A Wakimoto Type Realization of Toroidal $\mathfrak{sl}_{n+1}$

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Abstract. The authors construct a Wakimoto type realization of toroidal $\mathfrak{sl}_{n+1}$. The representation constructed in this paper utilizes non-commuting differential operators acting on the tensor product of two polynomial rings in infinitely many commuting variables.

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1 Introduction

Toroidal Lie algebras were first introduced in [24] as a natural generalization of affine algebras. Given a finite-dimensional simple Lie algebra $\mathfrak{a}$, a toroidal algebra is a central extension of $\mathfrak{a} \otimes \mathbb{C}[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$, where the $t_i$ are commuting variables. Toroidal algebras can be thought of as iterated loop algebras in many commuting variables. Such algebras can also be defined using generators and relations as we do here.

One motivation for the study of toroidal Lie algebras is for potential applications to mathematics and physics. For instance, one of the cocycles used in the construction of the toroidal extended affine Lie algebra is also used in Billig’s study of a magnetic hydrodynamics equation with asymmetric stress tensor (see [7] and [6]). In addition, Billig [5] and independently Iohara, Saito and Wakimoto [17] derive Hirota bilinear equations arising from both homogeneous and principal realizations of the vertex operator representations of 2-toroidal Lie algebras of type $A_l, D_l, E_l$. They derive the hierarchy of Hirota equations and present their soliton-type solutions. In [20], Kakei, Ikeda and Takasaki constructed the hierarchy associated to the $(2 + 1)$-dimensional nonlinear Schrödinger (NLS) equation and show how the representation theory of toroidal $\mathfrak{sl}_2$ can be used to derive the Hirota-type equations for $\tau$-functions. On the somewhat more mathematical side, in interesting work of Ginzburg, Kapranov and Vasserot [16] on Langland’s reciprocity for algebraic surfaces, Hecke operators are constructed for vector bundles on an algebraic surface. The main point of their paper is that under certain conditions, the corresponding

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algebra of Hecke operators is the homomorphic image of a quantum toroidal algebra. One should also note some of the recent work of Slodowy [27], Berman and Moody [3], Benkart and Zelmanov [1] on generalized intersection matrix algebras involve their relationship to toroidal Lie algebras. In addition, Wakimoto’s free field realization of affine \( \hat{\mathfrak{sl}}_2 \) and Feigin and Frenkel’s generalization to non-twisted affine algebras \( \hat{\mathfrak{g}} \) play a fundamental role in describing integral solutions to the Knizhnik-Zamolodchikov equations (see for example [28], [14], [21], [12], [25] and [26]).

The representation constructed here is similar to what is often called a “free field” representation, that is, our Lie algebra elements will be realized as formal power series of non-commuting differentiable operators \( a_n \) \((n \in \mathbb{Z})\) acting on a given vector space \( V \), where the formal power series associated with the Lie algebra become finite when applied to an element \( v \in V \). Our representation is constructed by first finding a representation of an infinite-dimensional Heisenberg like algebra, and then “inducing” to the full toroidal algebra. The free field representation in this paper is a generalization of the second author [9, 10] which were in turn motivated by the work of Feigin and Frenkel [13, 15] constructing free field realizations of affine Kac-Moody and \( W \)-algebras, as well as [4]. A completely different representation of a class of toroidal algebras given by free bosonic fields appears in [18]. Interestingly, some free field representations of toroidal lie algebras can be used to construct vertex algebras (see [2]) of a certain type, where all simple graded modules can be classified (see [23]).

Part of our motivation for studying Wakimoto type realizations of toroidal \( \hat{\mathfrak{sl}}_{n+1} \) is to gain insight into the role of 2-cocycles in a more general construction of free field realizations for universal central extensions of Lie algebras of the form \( \mathfrak{g} \otimes R \), where \( R \) is an algebra over the complex numbers. Another motivation is that they can often provide, in the generic setting, realizations in terms of partial differential operators of imaginary type Verma modules for toroidal Lie algebras. We plan to see how the realizations in this paper are related to these modules in future work.

