\mathbf{z}/\mathbf{w})$$

by the relation (R0). Now we take the Fourier $F^{\boldsymbol\lambda}_{\mathbf{z},\mathbf{w}}$ transform of the above, obtaining $[a_{ij}{}_{\boldsymbol\lambda}\kappa\cdot Da^*_{kl}] = [\kappa\cdot Da^*_{kl}{}_{\boldsymbol\lambda}a_{ij}] = \delta_{i,k}\delta_{j,l}\sum_{p=0}^{N}\kappa_p\lambda_p = \delta_{i,k}\delta_{j,l}\kappa\cdot\boldsymbol\lambda$. □

In addition, the following consequences of Lemma 5.1 are useful:

Lemma 5.2. *The following identities hold:*

$$\left[a_{mn}(\mathbf{w})a^*_{mn}(\mathbf{w})_{\boldsymbol\lambda} a_{js+1}(\mathbf{w})a^*_{js}(\mathbf{w})\right] = \delta_{mj}\begin{cases}-a_{j,s+1}(\mathbf{w})a^*_{js}(\mathbf{w}) & \text{if } n=s+1,\\ a_{j,s+1}(\mathbf{w})a^*_{js}(\mathbf{w}) & \text{if } n=s,\end{cases} \tag{8}$$

$$[a_{ij}(\mathbf{w})a^*_{ij}(\mathbf{w})_{\boldsymbol\lambda}\kappa Da^*_{mn}(\mathbf{w})] = \delta_{im}\delta_{jn}\kappa\cdot Da^*_{ij}(\mathbf{w}) + \delta_{im}\delta_{jn}a^*_{ij}(\mathbf{w})\kappa\cdot\boldsymbol\lambda. \tag{9}$$

Proof. We prove only the second relation (9) and leave the other to the reader. By Lemma 5.1(c) and by properties of $\delta(\mathbf{z}/\mathbf{w})$, one has

$$[a_{ij}(\mathbf{z})a^*_{ij}(\mathbf{z}),\kappa_w D_w a^*_{mn}(\mathbf{w})] = \delta_{im}\delta_{jn}a^*_{ij}(\mathbf{z})\kappa_w D_w\delta(\mathbf{z}/\mathbf{w})$$
$$= \delta_{im}\delta_{jn}\kappa_w D_w(a^*_{ij}(\mathbf{z})\delta(\mathbf{z}/\mathbf{w})) = \delta_{im}\delta_{jn}\kappa_w D_w(a^*_{ij}(\mathbf{w})\delta(\mathbf{z}/\mathbf{w}))$$
$$= \delta_{im}\delta_{jn}(\kappa_w D_w a^*_{ij}(\mathbf{w}))\delta(\mathbf{z}/\mathbf{w}) + \delta_{im}\delta_{jn}a^*_{ij}(\mathbf{w})\kappa_w D_w\delta(\mathbf{z}/\mathbf{w})).$$

Note that the formal multivariable in the series $a_{ij}(\mathbf{z})$ is not affected by the operator $\kappa(\mathbf{w})D_w$ which acts on series in $\mathbf{w}$. Now applying the transform $F^{\boldsymbol\lambda}_{\mathbf{z},\mathbf{w}}$ gives the result. □

5.2 Relations involving $H(\mathbf{z})$. The relations (T1) and (T2) involving the $H_i(\mathbf{z})$ are simpler to verify than those of type (T3) and (T4), so we begin with them. A reader familiar with other free field representations or vertex algebras can verify relation (T1) as an exercise (see also [9]). Because of the definition of a_{ij} and a^*_{ij} given in (7), there are no multiple contractions when computing out the operator product expansion for these terms. To further compress the notation, we sometimes omit the multivariable $\mathbf{w}$ in our computations when the variable is clear from the context.

Lemma 5.3. (T2) $[\rho(H_r)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(E_s)(\mathbf{w})] = A_{rs}\rho(E_s)(\mathbf{w})$.

Proof. First assume $r, s \neq 0$. If $|r-s| > 1$, observe that the indices of a_{ij} and a^*_{ij} that appear in $\rho(H_r)(\mathbf{z})$ and $\rho(E_s)(\mathbf{w})$ are disjoint and thus by Lemma 5.1(a) contribute nothing to the $\boldsymbol{\lambda}$-bracket $[\rho(H_r)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(E_s)(\mathbf{w})]$ (or equivalently to the commutator $[\rho(H_r)(\mathbf{z}), \rho(E_s)(\mathbf{w})]$). The remaining terms coming from the b_j have trivial commutator and thus $[\rho(H_r)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(E_s)(\mathbf{w})] = 0$.

Now assume $r = s$ (with $r, s \neq 0$). In this case, $\rho(E_r)(\mathbf{w})$ is equal to

$$\begin{aligned} &a_{r,r+1}a^*_{r,r+1}a^*_{r,r+1} + \sum_{j=r+2}^{n+1}\left(a_{rj}a^*_{rj} - a_{r+1,j}a^*_{r+1,j}\right)a^*_{r,r+1} \\ &+ \sum_{j=1}^{r-1} a_{jr}a^*_{j,r+1} - \sum_{j=r+2}^{n+1} a_{r+1,j}a^*_{rj} + a^*_{r,r+1}\Phi(b_r) + \kappa\cdot Da^*_{r,r+1}, \end{aligned}$$

and $\rho(H_r)(\mathbf{w})$ expands to

$$2a_{r,r+1}a^*_{r,r+1} + \sum_{i=1}^{r-1}\left(a_{i,r+1}a^*_{i,r+1} - a_{ir}a^*_{ir}\right) + \sum_{j=r+2}^{n+1}\left(a_{rj}a^*_{rj} - a_{r+1,j}a^*_{r+1,j}\right) + \Phi(b_r)$$

(where we have suppressed the variable $\mathbf{w}$). Now

$$\begin{aligned} &2\left[a_{r,r+1}a^*_{r,r+1}{}_{\boldsymbol{\lambda}}\rho(E_r)\right] \\ &= 2a_{r,r+1}a^*_{r,r+1}a^*_{r,r+1} + 2\sum_{j=r+2}^{n+1}\left(a_{r,j}a^*_{r,j} - a_{r+1,j}a^*_{r+1,j}\right)a^*_{r,r+1} \\ &\quad + 2a^*_{r,r+1}\Phi(b_r) + 2\kappa\cdot Da^*_{r,r+1} + 2a^*_{r,r+1}\sum_{l=0}^{N}\kappa_l\lambda_l. \end{aligned} \tag{10}$$

The second summation in $\rho(H_r)(\mathbf{w})$ contributes

$$\begin{aligned} &\sum_{i=1}^{r-1}\left[\left(a_{i,r+1}a^*_{i,r+1} - a_{ir}a^*_{ir}\right)_{\boldsymbol{\lambda}}\rho(E_r)\right] \\ &= \sum_{i=1}^{r-1}\left[\left(a_{i,r+1}a^*_{i,r+1} - a_{ir}a^*_{ir}\right)_{\boldsymbol{\lambda}} a_{ir}a^*_{i,r+1}\right] = 2\sum_{i=1}^{r-1} a_{ir}a^*_{i,r+1}. \end{aligned} \tag{11}$$

Now in the third summation in $\rho(H_r)(\mathbf{w})$, the index j is greater than or equal to $r+2$, and so commutes with all but the second and fourth terms of $\rho(E_r)(\mathbf{w})$ above, thus

$$\begin{aligned}
&\sum_{j=r+2}^{n+1}\left[\left(a_{rj}a_{rj}^* - a_{r+1,j}a_{r+1,j}^*\right)_{\boldsymbol{\lambda}}\rho(E_r)\right] \\
= &\sum_{j=r+2}^{n+1}\Big(\left[a_{rj}a_{rj}^*{}_{\boldsymbol{\lambda}}a_{rj}a_{rj}^*\right] + \left[a_{r+1,j}a_{r+1,j}^*{}_{\boldsymbol{\lambda}}a_{r+1,j}a_{r+1,j}^*\right]\Big) \\
&-\sum_{i=r+2}^{n+1}\sum_{j=r+2}^{n+1}\left[\left(a_{ri}a_{ri}^* - a_{r+1,i}a_{r+1,i}^*\right)_{\boldsymbol{\lambda}}a_{r+1,j}a_{rj}^*\right] \\
= &-2\sum_{j=r+2}^{n+1}a_{r+1,j}a_{rj}^* \qquad (12)
\end{aligned}$$

by Lemma 5.1(b) and (g). The last term in $\rho(H_r)(\mathbf{w})$ contributes

$$\left[\Phi(b_r)_{\boldsymbol{\lambda}}\rho(E_r)\right] = \left[\Phi(b_r)_{\boldsymbol{\lambda}}a_{r,r+1}^*\Phi(b_r)\right] = -2a_{r,r+1}^*\sum_{p=0}^{N}\kappa_p\lambda_p. \qquad (13)$$

The previous four calculations (10), (11), (12) and (13) sum up to give us the desired result $[\rho(H_r)_{\boldsymbol{\lambda}}\rho(E_r)] = 2\rho(E_r)$.