2 Notation and Preliminary Setting

All vector spaces are over the field of complex numbers \( \mathbb{C} \).

Let \( A_n = (A_{ij})_{i,j=0}^n \) be the indecomposable Cartan matrix of affine type \( A_n^1 \) with \( n \geq 2 \). Let \( \Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \) denote the simple roots, a basis for the set of roots denoted by \( \Delta \). Let \( Q \) be the root lattice, i.e., the free \( \mathbb{Z} \)-module with generators \( \alpha_0, \alpha_1, \ldots, \alpha_n \). The matrix \( A_n \) induces a symmetric bilinear form \((\cdot | \cdot)\) on \( Q \) satisfying \((\alpha_i | \alpha_j) = A_{ij}\). For \( 0 \leq i \leq n \), we set \( \check{\alpha}_i := \alpha_i \).

We review some of the calculus of formal series following [22], and we introduce a slightly modified form of the \( \lambda \)-bracket notation and Fourier transform of [19] which provides a very compressed notation, however many of the calculations are actually done in the more expanded form of [22]. As pointed out in [22], the formal calculus generalizes to several commuting variables, the case used here. Throughout this paper, \( z_i, w_i, x_i, y_i, \lambda_i \) will denote mutually commuting formal variables with \( i \) ranging over some index set. We use multi-index notation, for a positive integer \( k \), given an element \((m_0, m_1, \ldots, m_k) \in \mathbb{Z}^{k+1}\), we write \( \mathbf{m} = (m_0, m_1, \ldots, m_k) \), and define \( z^\mathbf{m} = z_0^{m_0}z_1^{m_1} \cdots z_k^{m_k} \). Denote by \( \mathbf{0} \) the \( k \)-tuple of all zeros, and by \( \mathbf{1} \) the \( k \)-tuple of all ones. Fix a decomposition of \( \mathbb{Z}^{k+1} = \mathbb{Z}_-^{k+1} \cup \{0\} \cup \mathbb{Z}_+^{k+1} \) into
three disjoint subsets such that $Z_{\pm}^{k+1}$ are sets closed under vector addition, i.e., for example, if $j, k \in Z_{\pm}^{k+1}$, then $j + k \in Z_{\pm}^{k+1}$. Define $m > 0$ if $m \in Z_{\pm}^{k+1}$ and $m < 0$ if $m \in Z_{\pm}^{k+1}$. Define the function $\theta$ by $\theta(m) = 1$ if $m > 0$, and 0 otherwise.

We work with formal series $a(z) = \sum_{n \in Z_{\pm}^{k+1}} a_n z^{-n}$ with $a_n \in \text{End}(V)$ for a vector space $V$ described below. The series in this paper are summable in the sense of [22], i.e., the coefficient of any monomial in the formal sum acts as a finite sum of operators when applied to any vector $v \in \text{End}(V)$. To simplify notation, we denote $\mathbb{C}[z_0, z_0^{-1}, z_1, z_1^{-1}, \ldots, z_k, z_k^{-1}]$ as $\mathbb{C}[z, z^{-1}]$. Define $\delta(z) := \sum_{n \in Z_{\pm}^{k+1}} z^n \in \mathbb{C}[z, z^{-1}]$. Similarly, $\delta(z/w) := \sum_{m \in Z_{\pm}^{k+1}} z^m w^{-m} \in \mathbb{C}[z, z^{-1}, w, w^{-1}]$ so that $\delta(z/w) = \prod_{i=0}^{k} \delta(z_i/w_i)$, where $\delta(z_i/w_i) = \sum_{k \in \mathbb{Z}} z^k w^{-k}$.

The following properties of $\delta$ hold (see [22, Proposition 2.1.8]) which we reproduce here in the multivariable setting.

**Proposition 2.1.**

(i) Let $f(z) \in V[z, z^{-1}]$. Then $f(z)\delta(z) = f(1)\delta(z)$.

(ii) Let $f(z, w) \in \text{End}(V)[z, z^{-1}, w, w^{-1}]$ such that $\lim_{z \to w} f(z, w)$ exists. Then in $\text{End}(V)[z, z^{-1}, w, w^{-1}]$, $f(z, w)\delta(z/w) = f(z, z)\delta(z/w) = f(w, w)\delta(z/w)$.