Now suppose $s = r+1$ so that $\rho(E_{r+1})(\mathbf{w})$ is equal to

$$\begin{aligned}
&a_{r+1,r+2}(a_{r+1,r+2}^*)^2 + \sum_{j=r+3}^{n+1}\left(a_{r+1,j}a_{r+1,j}^* - a_{r+2,j}a_{r+2,j}^*\right)a_{r+1,r+2}^* \\
&+\sum_{i=1}^{r}a_{i,r+1}a_{i,r+2}^* - \sum_{j=r+3}^{n+1}a_{r+2,j}a_{r+1,j}^* + a_{r+1,r+2}^*\Phi(b_{r+1}) + \kappa\cdot Da_{r+1,r+2}^*.
\end{aligned}$$

Then the first summand in $\rho(H_r)(\mathbf{w})$ contributes

$$2\left[a_{r,r+1}a_{r,r+1}^*{}_{\boldsymbol{\lambda}}\rho(E_{r+1})\right] = -2a_{r,r+1}a_{r,r+2}^*.$$

The second summation in $H_r(\mathbf{w})$ contributes

$$\begin{aligned}
&\sum_{i=1}^{r-1}\left[\left(a_{i,r+1}a_{i,r+1}^* - a_{ir}a_{ir}^*\right)_{\boldsymbol{\lambda}}\rho(E_{r+1})\right] \\
= &\sum_{i=1}^{r-1}\left[\left(a_{i,r+1}a_{i,r+1}^* - a_{ir}a_{ir}^*\right)_{\boldsymbol{\lambda}}a_{i,r+1}a_{i,r+2}^*\right] = -\sum_{i=1}^{r-1}a_{i,r+1}a_{i,r+2}^*.
\end{aligned}$$

The third summand contributes by Lemma 5.1

$$\begin{aligned}
&\sum_{j=r+2}^{n+1}\left[\left(a_{rj}a_{rj}^* - a_{r+1,j}a_{r+1,j}^*\right)_{\boldsymbol{\lambda}}\rho(E_{r+1})\right] \\
= &-a_{r+1,r+2}a_{r+1,r+2}^*a_{r+1,r+2}^* - \sum_{j=r+3}^{n+1}\left(a_{r+1,j}a_{r+1,j}^* - a_{r+2,j}a_{r+2,j}^*\right)a_{r+1,r+2}^* \\
&-a_{r,r+1}a_{r,r+2}^* - \sum_{j=r+3}^{n+1}a_{r+2,j}a_{r+1,j}^* - a_{r+1,r+2}^*\Phi(b_{r+1}) \\
&-\kappa\cdot Da_{r+1,r+2}^* - a_{r+1,r+2}^*\sum_{l=0}^{N}\kappa_l\lambda_i.
\end{aligned}$$

The last summand in $\rho(H_r)(\mathbf{w})$ has $\boldsymbol{\lambda}$-bracket with $\rho(E_{r+1})(\mathbf{w})$ equal to

$$\big[\Phi(b_r)_{\boldsymbol{\lambda}}\rho(E_{r+1})\big] = a^*_{r+1,r+2}[\rho(b_r)_{\boldsymbol{\lambda}}\rho(b_{r+1})] = a^*_{r+1,r+2}\sum_{p=0}^{N}\kappa_p\lambda_p.$$

Adding the previous four equations up, we get $[\rho(H_r)_{\boldsymbol{\lambda}}\rho(E_{r+1})] = -\rho(E_{r+1})$.

The final nontrivial case to consider is when $s = r-1$ (and $rs \neq 0$) so that $\rho(E_{r-1})(\mathbf{w})$ is equal to

$$\begin{aligned} &a_{r-1,r}(a^*_{r-1,r})^2 + \sum_{j=r+1}^{n+1}\left(a_{r-1,j}a^*_{r-1,j} - a_{rj}a^*_{rj}\right)a^*_{r-1,r} \\ &+ \sum_{j=1}^{r-2} a_{j,r-1}a^*_{jr} - \sum_{j=r+1}^{n+1} a_{r,j}a^*_{r-1,j} + a^*_{r-1,r}\Phi(b_{r-1}) + \kappa\cdot Da^*_{r-1,r}. \end{aligned}$$

Then $2\big[a_{r,r+1}a^*_{r,r+1}{}_{\boldsymbol{\lambda}}\rho(E_{r-1})\big] = -2a^*_{r-1,r+1}a_{r,r+1}$.

The second summation in $\rho(H_r)(\mathbf{w})$ contributes by Lemma 5.1

$$\begin{aligned} &\sum_{i=1}^{r-1}\big[\left(a_{i,r+1}a^*_{i,r+1} - a_{ir}a^*_{ir}\right)_{\boldsymbol{\lambda}}\rho(E_{r-1})\big] \\ &= -a_{r-1,r}a^*_{r-1,r}a^*_{r-1,r} - \sum_{j=r+1}^{n+1}\left(a_{r-1,j}a^*_{r-1,j} - a_{rj}a^*_{rj}\right)a^*_{r-1,r} \\ &\quad - \sum_{j=1}^{r-2} a_{j,r-1}a^*_{jr} + a_{r,r+1}a^*_{r-1,r+1} - a^*_{r-1,r}\Phi(b_{r-1}) - \kappa\cdot Da^*_{r-1,r} - \sum_{p=0}^{N}\kappa_p\lambda_p. \end{aligned}$$

The third summand contributes

$$\sum_{j=r+2}^{n+1}\big[\left(a_{rj}a^*_{rj} - a_{r+1,j}a^*_{r+1,j}\right)_{\boldsymbol{\lambda}}\rho(E_{r-1})\big] = \sum_{j=r+2}^{n+1} a_{r,j}a^*_{r-1,j}.$$

The last summation in $\rho(H_r)(\mathbf{w})$ has $\boldsymbol{\lambda}$-bracket with $\rho(E_{r-1})(\mathbf{w})$ that reduces to

$$\big[\Phi(b_r)_{\boldsymbol{\lambda}}\rho(E_{r-1})\big] = -\sum_{p=0}^{N}\kappa_p\lambda_p.$$

Summing the previous four equations gives $[\rho(H_r)(\mathbf{w}), \rho(E_{r-1})(\mathbf{w})] = -\rho(E_{r-1})(\mathbf{w})$.

We now consider the case of $s = 0$ and $r \neq 0$: Then $\rho(E_s)(\mathbf{w}) = \rho(E_0)(\mathbf{w}) = -a_{1,n+1}$ and hence $2\big[a_{r,r+1}a^*_{r,r+1}{}_{\boldsymbol{\lambda}}\rho(E_0)\big] = 0$.

The second summation in $\rho(H_r)(\mathbf{w})$ contributes

$$-\sum_{i=1}^{r-1}\big[\left(a_{i,r+1}a^*_{i,r+1} - a_{ir}a^*_{ir}\right)_{\boldsymbol{\lambda}}a_{1,n+1}\big] = \delta_{r,n}a_{1,n+1}.$$

The third summand contributes

$$-\sum_{j=r+2}^{n+1}\big[\left(a_{rj}a^*_{rj} - a_{r+1,j}a^*_{r+1,j}\right)_{\boldsymbol{\lambda}}a_{1,n+1}\big] = \delta_{r,1}a_{1,n+1}.$$

The last summation in $\rho(H_r)(\mathbf{z})$ has commutator with $\rho(E_0)(\mathbf{w})$ equal to 0, and hence does not contribute to the $\boldsymbol{\lambda}$-bracket. Summing the previous three equations, we get $[\rho(H_r)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(E_0)(\mathbf{w})] = -\rho(E_0)(\mathbf{w})$.

If $r = 0$, since $\rho(H_0) = -\sum_{r=1}^{n} \rho(H_r)$, we get

$$\begin{aligned}[]
[\rho(H_0)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(E_s)(\mathbf{w})] &= -\sum_{r=1}^{n} [\rho(H_r)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(E_s)(\mathbf{w})] \\
= -\sum_{r=1}^{n} A_{r,s}\rho(E_s)(\mathbf{w}) &= A_{0,s}\rho(E_s)(\mathbf{w}),
\end{aligned}$$

which holds for any s. This completes the proof of the lemma. □

Since our expression for $F_0(\mathbf{w})$ is quite different from that of $F_i(\mathbf{w})$ if $i \neq 0$, we will prove that case separately. First we consider the case:

Lemma 5.4. (T2) *For* $r, s \neq 0$, $[\rho(H_r)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(F_s)(\mathbf{w})] = -A_{rs}\rho(F_s)(\mathbf{w})$.

Proof. We assume $s, r \neq 0$ in

$$\begin{aligned}[]
[H_{r\boldsymbol{\lambda}}F_s] = \Big[&\sum_{i=1}^{r} a_{i,r+1}a^*_{i,r+1} - \sum_{i=1}^{r-1} a_{ir}a^*_{ir} + \sum_{j=r+1}^{n+1} a_{rj}a^*_{rj} \\
&- \sum_{j=r+2}^{n+1} a_{r+1,j}a^*_{r+1,j} + \Phi(b_r)_{\boldsymbol{\lambda}}a_{s,s+1} - \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}\Big]
\end{aligned}$$

(omitting the multivariable $\mathbf{w}$ as before). Using Lemma 5.2 and the fact that $\Phi(b_r)$ commutes with $a_{ij,\mathbf{m}}$ and $a^*_{ij,\mathbf{m}}$ gives

$$\begin{aligned}
&\Big[\sum_{i=1}^{r} a_{i,r+1}a^*_{i,r+1} - \sum_{i=1}^{r-1} a_{ir}a^*_{ir} + \sum_{j=r+1}^{n+1} a_{rj}a^*_{rj} - \sum_{j=r+2}^{n+1} a_{r+1,j}a^*_{r+1,j} + \Phi(b_r)_{\boldsymbol{\lambda}}a_{s,s+1}\Big] \\
&= -\delta_{r+1,s+1}a_{s,s+1} + \delta_{r,s+1}a_{s,s+1} - \delta_{r,s}a_{s,s+1} + \delta_{r+1,s}a_{s,s+1} = -A_{rs}a_{s,s+1}.
\end{aligned}$$

For the remaining component, we must show

$$\begin{aligned}
\Big[\Big(&\sum_{i=1}^{r} a_{i,r+1}a^*_{i,r+1} - \sum_{i=1}^{r-1} a_{ir}a^*_{ir}\Big) + \Big(\sum_{j=r+1}^{n+1} a_{rj}a^*_{rj} - \sum_{j=r+2}^{n+1} a_{r+1,j}a^*_{r+1,j}\Big)_{\boldsymbol{\lambda}} \\
&- \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}\Big] = A_{rs}\sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}. \qquad (14)
\end{aligned}$$

First note that by Lemma 5.1, $\Big[\sum_{i=1}^{r} a_{i,r+1}a^*_{i,r+1} - \sum_{i=1}^{r-1} a_{ir}a^*_{ir\boldsymbol{\lambda}} - \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}\Big] = 0$ unless $r = s$, $r + 1 = s$, or $r = s + 1$.