The formal residue for an element $f(z) \in V[z, z^{-1}]$ is $\text{Res}_z \sum_{n \in Z_{\pm}^{k+1}} a_n z^n = \sum_{n \in Z_{\pm}^{k+1}} a_n z^n$. Alternatively, we can define $\text{Res}_z$ as $\text{Res}_z \sum_{n \in Z_{\pm}^{k+1}} a_n z^n = a_{-1}$, the coefficient of $z^{-1}$. We introduce a slightly modified form of Kac's $\lambda$-bracket notation and Fourier transform (see [19]). For any $a(z, w) = \sum_{m, n} a_{m, n} z^m w^n$, we define the Fourier transform $F_{z, w}^\lambda a(z, w) = \text{Res}_z \cdots \text{Res}_z a(z, w) e^{\sum_{i=0}^{k} \lambda_i (z_i - w_i)}$.

For $j = (j_0, \ldots, j_k) \in \mathbb{N}^{k+1}$, set $j! = j_0! j_1! \cdots j_k!$, $\partial_{w_i}^{(j)} = 1/j! \partial_{w_i}^{j_i}$, and $\partial^{(j)} = \prod_{j=0}^{k} \partial_{w_i}^{(j_i)}$. We also write $\lambda^{(j)} := \lambda^{j}/j! = \prod_{i=0}^{k} \lambda_i^{j_i} / j!$. Using this notation allows us to compress many of the formal series we will encounter, due to the following identity

$$F_{z, w}^\lambda \partial^{(j)} \delta(z/w) = \lambda^{(j)}. \quad (1)$$

To prove identity (1), we recall a few properties shown in [19, Proposition 2.1]: For $j > 0$,

$$\text{Res}_z \partial_{w_i}^{(j)} \delta(z/w) = 0, \quad (z - w)\partial_{w_i}^{(j+1)} \delta(z/w) = \partial_{w_i}^{(j)} \delta(z/w), \quad (z - w)^{j+1} \partial_{w_i}^{(j)} \delta(z/w) = 0.$$

Thus,

$$F_{z, w}^\lambda \partial^{(j)} \delta(z/w) = \text{Res}_z \cdots \text{Res}_z e^{\sum_{i=0}^{k} \lambda_i (z_i - w_i)} \partial^{(j)} \delta(z/w) = \text{Res}_z \cdots \text{Res}_z \left( \prod_{i=0}^{k} \left( \sum_{k_i=0}^{\infty} \frac{1}{k_i!} \lambda_i^{k_i} (z_i - w_i)^{k_i} \right) \prod_{i=0}^{j} \partial_{w_i}^{(j_i)} \delta(z_i/w_i) \right).$$
Then we use frequently include

\[ \frac{\partial}{\partial z_i} \delta(z_i/w_i) = \delta(z_i/w_i) - z_i \delta(z_i/w_i) = 0. \]

\[ \partial \delta(z_i/w_i) = 0. \]

If \( a(z), b(w) \) and \( c(w) \) are formal distributions satisfying

\[ [a(z), b(w)] = \sum_{j \in \mathbb{N}^{k+1}} c(w) \delta^{(j)}(z/w), \]

we have \( F_{z,w}^\lambda[a(z), b(w)] = \sum_{j \in \mathbb{N}^{k+1}} c(w) \lambda^{(j)}. \) The \( \lambda \)-bracket is defined as

\[ [a(w)\lambda, b(w)] = \sum_{j \in \mathbb{N}^{k+1}} c(w) \lambda^{(j)}, \]

achieving the compressed notation. When the variables are clear from the context, we sometimes omit the formal multivariables \( z, w \). Properties of the \( \lambda \)-bracket that we use frequently include

\[ [a[b,c]] = [[a\lambda]b]_{\lambda+\eta}c + [b[a\lambda]c], \quad [a\lambda b]c + [a\lambda c] = [a\lambda bc]. \]  \hspace{1cm} (2)

2.1 The toroidal Lie algebra. Fix a positive integer \( N \). We define the toroidal Lie algebra \( \tau(A_n) \) by generators and relations. The generators are

\[ K_{m,j}, \quad H_i(m), \quad E_i(m), \quad F_i(m) \quad (0 \leq i \leq n, \ 0 \leq j \leq N, \ m \in \mathbb{Z}^{N+1}). \]

We will use generating functions to write the relations of the algebra. The generators of \( \tau(A_n) \) \((1 \leq i \leq n, \ 0 \leq s \leq N)\) have generating functions:

\[ K_s(z) = \sum_{m \in \mathbb{Z}^{N+1}} K_{m,0} z^{-m}, \quad H_i(z) = \sum_{m \in \mathbb{Z}^{N+1}} H_i(m) z^{-m}, \]

\[ E_i(z) = \sum_{m \in \mathbb{Z}^{N+1}} E_i(m) z^{-m}, \quad F_i(z) = \sum_{m \in \mathbb{Z}^{N+1}} E_i(m) z^{-m}. \]

Let \( \partial_{z_i} = \frac{\partial}{\partial z_i} \) denote the formal differentiation. Define the operator \( D_{z_i} = \partial_{z_i} \), and

\[ D := \sum_{s=0}^{N} D_{z_i} \] (the indeterminate in use is understood in the context of the formula).

Then

\[ K(z) \cdot D = \sum_{i=0}^{N} K_i(z) \frac{\partial}{\partial z_i}, \quad K(z) = \sum_{i=0}^{N} K_i(z) = \sum_{i=0}^{N} \sum_{m \in \mathbb{Z}^{N+1}} K_{m,i} z^{-m} z_i, \]

\[ D \cdot K(z) = \sum_{s=0}^{N} \sum_{m \in \mathbb{Z}^{N+1}} m_s K_{m,s} z^m. \]

The relations of the toroidal algebra \( \tau(A_n) \) are given by the generating functions:

\( (R0) \) \( K(z) \) is central, \( D \cdot K(z) = 0; \)

\( (R1) \) \([H_i(z), H_j(w)] = A_{ij} K(w) \cdot D \delta(z/w); \)

\( (R2) \) \([H_i(z), E_j(w)] = A_{ij} E_j(w) \delta(z/w), \ [H_i(z), F_j(w)] = -A_{ij} F_j(w) \delta(z/w); \)

\( (R3) \) \([E_i(z), F_j(w)] = -\delta_{i,j} [H_i(w) + \frac{2}{\lambda} K(w) \cdot D] \delta(z/w); \)

\( (R4) \) \([E_i(z), E_j(w)] = 0 = [F_i(z), F_j(w)], \ \text{ad} E_i^{-A_{ij}+1}(z) E_j(w) = 0 \) for \( i \neq j, \)

\[ \text{ad} F_i^{-A_{ij}+1}(z) F_j(w) = 0 \] for \( i \neq j. \)
3 The Toroidal Heisenberg Algebra

Define the toroidal Heisenberg algebra $\mathfrak{B}$ as the Lie algebra with generators $b_i(r)$ ($1 \leq i \leq n$) and $K_{r,p}$ ($0 \leq p \leq N$, $r \in \mathbb{Z}^{n+1}$) which satisfy the following relations:

$$[b_i(r), b_j(s)] = A_{ij} \sum_{p=0}^{N} r_p K_{r+s,p} \quad \text{and} \quad \sum_{p=0}^{N} r_p K_{r,p} = 0 \quad \forall \, r \in \mathbb{Z}^{N+1}. \quad (3)$$

Here $A_{ij}$ denotes the $(i,j)$-th entry of the Cartan matrix $A_n$, where we have deleted the first row and column. If we set

$$b_0(m) := -\sum_{i=1}^{n} b_i(m), \quad (4)$$

then one can check that the first equality in (3) is also satisfied for $i = 0$ or $j = 0$.