Equation (8) of Lemma 5.2 allows us to compute each case. Suppose $r = s$, then

$$\begin{aligned}
&\Big[\sum_{i=1}^{r} a_{i,r+1}a^*_{i,r+1} - \sum_{i=1}^{r-1} a_{ir}a^*_{ir\boldsymbol{\lambda}} - \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}\Big] \\
&= \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js} + \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js} = 2\sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}.
\end{aligned}$$

Suppose $r = s - 1$, then $\Big[-\sum_{i=1}^{r-1} a_{ir}a^*_{ir\boldsymbol{\lambda}} - \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}\Big] = 0$ and

$$\Big[\sum_{i=1}^{r} a_{i,r+1}a^*_{i,r+1\boldsymbol{\lambda}} - \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}\Big] = -\sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}.$$

Similarly, if $r = s+1$, then $\Big[\sum_{i=1}^{r} a_{i,r+1}a^*_{i,r+1}{}_{\boldsymbol\lambda} - \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}\Big] = 0$ and

$$\Big[-\sum_{i=1}^{r-1} a_{ir}a^*_{ir}{}_{\boldsymbol\lambda} - \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}\Big] = -\sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}.$$

We have shown

$$\Big[\Big(\sum_{i=1}^{r} a_{i,r+1}a^*_{i,r+1} - \sum_{i=1}^{r-1} a_{ir}a^*_{ir}\Big)_{\boldsymbol\lambda} - \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}\Big] = A_{rs}\sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}. \tag{15}$$

Applying Lemmas 5.1 and 5.2 and splitting into cases $r > s-1$, $r = s-1$ and $1 \le r \le s-1$, a straightforward computation shows the remaining component satisfies

$$\Big[\sum_{i=r+1}^{n+1} a_{ri}a^*_{ri} - \sum_{i=r+2}^{n+1} a_{r+1,i}a^*_{r+1,i}{}_{\boldsymbol\lambda} - \sum_{j=1}^{s-1} a_{j,s+1}a^*_{js}\Big] = 0 \tag{16}$$

for all $r, s \neq 0$. Equations (15) and (16) give (14) and the desired result. □

The case of $[H_0(\mathbf{w})_{\boldsymbol\lambda} F_i(\mathbf{w})]$ is similar to the above and is left to the reader. Next we consider the case of $F_0(\mathbf{w})$.

Lemma 5.5. (T2) *For all* $0 \le k \le n$, $[\rho(H_k)(\mathbf{w})_{\boldsymbol\lambda}\rho(F_0)(\mathbf{w})] = -A_{k0}F_0(\mathbf{w})$.

Proof. To simplify the computation for $\rho(F_0)(\mathbf{w})$, one should note that for all positive integers s, t, i, and fixed $(i+1)$-tuple $\mathbf{q} = (q_1, q_2, \ldots, q_{i+1}) \in \mathbf{Z}^{i+1}$ with $1 = q_1 < q_2 < \cdots < q_i$, it follows immediately from Lemma 5.1 that

$$\Big[a_{st}a^*_{st}{}_{\boldsymbol\lambda}\prod_{l=1}^{i} a^*_{q_l q_{l+1}}\Big] = \prod_{l=1}^{i}\delta_{sq_l}\delta_{tq_{l+1}}a^*_{q_l q_{l+1}}. \tag{17}$$

In other words, the expression $\big[a_{st}a^*_{st\lambda}\prod_{l=1}^{i} a^*_{q_l q_{l+1}}\big]$ is zero unless s and t appear as consecutive integers in the increasing sequence $q_1 < q_2 < \cdots < q_i$ in which case $[a_{st}a^*_{st\lambda}\cdot]$ acts as an identity operator. From this observation and noting that the expression $\rho(F_0)(\mathbf{w})$ contains sums of strings of such products, one is motivated to arrange terms of $\rho(H_k)(\mathbf{w})$ to promote cancellation, writing (where we suppress the multivariable):

$$\rho(H_k) = \sum_{i=1}^{k} a_{i,k+1}a^*_{i,k+1} - \sum_{j=k+2}^{n+1} a_{k+1,j}a^*_{k+1,j} - \sum_{i=1}^{k-1} a_{ik}a^*_{ik} + \sum_{j=k+1}^{n+1} a_{kj}a^*_{kj} + \Phi(b_k).$$

Now consider the $\boldsymbol\lambda$ (or equivalently the commutators) of components of $\rho(H_k)(\mathbf{w})$ and $\rho(F_0)(\mathbf{w})$, and will show that $[\rho(H_k)(\mathbf{w})_{\boldsymbol\lambda}\rho(F_0)(\mathbf{w})]$ is zero except in cases of $k = 0, 1, n$. Let

$$A := -\sum_{1\le r<n+1}\sum_{m=r+1}^{n+1} a_{rm}\sum_{\substack{\mathbf{q}\\ m=q_i>q_{i-1}\cdots>q_1=1}}\prod_{l=1}^{i-1} a^*_{q_l q_{l+1}}a^*_{r,n+1},$$

$$B := -\sum_{1\le r<n+1} \Phi(b_r) \sum_{\substack{\mathbf{q} \\ r\ge q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}},$$
$$C := -\sum_{\substack{\mathbf{q} \\ r=q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} \kappa \cdot D a^*_{r,n+1},$$

so $\rho(F_0)(\mathbf{w}) = A + B + C$. Because it is simpler, we first consider the second component B. Fix r with $1 \le r < n+1$, recall $q_{i+1} = n+1$, and fix k with $1 \le k < n$. In a fixed $\mathbf{q} \in \mathbf{Z}^{i+1}$ as above, if none of the $q_j = k+1$ for $1 \le j \le i$, then by (17), $\Big[\Big(\sum_{i=1}^{k} a_{i,k+1}a^*_{i,k+1} - \sum_{j=k+2}^{n+1} a_{k+1,j}a^*_{k+1,j}\Big)_{\boldsymbol{\lambda}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}}\Big] = 0$.

On the other hand, if $q_t = k+1$ for some $1 < t < i+1$ (unique because of the conditions on $\mathbf{q}$), then (17) shows

$$\Big[\Big(\sum_{i=1}^{k} a_{i,k+1}a^*_{i,k+1} - \sum_{j=k+2}^{n+1} a_{k+1,j}a^*_{k+1,j}\Big)_{\boldsymbol{\lambda}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}}\Big] = \prod_{l=1}^{i} a^*_{q_l q_{l+1}} - \prod_{l=1}^{i} a^*_{q_l q_{l+1}} = 0.$$

Since $\Phi(b_r)(\mathbf{z})$ commutes with the operators $a_{ij}(\mathbf{w})$ and $a^*_{ij}(\mathbf{w})$, summing over $r, \mathbf{q}$ with $r \ge q_i$, we have shown: For $1 \le k < n$,

$$\Big[\sum_{i=1}^{k} a_{i,k+1}a^*_{i,k+1} - \sum_{j=k+2}^{n+1} a_{k+1,j}a^*_{k+1,j\,\boldsymbol{\lambda}} B\Big] = 0. \tag{18}$$

A similar argument shows that for $1 < k \le n$,

$$\Big[\sum_{j=k+1}^{n+1} a_{kj}a^*_{kj} - \sum_{i=1}^{k-1} a_{ik}a^*_{ik\,\lambda} B\Big] = 0. \tag{19}$$

Adding equations (18) and (19) shows $[\rho(H_k)(\mathbf{w})_\lambda B] = 0$ for $1 < k < n$.

Now consider the first component A of our realization of $F_0(\mathbf{w})$. Note that the last term $a^*_{r,n+1}$ may also appear in the product $\prod_{l=1}^{i-1} a^*_{q_l q_{l+1}}$. Assume $k \ne n$, apply (17) and (2) for simplifying the brackets:

$$\begin{aligned}
&\sum_{i,j} \Big[a_{i,k+1}a^*_{i,k+1} - a_{k+1,j}a^*_{k+1,j\,\boldsymbol{\lambda}} A\Big] \\
&= \sum_{i,j}\sum_{r<m} \Big(a_{i,k+1}\delta_{i,r}\delta_{k+1,m} \sum_{\substack{\mathbf{q},\, m=q_i \\ r\ge q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{r,n+1} \\
&\quad - a_{k+1,j}\delta_{k+1,r}\delta_{j,m} \sum_{\substack{\mathbf{q},\, m=q_i \\ r\ge q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{r,n+1}\Big) \\
&\quad + \sum_{r<m} \Big(-a_{rm}\delta_{k+1,m} \sum_{\substack{\mathbf{q} \\ m=q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{r,n+1} \\
&\quad + a_{r,m} \sum_{\substack{\mathbf{q} \\ m=q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} [a_{k+1,j}a^*_{k+1,j\,\boldsymbol{\lambda}} a^*_{r,n+1}]\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{r<m} \Big(a_{r,k+1}\delta_{mk+1} \sum_{\substack{\mathbf{q},\, m=q_i \\ r\geq q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{r,n+1} \\
&\quad - a_{k+1,m}\delta_{k+1,r} \sum_{\substack{\mathbf{q},\, m=q_i \\ r\geq q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{r,n+1} \Big) \\
&\quad + \sum_{r<m} \Big(- a_{rm}\delta_{k+1,m} \sum_{\substack{\mathbf{q} \\ m=q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{r,n+1} \\
&\quad + \delta_{r,k+1} a_{rm} \sum_{\substack{\mathbf{q} \\ m=q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{r,n+1} \Big) \\
&= 0.
\end{aligned}
$$

Thus, for $1\leq k<n$ we have shown

$$
\Big[\sum_{i=1}^{k} a_{i,k+1}a^*_{i,k+1} - \sum_{j=k+2}^{n+1} a_{k+1,j}a^*_{k+1,j}{}_{\boldsymbol{\lambda}} A \Big] = 0. \tag{20}
$$

Using a similar argument, one can also show for $1<k\leq n$,

$$
\Big[\sum_{i=1}^{k-1} -a_{ik}a^*_{ik} + \sum_{j=k+1}^{n+1} a_{kj}a^*_{kj}{}_{\boldsymbol{\lambda}} A \Big] = 0. \tag{21}
$$

We have shown $[H_k(\mathbf{w})_{\boldsymbol{\lambda}} A] = 0$ for $1<k<n$.