3.1 Representation of the Heisenberg algebra. We define a polynomial ring over indeterminates indexed by $0 \leq i \leq n+1$ and $k \in \mathbb{Z}^{N+1}$:

$$\mathbb{C}[y] := \mathbb{C}[y_i(k) \mid 0 < k \in \mathbb{Z}^{N+1}, 1 \leq i \leq n].$$

For fixed $\kappa_{m,p} \in \mathbb{C}$, $0 \leq p \leq N$ and $\lambda_i \in \mathbb{C}$, we define a map $\Phi : \mathfrak{B} \to \text{End} \mathbb{C}[y]$ below by an action on the generators. The construction of the map is similar to that appearing in [10]. The motivation for the definition of $\Phi$ uses heuristic ideas about how the toroidal Lie algebra “should” act on sections of certain (not well defined) line bundles. For readers who are interested in this heuristic type of construction, one could consult [13], [4] and [11]. The resulting map $\Phi$ is twisted as in [10] so that $\Phi(b_0(m))$ is a well defined element of $\text{End} \mathbb{C}[y]$. The definition of $\Phi(b_0(m))$ follows from the definition (4).

**Proposition 3.1.** (Realization of the Toroidal Heisenberg Algebra) Fix $\kappa_{m,p} \in \mathbb{C}$, $0 \leq p \leq N$ and $\lambda_i \in \mathbb{C}$, where $0 \leq i \leq n$. Assume

$$\sum_{p=0}^{N} m_p \kappa_{m,p} = 0 \quad \text{for all } m, \quad (5)$$

$$\sum_{p=0}^{N} m_p \kappa_{-m-n,p} = 0 \quad \text{for } m > 0 \text{ and } n > 0. \quad (6)$$

Then the map $\Phi : \mathfrak{B} \to \text{End} \mathbb{C}[y]$ given by

$$\Phi(b_i(m)) = \theta(-m) \sum_{p=0}^{N} \sum_{s>0} \left( \partial_{y_{i-1}}(s) - \partial_{y_{i}}(s) \right) m_p \kappa_{-m-s,p} \quad + \theta(m) \sum_{p=0}^{N} \sum_{s>0} \left( \partial_{y_{i-1}}(s) - 2\partial_{y_{i}}(s) + \partial_{y_{i+1}}(s) \right) m_p \kappa_{-m-s,p}$$

$$+ \theta(-m) \lambda_{i} \delta_{m,0} \lambda_{i},$$

$$\Phi(K_{m+n,p}) = -\kappa_{-m-n,p}$$

for $1 \leq i \leq n$ and $m,n \in \mathbb{Z}^{N+1}$ defines a representation $\mathfrak{B}$ on $\mathbb{C}[y]$. 


Proof. For $1 < i, j \leq n$, we have

$$\left[ \Phi(b_i(m)), \Phi(b_j(n)) \right]$$

$$\begin{align*}
= & \theta(-n)\theta(-m)(\delta_{j, i-1} - \delta_{j, i}) \sum_{p=0}^{N} m_p \kappa_{-m - n, p} \\
+ & \theta(-n)\theta(m)(\delta_{j, i-1} - 2\delta_{i, j} + \delta_{j, i+1}) \sum_{p=0}^{N} m_p \kappa_{-m - n, p} \\
- & \theta(-m)\theta(-n)(\delta_{j, i-1} - \delta_{i, j}) \sum_{q=0}^{N} n_q \kappa_{-m - n, q} \\
- & \theta(-m)\theta(m)(\delta_{j, i-1} - 2\delta_{i, j} + \delta_{j, i+1}) \sum_{q=0}^{N} n_q \kappa_{-m - n, q} \\
= & (\theta(n)\theta(-m) + \theta(-n)\theta(m) + \theta(-m)\theta(-n)) (\delta_{j, i-1} - 2\delta_{i, j} + \delta_{j, i+1}) \sum_{p=0}^{N} m_p \kappa_{-m - n, p} \\
= & (\delta_{j, i-1} - 2\delta_{i, j} + \delta_{j, i+1}) \sum_{p=0}^{N} m_p \kappa_{-m - n, p} = -A_{ij} \sum_{p=0}^{N} m_p \kappa_{-m - n, p},
\end{align*}$$

where in the last two equalities we used the hypotheses (5) and (6), respectively. The remaining relations are also straightforward. □

4 Main Result, the Representation of the Toroidal Algebra

Let $i, j \leq n + 1$, $m \in \mathbb{Z}^{N+1}$ and

$$\mathbb{C}[x] := \mathbb{C}[x_{ij}(m) \mid 0 < i < j \leq n + 1, m \in \mathbb{Z}^{N+1}] .$$

The elements $x_{ij}(m)$ act via multiplication on the ring $\mathbb{C}[x]$, and hence on the ring $\mathbb{C}[x] \otimes \mathbb{C}[y]$ (as $x_{ij}(m) \otimes 1$). Define the following differential operators acting on the polynomial ring $\mathbb{C}[x] \otimes \mathbb{C}[y]$:

$$a_{ij, m} := -x_{ij}(m), \quad a_{ij, m}^* := \frac{\partial}{\partial x_{ij}(-m)} . \quad (7)$$

With corresponding generating functions

$$a_{ij}(z) = \sum_{m \in \mathbb{Z}^{N+1}} a_{ij, m} z^{-m}, \quad a_{ij}(z) = \sum_{m \in \mathbb{Z}^{N+1}} a_{ij, m}^* z^{-m}, \quad \kappa_i(z) = \sum_{m \in \mathbb{Z}^{N+1}} \kappa_{m, i} z^m,$$

define the operators

$$\kappa(z) \cdot D = \sum_{i=0}^{N} \kappa_i(z) \frac{\partial}{\partial z_i}, \quad \kappa(z) = \sum_{i=0}^{N} \kappa_i(z) = \sum_{i=0}^{N} \sum_{m} \kappa_{m, i} z^m z_i.$$

Note that $\kappa(z) \cdot D$ is a weighted version of Euler’s differential operator. The operators $\Phi(b_i)$ commute with the $a_{ij, m}, a_{ij, m}^*$ and act on $\mathbb{C}[x] \otimes \mathbb{C}[y]$ as $1 \otimes \Phi(b_i)$.

**Theorem 4.1. (Realization)** Let $\kappa_{m, i}$ be fixed complex numbers satisfying conditions (5) and (6) and fix $\lambda_i \in \mathbb{C}$ for $0 \leq i \leq n$. Then the generating functions given below

$$\rho(F_r)(z) = a_{r, r+1}(z) - \sum_{j=1}^{r-1} a_{j, r+1}(z) a_{jr}^*(z),$$

are realizations of the elements $\kappa_{m, i}$. The remaining elements $\delta_{j, i} - \delta_{i, j}$ are represented by the operators $A_{ij}$.
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\[ \rho(H_r)(z) = 2a_{r,r+1}(z)a^*_{r,r+1}(z) + \sum_{i=1}^{r-1} (a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z)) \]
\[ + \sum_{j=r+2}^{n+1} (a_{r,j}(z)a^*_{r,j}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z)) + \Phi(h_r)(z), \]
\[ \rho(E_r)(z) = a_{r,r+1}(z)a^*_{r,r+1}(z) \]
\[ - \sum_{j=r+2}^{n+1} a_{r+1,j}(z)a^*_{r,j}(z) + \sum_{j=r+2}^{n+1} a_{jr}(z)a^*_{j,r+1}(z) \]
\[ + \sum_{j=r+2}^{n+1} (a_{r,j}(z)a^*_{r,j}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z)) a^*_{r,r+1}(z) \]
\[ + a^*_{r,r+1}(z)\Phi(b_r)(z) + \kappa \cdot Da^*_{r,r+1}(z), \]