Using Lemma 5.2 and equation (17) as above, for $1\leq k<n$, one obtains

$$
\begin{aligned}
&\sum_{r=1}^{n} \Big[\sum_{i=1}^{k} a_{i,k+1}a^*_{i,k+1} - \sum_{j=k+2}^{n+1} a_{k+1,j}a^*_{k+1,j}{}_{\boldsymbol{\lambda}} C \Big] \\
&= \sum_{\substack{\mathbf{q} \\ k+1=q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{k+1,n+1} \kappa \cdot \boldsymbol{\lambda},
\end{aligned} \tag{22}
$$

and for k with $1<k\leq n$,

$$
\sum_{r=1}^{n} \Big[\sum_{i=1}^{k-1} -a_{ik}a^*_{ik} + \sum_{j=k+1}^{n+1} a_{kj}a^*_{kj}{}_{\boldsymbol{\lambda}} C \Big] = - \sum_{\substack{\mathbf{q} \\ k=q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{k,n+1} \kappa \cdot \boldsymbol{\lambda}. \tag{23}
$$

The last term $\Phi(b_k)(\mathbf{z})$ appearing in $H_k(\mathbf{z})$ commutes with all of the operators $a_{ij}(\mathbf{w}), a^*_{ij}(\mathbf{w})$, so all that remains is to compute for $1\leq k<n$:

$$
\begin{aligned}
&\Big[\Phi(b_k)(\mathbf{w})_{\boldsymbol{\lambda}} - \sum_{1\leq r<n+1} \Phi(b_r)(\mathbf{w}) \sum_{\substack{\mathbf{q} \\ r\geq q_i>\cdots>q_1=1}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}} \Big] \\
&= \sum_{1\leq r<n+1} A_{kr} \sum_{\substack{\mathbf{q} \\ r\geq q_i>\cdots>q_1=1}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}} \kappa \cdot \boldsymbol{\lambda} \\
&= (-1) \sum_{\substack{\mathbf{q} \\ k-1\geq q_i>\cdots>q_1=1}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}} \kappa \cdot \boldsymbol{\lambda} + 2 \sum_{\substack{\mathbf{q} \\ k\geq q_i>\cdots>q_1=1}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}} \kappa \cdot \boldsymbol{\lambda} \\
&\quad + (-1) \sum_{\substack{\mathbf{q} \\ k+1\geq q_i>\cdots>q_1=1}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}} \kappa \cdot \boldsymbol{\lambda}
\end{aligned}
$$

$$= \sum_{\substack{\mathbf{q} \\ k=q_i>\cdots>q_1=1}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}} \kappa \cdot \boldsymbol{\lambda} + (-1) \sum_{\substack{\mathbf{q} \\ k+1=q_i>\cdots>q_1=1}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}} \kappa \cdot \boldsymbol{\lambda}, \tag{24}$$

where we collect partitions in the last equality.

Now for $k \neq 0, 1, n$, we have shown

$$[\rho(H_k)(\mathbf{w})_{\boldsymbol{\lambda}} \rho(F_0)(\mathbf{w})] = (20) + (21) + (18) + (19) + (22) + (23) + (24) = 0. \quad \square$$

Now we consider the case $k = 1$, where

$$H_1 = a_{1,2}a^*_{1,2} - \sum_{j=3}^{n+1} a_{2,j}a^*_{2,j} + \sum_{j=2}^{n+1} a_{1j}a^*_{1j} + \Phi(b_1).$$

Equations (18) and (20) hold for $k = 1$ as does equation (22), so

$$\begin{aligned} &\Big[a_{1,2}a^*_{1,2} - \sum_{j=3}^{n+1} a_{2,j}a^*_{2,j\,\boldsymbol{\lambda}} F_0\Big] \\ &= - \sum_{\substack{\mathbf{q} \\ 2=q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{2,n+1} \kappa \cdot \boldsymbol{\lambda} = a^*_{12} a^*_{2,n+1} \kappa \cdot \boldsymbol{\lambda}. \end{aligned} \tag{25}$$

Furthermore, since in all of our $\mathbf{q}$, $q_1 = 1$, equations (17) and (9) give

$$\begin{aligned} \Big[\sum_{j=2}^{n+1} a_{1j}a^*_{1j\,\boldsymbol{\lambda}} F_0\Big] &= A + B + \Big[\sum_{j=2}^{n+1} a_{1j}a^*_{1j\,\boldsymbol{\lambda}} - \sum_r \sum_{\substack{\mathbf{q} \\ r=q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} \kappa \cdot D a^*_{r,n+1}\Big] \\ &= F_0 - a^*_{1,n+1} \kappa \cdot \boldsymbol{\lambda} \end{aligned} \tag{26}$$

Finally,

$$\begin{aligned} [\Phi(b_1)_{\boldsymbol{\lambda}} F_0] &= 2a^*_{1,n+1}\kappa \cdot \boldsymbol{\lambda} - 1(a^*_{12}a^*_{2,n+1} + a^*_{1,n+1})\kappa \cdot \boldsymbol{\lambda} \\ &= -a^*_{12}a^*_{2,n+1}\kappa \cdot \boldsymbol{\lambda} + a^*_{1,n+1}\kappa \cdot \boldsymbol{\lambda}. \end{aligned} \tag{27}$$

Summing equations (27), (25) and (26) yields $[\rho(H_1)(\mathbf{w})_{\boldsymbol{\lambda}} \rho(F_0)(\mathbf{w})] = \rho(F_0)(\mathbf{w})$ as desired.

Now consider $H_n = \sum_{i=1}^{n} a_{i,n+1}a^*_{i,n+1} - \sum_{i=1}^{n-1} a_{in}a^*_{in} + a_{n,n+1}a^*_{n,n+1} + \Phi(b_n)$, writing $F_0(\mathbf{w}) = A + B + C$ as above, and recalling our assumption that in all the $(i+1)$-tuples $\mathbf{q}$, the term $\mathbf{q}_{i+1} = n + 1$, a straightforward computation using (17) shows

$$\Big[\sum_{i=1}^{n} a_{i,n+1}a^*_{i,n+1\,\boldsymbol{\lambda}} A\Big] = A, \tag{28}$$

$$\Big[\sum_{i=1}^{n} a_{i,n+1}a^*_{i,n+1\,\boldsymbol{\lambda}} B\Big] = B. \tag{29}$$

Furthermore, using Lemmas 5.1, 5.2 and collecting partitions, one obtains

$$\Big[\sum_{i=1}^{n} a_{i,n+1}a^*_{i,n+1\,\boldsymbol{\lambda}} C\Big] = C - \sum_{\substack{\mathbf{q} \\ n\geq q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i} a^*_{q_l q_{l+1}} \kappa \cdot \boldsymbol{\lambda}.$$

Equations (19), (21) and (23) hold for $k = n$ and show

$$\Big[-\sum_{i=1}^{n-1} a_{in}a_{in}^* + a_{n,n+1}a_{n,n+1}^*{}_{\boldsymbol{\lambda}} A\Big] = 0, \tag{30}$$

$$\Big[-\sum_{i=1}^{n-1} a_{in}a_{in}^* + a_{n,n+1}a_{n,n+1}^*{}_{\boldsymbol{\lambda}} B\Big] = 0, \tag{31}$$

$$\Big[-\sum_{i=1}^{n-1} a_{in}a_{in}^* + a_{n,n+1}a_{n,n+1}^*{}_{\boldsymbol{\lambda}} C\Big] = -\sum_{n=q_i>q_{i-1}>\cdots q_1} \prod_{l=1}^{i} a_{q_lq_{l+1}}^*\kappa\cdot\boldsymbol{\lambda}, \tag{32}$$

$$\begin{aligned}\big[\Phi(b_n)_{\boldsymbol{\lambda}}\rho(F_0)\big] = {} & (-1)\sum_{\substack{\mathbf{q}\\ n-1\geq q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i} a_{q_lq_{l+1}}^*\kappa\cdot\boldsymbol{\lambda} \\ & + 2\sum_{\substack{\mathbf{q}\\ n\geq q_i>q_{i-1}\cdots>q_1=1}} \prod_{l=1}^{i} a_{q_lq_{l+1}}^*\kappa\cdot\boldsymbol{\lambda}.\end{aligned} \tag{33}$$

Summing equations (28)–(33), we have

$$\begin{aligned} & [\rho(H_n)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(F_0)(\mathbf{w})] \\ = {} & A + B + C - \sum_{\substack{\mathbf{q}\\ n\geq q_i>\cdots>q_1=1}} \prod_{l=1}^{i} a_{q_lq_{l+1}}^*\kappa\cdot\boldsymbol{\lambda} + 2\sum_{\substack{\mathbf{q}\\ n\geq q_i>\cdots>q_1=1}} \prod_{l=1}^{i} a_{q_lq_{l+1}}^*\kappa\cdot\boldsymbol{\lambda} \\ & -\Big(\sum_{\substack{\mathbf{q}\\ n=q_i>\cdots>q_1}} \prod_{l=1}^{i} a_{q_lq_{l+1}}^*\kappa\cdot\boldsymbol{\lambda} + \sum_{\substack{\mathbf{q}\\ n-1\geq q_i>\cdots>q_1=1}} \prod_{l=1}^{i} a_{q_lq_{l+1}}^*\kappa\cdot\boldsymbol{\lambda}\Big) \\ = {} & F_0(\mathbf{w}). \end{aligned}$$

The case of $k = 0$ follows from the above, and is left to the reader.

5.3 Relations involving $E_k(\mathbf{w}), F_r(\mathbf{w})$, and Serre relations. We will prove a selection of the relations involving the elements $E_k(\mathbf{w}), F_r(\mathbf{w})$ including the Serre type relations.

Lemma 5.6. (T3) $[\rho(E_s)(\mathbf{w})_{\boldsymbol{\lambda}}\rho(F_r)(\mathbf{w})] = -\delta_{r,s}\Big(\rho(H_r)(\mathbf{w}) + \frac{2}{A_{rs}}\sum_{l=0}^{N}\rho(K_l)(\mathbf{w})\lambda_l\Big)$.

Proof. For $r \neq 0$ and $s \neq 0$, the proof is similar to that in [9, Lemma 3.4], where $-\gamma b_r(z) - \frac{1}{2}\big(b_{r-1}^+(z) + b_{r+1}^+(z)\big)$ is replaced by $\Phi(b_r)$ and $-\frac{\gamma}{2}\dot{a}_{r,r+1}^*(z)$ is replace by $\kappa\cdot Da_{r,r+1}^*$. We refer the interested reader to that paper for the proof.

It is also straightforward to check that

$$[\rho(E_0)_{\boldsymbol{\lambda}}\rho(F_r)] = -\delta_{0,r}\Big(\rho(H_r) + \frac{2}{A_{r0}}\sum_{l=0}^{N}\rho(K_l)\lambda_l\Big).$$

The case $[\rho(E_s)_{\boldsymbol{\lambda}}\rho(F_0)]$ with $s > 0$ is shown via lengthy calculation made available in [8]. □

We are now left with the Serre relations:

Lemma 5.7. (T4) *Let* $\rho(E_r)(\mathbf{w})$, $\rho(F_r)(\mathbf{w})$, $\rho(H_r)(\mathbf{w})$ *be defined as in Theorem 4.1, then the following relations hold:*

$$\begin{aligned}&[\rho(F_r)(\mathbf{w})_{\boldsymbol\lambda}\rho(F_s)(\mathbf{w})]\\ =\ &[\rho(E_r)(\mathbf{w})_{\boldsymbol\lambda}\rho(E_s)(\mathbf{w})]=0 \quad \textit{if } A_{rs}\neq -1,\\ &[\rho(F_r)(\mathbf{w})_{\boldsymbol\lambda}[\rho(F_r)(\mathbf{w})_{\boldsymbol\mu}\rho(F_s)(\mathbf{w})]]\\ =\ &[\rho(E_r)(\mathbf{w})_{\boldsymbol\lambda}[\rho(E_r)(\mathbf{w})_{\boldsymbol\mu}\rho(E_s)(\mathbf{w})]]=0 \quad \textit{if } A_{rs}=-1.\end{aligned}$$

Proof. As in the previous lemmas, we first assume $rs\neq 0$. In this case, the proof is exactly the same as in [9, Lemma 3.5] with the exception of a sign change in the formulation of $\rho(F_r)$.

Now suppose $r=0$, by expanding all $\boldsymbol\lambda$-brackets and collecting over partitions along with other simplifications, one obtains for $1<s\leq n$:

$$\begin{aligned}&[\rho(F_0)_{\boldsymbol\lambda}\rho(F_s)]\\ =\ &\delta_{s,n}a_{n,n+1}\sum_{\substack{\mathbf{q}\\ q_i=n+1}}\prod_{k=1}^{i-1}a^*_{q_kq_{k+1}}-\delta_{s,n}\sum_{l=1}^{n-1}\sum_{l<j\leq n+1}a_{lj}\sum_{\substack{\mathbf{q}\\ j=q_i;\, l\geq q_{i-1}}}\prod_{m=1}^{i-1}a^*_{q_mq_{m+1}}a^*_{ln}\\ &-\delta_{s,n}\sum_{\mathbf{q};\, q_i=n}\kappa\cdot(D+\boldsymbol\lambda)\prod_{j=1}^{i-1}a^*_{q_jq_{j+1}}\\ &-\delta_{s,n}\sum_{l=1}^{n-1}\sum_{1\leq r<n}\sum_{\substack{\mathbf{q}\\ r\geq q_i,\,\exists a,\,(q_a,q_{a+1})=(l,n+1)}}\Big(\prod_{k=1,k\neq a}^{i}a^*_{q_kq_{k+1}}\Big)a^*_{ln}\Phi b_r\\ &+\delta_{s,n}\sum_{l=1}^{n-1}\sum_{\mathbf{q};\, q_i=l}\kappa\cdot(D+\boldsymbol\lambda)\Big(\prod_{j=1}^{i-1}a^*_{q_jq_{j+1}}\Big)a^*_{ln}.\end{aligned}\tag{34}$$

This proves the Serre relation for $s\neq 0,1,n$. Consider the case $s=n$. We want to show $[[\rho(F_0)_{\boldsymbol\lambda}\rho(F_n)]_{\boldsymbol\mu}\rho(F_n)]=0$.

To prove this, first recall $\rho(F_n)=a_{n,n+1}-\sum_{p=1}^{n-1}a_{p,n+1}a^*_{p,n}$. Now using (34),

$$\begin{aligned}&[[\rho(F_0)_{\boldsymbol\lambda}\rho(F_n)]_{\boldsymbol\mu}a_{n,n+1}]\\ =\ &\Big[\Big(a_{n,n+1}\sum_{\substack{\mathbf{q}\\ q_i=n+1}}\prod_{k=1}^{i-1}a^*_{q_kq_{k+1}}\\ &-\sum_{l=1}^{n-1}\sum_{l<j\leq n+1}a_{lj}\sum_{r,\,\mathbf{q};\, j=q_i;\, l\geq q_{i-1}}\prod_{m=1}^{i-1}a^*_{q_mq_{m+1}}a^*_{ln}\Big)_{\boldsymbol\mu}a_{n,n+1}\Big]\\ =\ &-a_{n,n+1}\sum_{\substack{\mathbf{q}\\ q_{i-1}=n}}\prod_{k=1}^{i-2}a^*_{q_kq_{k+1}},\end{aligned}$$

whereas

$$\begin{aligned}&-\sum_{p=1}^{n-1}[[\rho(F_0)_{\boldsymbol\lambda}\rho(F_n)]_{\boldsymbol\mu}a_{p,n+1}a^*_{p,n}]\\ =\ &-\sum_{p=1}^{n-1}\Big[\Big(a_{n,n+1}\sum_{\substack{\mathbf{q}\\ q_i=n+1}}\prod_{k=1}^{i-1}a^*_{q_kq_{k+1}}\\ &-\sum_{1\leq l<j\leq n+1}a_{lj}\sum_{\mathbf{q};\, j=q_i;\, l\geq q_{i-1}}\prod_{m=1}^{i-1}a^*_{q_mq_{m+1}}a^*_{ln}\Big)_{\boldsymbol\mu}a_{p,n+1}a^*_{p,n}\Big]\end{aligned}$$

$$= a_{n,n+1}\sum_{p=1}^{n-1}\sum_{\substack{\mathbf{q}\\ q_{i-1}=p}}\prod_{k=1}^{i-2}a^*_{q_kq_{k+1}}a^*_{p,n}+\sum_{l=1}^{n-1}a_{l,n+1}\sum_{\mathbf{q};\,q_i=n;\,l\geq q_{i-1}}\prod_{m=1}^{i-1}a^*_{q_mq_{m+1}}a^*_{l,n}$$
$$-\sum_{l=1}^{n-1}a_{l,n+1}\sum_{p=1}^{l}\sum_{\mathbf{q};\,n+1=q_i;\,l\geq q_{i-1}=p}\prod_{m=1}^{i-2}a^*_{q_mq_{m+1}}a^*_{p,n}a^*_{l,n}.$$

Hence, $[[\rho(F_0)_{\boldsymbol\lambda}\rho(F_n)]_{\boldsymbol\mu}\rho(F_n)]=0$.

For $s=1$ we get $\rho(F_1)=a_{1,2}$ and hence

$$\begin{aligned}&[\rho(F_0)_{\boldsymbol\lambda}\rho(F_1)]\\ &=\sum_{1\leq r<j\leq n+1}a_{rj}\sum_{\mathbf{q};\,j=q_i,\,r\geq q_{i-1},\,q_2=2}\prod_{j=2}^{i-1}a^*_{q_jq_{j+1}}a^*_{r,n+1}\\ &\quad+\sum_{r<n+1}\sum_{\substack{\mathbf{q}\\ r=q_i;\,q_2=2}}\prod_{j=2}^{i}a^*_{q_jq_{j+1}}\Phi b_r+\sum_{1\leq r\leq n}\sum_{\substack{\mathbf{q}\\ q_2=2;\,q_i=r}}\prod_{j=2}^{i-1}a^*_{q_jq_{j+1}}\kappa\cdot Da^*_{r,n+1}.\end{aligned}$$

Thus, $[[\rho(F_0)_{\boldsymbol\lambda}\rho(F_1)]_{\boldsymbol\mu}\rho(F_1)]=0$.

Next up is the calculation for $[\rho(F_0)_{\boldsymbol\lambda}[\rho(F_0)_{\boldsymbol\mu}\rho(F_1)]]$:

For a partition $\mathbf{q}=(1=q_1,q_2,\ldots,q_i,n+1)$, recall that we set $l(\mathbf{q})=i$. We now write $[\rho(F_0)_{\boldsymbol\lambda}\rho(F_1)]=A_{01}+B_{01}+C_{01}$, where

$$\begin{aligned}A_{01}&=\sum_{1\leq r<j\leq n+1}a_{rj}\sum_{\mathbf{q};\,j=q_{l(\mathbf{q})};\,r\geq q_{l(\mathbf{p})-1},\,q_2=2}\prod_{j=2}^{l(\mathbf{q})-1}a^*_{q_jq_{j+1}}a^*_{r,n+1}\\ B_{01}&=\sum_{r\leq n}\sum_{\substack{\mathbf{q}\\ r\geq q_{l(\mathbf{q})};\,q_2=2}}\prod_{j=2}^{l(\mathbf{q})}a^*_{q_jq_{j+1}}\Phi b_r\\ C_{01}&=\sum_{1\leq r\leq n}\sum_{\substack{\mathbf{q}\\ q_2=2;\,q_{l(\mathbf{q})}=r}}\prod_{j=2}^{l(\mathbf{q})-1}a^*_{q_jq_{j+1}}\kappa\cdot Da^*_{r,n+1},\end{aligned}$$

and $F_0=A+B+C$, where

$$\begin{aligned}A&=\sum_{1\leq r<j\leq n+1}-a_{rj}\sum_{\mathbf{q};\,j=q_{l(\mathbf{q})};\,r\geq q_{l(\mathbf{p})-1}}\prod_{l=1}^{l(\mathbf{q})-1}a^*_{q_lq_{l+1}}a^*_{r,n+1}\\ B&=-\sum_{1\leq r<n+1}\sum_{r\geq q_{l(\mathbf{q})},\,\mathbf{q}}\prod_{j=1}^{l(\mathbf{q})}a^*_{q_jq_{j+1}}\Phi(b_r)\\ C&=-\sum_{1\leq r<n+1}\sum_{r=q_{l(\mathbf{q})},\,\mathbf{q}}\prod_{j=1}^{l(\mathbf{q})-1}a^*_{q_jq_{j+1}}\kappa\cdot Da^*_{r,n+1}.\end{aligned}$$

Then $[F_0{}_{\boldsymbol\lambda}[F_0{}_{\boldsymbol\mu}F_1]]=[A_{\boldsymbol\lambda}A_{01}]+[B_{\boldsymbol\lambda}B_{01}]+[A_{\boldsymbol\lambda}B_{01}]+[B_{\boldsymbol\lambda}A_{01}]+[A_{\boldsymbol\lambda}C_{01}]+[C_{\boldsymbol\lambda}A_{01}]$.

Now we calculate each summand above, and simplify

$$\begin{aligned}&[A_{\boldsymbol\lambda}A_{01}]\\ &=-\sum_{\substack{1\leq r<j\leq n+1\\ 1\leq s<k\leq n+1}}\sum_{\substack{\mathbf{q};\,j=q_{l(\mathbf{q})};\,r\geq q_{l(\mathbf{q})-1}\\ \mathbf{p};\,k=p_{l}(\mathbf{p});\,s\geq p_{l(\mathbf{p})-1},\,p_2=2}}\left[a_{rj}\prod_{l=1}^{l(\mathbf{q})-1}a^*_{q_lq_{l+1}}a^*_{r,n+1}{}_{\boldsymbol\lambda}a_{sk}\prod_{\xi=2}^{l(\mathbf{p})-1}a^*_{p_\xi p_{\xi+1}}a^*_{s,n+1}\right]\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{\substack{1\leq r<j\leq n+1\\ 1\leq s<k\leq n+1}} \sum_{\substack{\mathbf{q};\, j=q_{l(\mathbf{q})};\, r\geq q_{l(\mathbf{q})-1}\\ \mathbf{p};\, k=p_{l(\mathbf{p})};\, s\geq p_{l(\mathbf{p})-1},\, p_2=2}} a_{sk}\Bigl(\Bigl[a_{rj}{}_{\boldsymbol{\lambda}} \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}}\Bigr] a^*_{s,n+1} \\
&\quad + \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}} [a_{rj}{}_{\boldsymbol{\lambda}} a^*_{s,n+1}]\Bigr) \prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} a^*_{r,n+1} \\
&\quad - \sum_{\substack{1\leq r<j\leq n+1\\ 1\leq s<k\leq n+1}} \sum_{\substack{\mathbf{q};\, j=q_{l(\mathbf{q})};\, r\geq q_{l(\mathbf{q})-1}\\ \mathbf{p};\, k=p_{l(\mathbf{p})};\, s\geq p_{l(\mathbf{p})-1},\, p_2=2}} a_{rj}\Bigl(\Bigl[\prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}}{}_{\boldsymbol{\lambda}} a_{sk}\Bigr] a^*_{r,n+1} \\
&\quad + \prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} [a^*_{r,n+1}{}_{\boldsymbol{\lambda}} a_{sk}]\Bigr) \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}} a^*_{s,n+1}.
\end{aligned}
$$

Re-indexing the above gives

$$
\begin{aligned}
&= -\sum_{\substack{1\leq r<j\leq n+1\\ 1\leq s<k\leq n+1}} \sum_{\substack{\mathbf{q};\, j=q_{l(\mathbf{q})};\, r\geq q_{l(\mathbf{q})-1}\\ \mathbf{p};\, k=p_{l(\mathbf{p})};\, s\geq p_{l(\mathbf{p})-1},\, p_2=2}} a_{sk}\Bigl(\Bigl[a_{rj}{}_{\boldsymbol{\lambda}} \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}}\Bigr] a^*_{s,n+1} \\
&\quad + \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}} [a_{rj}{}_{\boldsymbol{\lambda}} a^*_{s,n+1}]\Bigr) \prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} a^*_{r,n+1} \\
&\quad - \sum_{\substack{1\leq s<k\leq n+1\\ 1\leq r<j\leq n+1}} \sum_{\substack{\mathbf{q};\, k=q_{l(\mathbf{q})};\, s\geq q_{l(\mathbf{q})-1}\\ \mathbf{p};\, j=p_{l(\mathbf{p})};\, r\geq p_{l(\mathbf{p})-1},\, p_2=2}} a_{sk}\Bigl(\Bigl[\prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}}{}_{\boldsymbol{\lambda}} a_{rj}\Bigr] a^*_{s,n+1} \\
&\quad + \prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} [a^*_{s,n+1}{}_{\boldsymbol{\lambda}} a_{rj}]\Bigr) \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}} a^*_{r,n+1} \\
&= -\sum_{1\leq s<k\leq n+1} \sum_{\substack{\mathbf{q};\, n+1=q_{l(\mathbf{q})};\, s\geq q_{l(\mathbf{q})-1}\\ \mathbf{p};\, k=p_{l(\mathbf{p})};\, s\geq p_{l(\mathbf{p})-1},\, p_2=2}} a_{sk} \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}} \prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} a^*_{s,n+1} \\
&\quad + \sum_{1\leq s<k\leq n+1} \sum_{\substack{\mathbf{q};\, k=q_{l(\mathbf{q})};\, s\geq q_{l(\mathbf{q})-1}\\ \mathbf{p};\, n+1=p_{l(\mathbf{p})};\, s\geq p_{l(\mathbf{p})-1},\, p_2=2}} a_{sk} \prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}} a^*_{s,n+1} \\
&\quad - \sum_{2\leq r\leq s, j\leq k\leq n+1} \sum_{\substack{\mathbf{q};\, j=q_{l(\mathbf{q})};\, r\geq q_{l(\mathbf{q})-1},\, \exists t;\, 2\leq t\leq l(\mathbf{p})-1;\, (r,j)=(p_t,p_{t+1})\\ \mathbf{p};\, k=p_{l(\mathbf{p})};\, s\geq p_{l(\mathbf{p})-1},\, p_2=2}} a_{sk} \\
&\quad \cdot \prod_{\xi=2,\xi\neq t}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}} a^*_{s,n+1} \prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} a^*_{r,n+1} \\
&\quad + \sum_{\substack{1\leq r\leq s<k\leq n+1\\ 1\leq r<j\leq n+1}} \sum_{\substack{\mathbf{q};\, k=q_{l(\mathbf{q})};\, s\geq q_{l(\mathbf{q})-1},\, \exists t\leq l(\mathbf{q})-1;\, (r,j)=(q_t,q_{t+1})\\ \mathbf{p};\, j=p_{l(\mathbf{p})};\, r\geq p_{l(\mathbf{p})-1},\, p_2=2}} a_{sk} \\
&\quad \cdot \prod_{l=1,l\neq t}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} a^*_{s,n+1} \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}} a^*_{r,n+1}.
\end{aligned}
$$

To show that the above summation is zero reduces to showing the following are zero:

$$
\begin{aligned}
I_1 &:= -\sum_{\substack{\mathbf{q};\, n+1=q_{l(\mathbf{q})};\, s\geq q_{l(\mathbf{q})-1}\\ \mathbf{p};\, k=p_{l(\mathbf{p})};\, s\geq p_{l(\mathbf{p})-1},\, p_2=2}} \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}} \prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} \\
&\quad + \sum_{1\leq r<j\leq n+1} \sum_{\substack{\mathbf{q};\, k=q_{l(\mathbf{q})};\, s\geq q_{l(\mathbf{q})-1},\, \exists t\leq l(\mathbf{q})-1;\, (r,j)=(q_t,q_{t+1})\\ \mathbf{p};\, j=p_{l(\mathbf{p})};\, r\geq p_{l(\mathbf{p})-1},\, p_2=2}} \prod_{l=1,l\neq t}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} \prod_{\xi=2}^{l(\mathbf{p})-1} a^*_{p_\xi p_{\xi+1}} a^*_{r,n+1}.
\end{aligned}
$$

Note that the first summation is over all partitions $\mathbf{p}$ and $\mathbf{q}$ and has summands of the form $a^*_{2p_3}\cdots a^*_{p_{l(\mathbf{p})-1}k}a^*_{1,q_2}\cdots a^*_{q_{l(\mathbf{q})-1}n+1}$ with $p_{l(\mathbf{p})-1}\le s$ and $q_{l(\mathbf{q})-1}\le s$. The second summation is over all partitions $\mathbf{p}$ and $\mathbf{q}$ and has summands of the form

$$\begin{aligned}&a^*_{1,q_2}\cdots a^*_{q_{t-1},r}a^*_{j,q_{t+2}}\cdots a^*_{q_{l(\mathbf{q})-1},k}a^*_{2,p_3}\cdots a^*_{p_{l(\mathbf{p})-1},j}a^*_{r,n+1}\\ &=(a^*_{2,p_3}\cdots a^*_{p_{l(\mathbf{p})-1},j}a^*_{j,q_{t+2}}\cdots a^*_{q_{l(\mathbf{q})-1},k})(a^*_{1,q_2}\cdots a^*_{q_{t-1},r}a^*_{r,n+1}),\end{aligned}$$

where $s\ge q_{l(\mathbf{q})-1}$, $s\ge r\ge p_{l(\mathbf{p})-1}$ and $r<j\le k$. But these two sets of partitions are the same, so $I_1=0$.

Similarly, if we look at the partitions for the summands of

$$\begin{aligned}I_2:=&\sum_{\substack{\mathbf{q};\,k=q_{l(\mathbf{q})};\,s\ge q_{l(\mathbf{q})-1}\\ \mathbf{p};\,n+1=p_{l(\mathbf{p})};\,s\ge p_{l(\mathbf{p})-1},\,p_2=2}}\prod_{l=1}^{l(\mathbf{q})-1}a^*_{q_lq_{l+1}}\prod_{\xi=2}^{l(\mathbf{p})-1}a^*_{p_\xi p_{\xi+1}}\\ &-\sum_{2\le r<j\le n+1}\sum_{\substack{\mathbf{q};\,j=q_{l(\mathbf{q})};\,r\ge q_{l(\mathbf{q})-1}\\ \exists t;\,2\le t\le l(\mathbf{p})-1;\,(r,j)=(p_t,p_{t+1})\\ \mathbf{p};\,k=p_{l(\mathbf{p})};\,s\ge p_{l(\mathbf{p})-1},\,p_2=2}}\prod_{\xi=2,\xi\ne t}^{l(\mathbf{p})-1}a^*_{p_\xi p_{\xi+1}}\prod_{l=1}^{l(\mathbf{q})-1}a^*_{q_lq_{l+1}}a^*_{r,n+1},\end{aligned}$$

collecting partitions shows that $I_2=0$. Hence, $[A_{\boldsymbol\lambda}A_{01}]=0$.

One can also obtain:

$$\begin{aligned}&[B_{\boldsymbol\lambda}B_{01}]\\ &=-\sum_{1\le r<n}\Big(\sum_{\mathbf{q};\,r\ge q_{l(\mathbf{q})}}\kappa\cdot(\boldsymbol\lambda+D)\Big(\prod_{j=1}^{l(\mathbf{q})}a^*_{q_jq_{j+1}}\Big)\Big)\Big(\sum_{\mathbf{p};\,r+1=p_{l(\mathbf{p})},\,p_2=2}\prod_{l=2}^{l(\mathbf{p})}a^*_{p_lp_{l+1}}\Big)\\ &\quad+2\Big(\sum_{\mathbf{q};\,n\ge q_{l(\mathbf{q})}}\kappa\cdot(\boldsymbol\lambda+D)\Big(\prod_{j=1}^{l(\mathbf{q})}a^*_{q_jq_{j+1}}\Big)\Big)\Big(\sum_{\mathbf{p};\,n\ge p_{l(\mathbf{p})},\,p_2=2}\prod_{l=2}^{l(\mathbf{p})}a^*_{p_lp_{l+1}}\Big)\\ &\quad-\sum_{1\le r<n}\Big(\sum_{\mathbf{q};\,r+1=q_{l(\mathbf{q})}}\kappa\cdot(\boldsymbol\lambda+D)\Big(\prod_{j=1}^{l(\mathbf{q})}a^*_{q_jq_{j+1}}\Big)\Big)\Big(\sum_{\mathbf{p};\,r\ge p_{l(\mathbf{p})},p_2=2}\prod_{l=2}^{l(\mathbf{p})}a^*_{p_lp_{l+1}}\Big),\end{aligned}\tag{35}$$

and the following identities

$$\begin{aligned}[A_{\boldsymbol\lambda}B_{01}]=&-\sum_{1\le r<j\le n+1}\sum_{\substack{\mathbf{q}\\ j=q_{l(\mathbf{q})};\,r\ge q_{l(\mathbf{q})-1}}}\sum_{s=1}^{n}\sum_{\substack{\mathbf{p}\\ s\ge p_{l(\mathbf{p})},\,p_2=2}}\\ &\prod_{l=1}^{l(\mathbf{q})-1}a^*_{q_lq_{l+1}}\Big[a_{rj}\boldsymbol\lambda\prod_{j=2}^{l(\mathbf{p})}a^*_{p_jp_{j+1}}\Big]a^*_{r,n+1}\Phi(b_s)\\ =&-\sum_{s=1}^{n}\sum_{2\le r<j\le n+1}\sum_{\substack{\mathbf{q}\\ j=q_{l(\mathbf{q})};\,r\ge q_{l(\mathbf{q})-1}}}\sum_{\substack{\mathbf{p}\\ s\ge p_{l(\mathbf{p})},\,p_2=2}}\\ &\prod_{l=1}^{l(\mathbf{q})-1}a^*_{q_lq_{l+1}}\Big[a_{rj}\boldsymbol\lambda\prod_{j=2}^{l(\mathbf{p})}a^*_{p_jp_{j+1}}\Big]a^*_{r,n+1}\Phi(b_s),\\ [B_{\boldsymbol\lambda}A_{01}]=&-\sum_{1\le r<j\le n+1}\sum_{\substack{\mathbf{q}\\ j=q_{l(\mathbf{q})};\,r\ge q_{l(\mathbf{q})-1},\,q_2=2}}\sum_{s=1}^{n}\sum_{\mathbf{p};\,s\ge p_{l(\mathbf{p})}}\\ &\prod_{l=2}^{l(\mathbf{q})-1}a^*_{q_lq_{l+1}}\Big[\prod_{j=1}^{l(\mathbf{p})}a^*_{p_jp_{j+1}}\boldsymbol\lambda a_{rj}\Big]a^*_{r,n+1}\Phi(b_s)\end{aligned}$$

$$= -\sum_{s=1}^{n} \sum_{2\le r<j\le n+1} \sum_{\substack{\mathbf{q} \\ j=q_{l(\mathbf{q})};\, r\ge q_{l(\mathbf{q})-1},\, q_2=2}} \sum_{\substack{\mathbf{p} \\ s\ge p_{l(\mathbf{p})}}} \prod_{l=2}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} \Big[\prod_{j=1}^{l(\mathbf{p})} a^*_{p_j p_{j+1}} {}_{\boldsymbol\lambda} a_{rj}\Big] a^*_{r,n+1}\Phi(b_s).$$

Computing the $\boldsymbol\lambda$-brackets above, then collecting and re-labelling partitions yields $[A_{\boldsymbol\lambda}B_{01}] + [B_{\boldsymbol\lambda}A_{01}] = 0$.

Next we calculate

$$\begin{aligned}
&[A_{\boldsymbol\lambda}C_{01}] \\
&= -\sum_{1\le r<j\le n+1} \Big[a_{rj} \sum_{\mathbf{q};\, j=q_{l(\mathbf{q})};\, r\ge q_{l(\mathbf{q})-1}} \prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} a^*_{r,n+1} {}_{\boldsymbol\lambda} \sum_{1\le s<n+1} \sum_{\mathbf{p};\, s=p_{l(\mathbf{p})},\, p_2=2} \prod_{k=2}^{l(\mathbf{p})-1} a^*_{p_k p_{k+1}} \kappa\cdot D a^*_{s,n+1}\Big] \\
&= -\sum_{1\le r<j\le s<n+1} \sum_{\substack{\mathbf{q};\, j=q_{l(\mathbf{q})}; \\ r\ge q_{l(\mathbf{q})-1}}} \sum_{\substack{\mathbf{p},s=p_{l(\mathbf{p})},p_2=2 \\ \exists t:(p_t,p_{t+1})=(r,j)}} \Big(\prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}}\Big) a^*_{r,n+1}\kappa\cdot D a^*_{s,n+1} \prod_{k=2,k\ne t}^{l(\mathbf{p})-1} a^*_{p_k p_{k+1}} \\
&\quad - \sum_{1\le s<n+1} \sum_{\mathbf{q};\, n+1=q_{l(\mathbf{q})};\, s\ge q_{l(\mathbf{q})-1}} \sum_{\mathbf{p};\, s=p_{l(\mathbf{p})},\, p_2=2} \kappa\cdot(\boldsymbol\lambda+D)\Big(\Big(\prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}}\Big)a^*_{s,n+1}\Big) \prod_{k=2}^{l(\mathbf{p})-1} a^*_{p_k p_{k+1}}.
\end{aligned}$$

While on the other hand, we have

$$\begin{aligned}
&[C_{\boldsymbol\lambda}A_{01}] \\
&= -\Big[\sum_{1\le s<n+1} \sum_{\substack{\mathbf{p} \\ s=p_{l(\mathbf{p})}}} \prod_{k=1}^{l(\mathbf{p})-1} a^*_{p_k p_{k+1}} \kappa\cdot D a^*_{s,n+1} {}_{\boldsymbol\lambda} \sum_{1\le r<j\le n+1} a_{rj} \sum_{\substack{\mathbf{q} \\ j=q_{l(\mathbf{q})};\, r\ge q_{l(\mathbf{q})-1},\, q_2=2}} \prod_{l=2}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}} a^*_{r,n+1}\Big] \\
&= \sum_{1\le r<j\le s<n+1} \sum_{\substack{\mathbf{q};\, j=q_{l(\mathbf{q})} \\ r\ge q_{l(\mathbf{q})-1},\, q_2=2}} \sum_{\substack{\mathbf{p},\, s=p_{l(\mathbf{p})} \\ \exists t:(p_t,p_{t+1})=(r,j)}} \Big(\prod_{l=2}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}}\Big) a^*_{r,n+1}\kappa\cdot D a^*_{s,n+1} \\
&\quad \prod_{k=1,l\ne t}^{l(\mathbf{p})-1} a^*_{p_k p_{k+1}} - \sum_{1\le s<n+1} \sum_{\substack{\mathbf{q};\, n+1=q_{l(\mathbf{q})} \\ s\ge q_{l(\mathbf{q})-1},\, q_2=2}} \sum_{\mathbf{p},\, s=p_{l(\mathbf{p})}} \prod_{l=2}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}}\, a^*_{s,n+1}\kappa\cdot(\boldsymbol\lambda+D) \\
&\quad \prod_{k=1}^{l(\mathbf{p})-1} a^*_{p_k p_{k+1}}.
\end{aligned}$$

For the factors with $\mathbf{q}$; $n+1 = q_{l(\mathbf{q})}$; $s \ge q_{l(\mathbf{q})-1}$, $\mathbf{p}$, $s = p_{l(\mathbf{p})}$, $p_2 = 2$, one applies the following type of calculation:

$$\begin{aligned}
&-\kappa\cdot(\boldsymbol\lambda+D)\Big(\prod_{l=1}^{l(\mathbf{q})-1} a^*_{q_l q_{l+1}}\Big) a^*_{s,n+1} \prod_{k=2}^{l(\mathbf{p})-1} a^*_{p_k p_{k+1}} \\
&= -\kappa\cdot(\boldsymbol\lambda+D)\big(a^*_{1q_2}\cdots a^*_{q_{l(\mathbf{q})-1},n+1}\big) a^*_{s,n+1} a^*_{2p_3}\cdots a^*_{l(\mathbf{p})-1,s}
\end{aligned}$$

$$
\begin{aligned}
&= -\kappa \cdot (\boldsymbol{\lambda}+D)\big(a^*_{1q_2}\cdots a^*_{q_{l(\mathbf{q})-1},n+1}\big)a^*_{2p_3}\cdots a^*_{l(\mathbf{p})-1,s}a^*_{s,n+1}\\
&= -\kappa \cdot (\boldsymbol{\lambda}+D)\big(a^*_{1q_2}\cdots a^*_{q_{l(\mathbf{q})-1},n+1}\big)a^*_{2p_3}\cdots a^*_{l(\mathbf{p}),n+1}\\
&= -\kappa \cdot (\boldsymbol{\lambda}+D)\Big(\prod_{l=1}^{l(\mathbf{q}')} a^*_{q'_l q'_{l+1}}\Big)\prod_{k=2}^{l(\mathbf{p})} a^*_{p_k p_{k+1}},
\end{aligned}
$$

where $\mathbf{q}' := (1, q_2, \ldots, q_{l(\mathbf{q})-1}, n+1)$ (so $q'_{l(\mathbf{q}')+1} = n+1$ and $q'_{l(\mathbf{q}')} = q_{l(\mathbf{q})-1}$) and $l(\mathbf{q}') \le s = l(\mathbf{p})$. The other factors can also be simplified, and as a consequence we obtain

$$
\begin{aligned}
&[A_{\boldsymbol{\lambda}}C_{01}] + [C_{\boldsymbol{\lambda}}A_{01}]\\
&= -\sum_{1\le s<n+1}\sum_{\mathbf{q};\, s\ge q_{l(\mathbf{q})}} \kappa\cdot(\boldsymbol{\lambda}+D)\Big(\prod_{l=1}^{l(\mathbf{q})} a^*_{q_l q_{l+1}}\Big)\Big(\sum_{\mathbf{p},\, s=p_{l(\mathbf{p})},\, p_2=2}\ \prod_{k=2}^{l(\mathbf{p})} a^*_{p_k p_{k+1}}\Big)\\
&\quad -\sum_{1\le s<n+1}\sum_{\mathbf{q};\, s= q_{l(\mathbf{q})}} \kappa\cdot(\boldsymbol{\lambda}+D)\Big(\prod_{l=1}^{l(\mathbf{q})} a^*_{q_l q_{l+1}}\Big)\Big(\sum_{\mathbf{p},\, s\ge p_{l(\mathbf{p})},\, p_2=2}\ \prod_{k=2}^{l(\mathbf{p})} a^*_{p_k p_{k+1}}\Big). \qquad (36)
\end{aligned}
$$

Now if we set

$$
\mathcal{B}(t) := \sum_{\mathbf{q};\, t=q_{l(\mathbf{q})}} \prod_{l=1}^{l(\mathbf{q})} a^*_{q_l q_{l+1}} \quad \text{and} \quad \mathcal{C}(t) := \sum_{\mathbf{p},\, t=p_{l(\mathbf{p})},\, p_2=2}\ \prod_{k=2}^{l(\mathbf{p})} a^*_{p_k p_{k+1}},
$$

then by induction one can show that

$$
\begin{aligned}
&\sum_{s=1}^{n}\Big(\Big(\sum_{q=1}^{s}\mathcal{B}(q)\Big)\mathcal{C}(s)\Big) + \sum_{s=1}^{n}\Big(\mathcal{B}(s)\Big(\sum_{q=1}^{s}\mathcal{C}(q)\Big)\Big)\\
&= 2\Big(\Big(\sum_{q=1}^{n}\mathcal{B}(q)\Big)\Big(\sum_{q=1}^{n}\mathcal{C}(q)\Big)\Big)\\
&\quad -\sum_{s=1}^{n-1}\Big(\Big(\sum_{q=1}^{s}\mathcal{B}(q)\Big)\mathcal{C}(s+1)\Big) - \sum_{s=1}^{n-1}\Big(\mathcal{B}(s+1)\Big(\sum_{q=1}^{s}\mathcal{C}(q)\Big)\Big)
\end{aligned}
$$

(the identity above holds for any elements $\mathcal{B}(q)$ and $\mathcal{C}(s)$ in an algebra with coefficients in $\mathbb{Z}$). Applying this identity to the sum of (35) and (36), we show

$$
[B_{\boldsymbol{\lambda}}B_{01}] + [A_{\boldsymbol{\lambda}}C_{01}] + [C_{\boldsymbol{\lambda}}A_{01}] = 0.
$$

This completes the proof that $[\rho(F_{0\boldsymbol{\lambda}})[\rho(F_0)_{\boldsymbol{\mu}}\rho(F_1)]] = 0$.

The remaining relations $[\rho(F_0)_{\boldsymbol{\lambda}}[\rho(F_0)_{\boldsymbol{\mu}}\rho(F_n)]] = 0$ and $[\rho(F_0)_{\boldsymbol{\lambda}}\rho(F_0)] = 0$ are proven in a similar manner, where one applies to the following formal identities:

$$
\begin{aligned}
&\sum_{r=1}^{n}\Big(\sum_{s=1}^{n-1}(-\delta_{r,s-1}+2\delta_{rs}-\delta_{r,s+1})\Big(\sum_{t=1}^{r}\Big(\sum_{v=1}^{s}\mathcal{B}(t)\mathcal{C}(v)\Big)\Big)\Big)\\
&= \sum_{t=1}^{n-1} B(t)C(t) - B(n)\Big(\sum_{t=1}^{n-1} C(t)\Big)
\end{aligned}
$$

and

$$
\Big(\sum_{r=1}^{n-1}\Big(\sum_{q=1}^{r}\mathcal{A}(q)\Big)\mathcal{A}(r+1)\Big) - \Big(\sum_{q=1}^{n}\mathcal{A}(q)\Big)^2 + \sum_{s=1}^{n}\Big(\Big(\sum_{q=1}^{s}\mathcal{A}(q)\Big)\mathcal{A}(s)\Big) = 0.
$$

For the Serre type relations for the E_r, the calculations are the same as those in [9, Lemma 3.5], where $-\gamma b_r(z) - \frac{1}{2}\left(b^+_{r-1}(z) + b^+_{r+1}(z)\right)$ is replaced by $\Phi(b_r)$ and $-\frac{\gamma}{2}\dot{a}^*_{r,r+1}(z)$ is replace by $\kappa \cdot Da^*_{r,r+1}$. We refer the interested reader to that paper for the proof. $\square$

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