for $1 \leq r \leq n$, together with

\[ \rho(E_0)(z) = -a^*_{1,n+1}(z), \]
\[ \rho(H_0)(z) = -\sum_{r=1}^{n} \rho(H_r)(z) \]
\[ = -\sum_{r=1}^{n} a_{r,n+1}(z)a^*_{r,n+1}(z) - \sum_{z=2}^{n+1} (z)a^*_{1r}(z) + \Phi(h_1)(z), \]
\[ \rho(F_0)(z) = \sum_{1 \leq r < j \leq n+1} -a_{r,j}(z) \sum_{qj \geq q_i \geq 1} a^*_{qj,q_i+1}(z)a^*_{r,q_i+1}(z) \]
\[ - \sum_{1 \leq r < n+1 \leq n+1} \prod_{q,j \geq 1} a^*_{qj,q_i+1}(z)\Phi(h_r)(z) \]
\[ - \sum_{1 \leq r < n+1 \leq n+1} \prod_{q,j \geq 1} a^*_{qj,q_i+1}(z)\kappa \cdot Da^*_{r,n+1}(z), \]

define an action of the generators $E_r(m)$, $F_r(m)$ and $H_r(m)$ and a representation of $\tau(A_4)$ on the Fock space $C[x] \otimes C[y]$. In the partitions above, $1 = q_1 < q_2 < \cdots < q_i$, $q_{i+1} = n+1$. In addition, $K_m \otimes_l$ acts as left multiplication by $-\kappa^{-m} l$.

Note that one should also have $\rho(E_{ik}) = -a_{kl} + \sum_{j=1}^{k-1} a_{jl}a^*_{jk}$ for $k < l$, but we do not need to know this general formula, so we do not determine whether it is always true.

5 Proof of the Main Result

We should point out that the proof requires very lengthy (at least to us) calculations. We have selected representative portions of the calculations to include here, from an original manuscript of over one hundred pages, a version of which is available (see [8]). Calculations similar to those omitted can be found in [9] and [10], students may also wish to specialize to the special cases of type $A_2$ and $A_3$ especially the latter which is a good guide for the general setting of $n \geq 2$.

Let $\Phi(b_r) := \Phi(b_r)(z) = \sum_{m} \Phi(b_r)(m)z^{-m}$, then we can write the last calculation in the proof of Proposition 3.1 as $[\Phi(b_r)\chi \Phi(b_s)] = A_{rs} \sum_{l=0}^{N} \rho(K_l)\lambda_l = -A_{rs}\kappa \cdot \lambda$. Set
The relations (R1)–(R4) will follow if the following identities are satisfied:

\[ \rho(H_i)(w)\lambda(v)(F_j)(w) = A_{ij} \sum_{l=0}^{N} \rho(K_l)(w)\lambda_l \] (0 ≤ i, j ≤ n);

\[ \rho(H_i)(w)\lambda(v)(E_j)(w) = A_{ij} \rho(E_j)(w), \quad \rho(H_i)(w)\lambda(v)(F_j)(w) = -A_{ij} \rho(F_j)(w); \]

\[ \rho(E_i)(w)\lambda(v)(E_j)(w) = -\delta_{i,j} \left( \rho(H_i)(w) + \frac{1}{N} \sum_{l=0}^{N} \rho(K_l)(w)\lambda_l \right); \]

\[ \rho(E_i)(w)\lambda(v)(E_j)(w) = 0 = \rho(F_i)(w)\lambda(v)(F_j)(w) \] if |i - j| ≠ 1.

Proof. We demonstrate how to write the relation (R1) in \( \lambda \)-bracket form:

\[
\sum_{m,n} [H_i(m), H_j(n)] z^{-m} w^{-n} = A_{ij} \sum_{l=0}^{N} K_l(w) \partial_{w^l} \delta(z/w) = A_{ij} \sum_{l=0}^{N+1} c^l(w) \partial^{(j)} \delta(z/w),
\]

where \( c^l(w) \) is defined as follows: we take \( \epsilon_l \) to be the \( N \)-tuple with 1 in the \( l \)-th position and zeros elsewhere, and define \( c^l(w) = K_1(w) \) and \( c^l(w) = 0 \) if \( j \neq \epsilon_l \) for some \( 0 \leq l \leq N \). Applying \( F^\lambda \) gives the result.

\[ \square \]

5.1 Preliminary lemmas. We have the following identities for \( a_{i,j,m} \) and \( a_{i,j,m}^* \) as in (7), whose proofs carry over from \([10, \text{Lemma 4.1}]\). The identities are written which we write in terms of the \( \lambda \)-bracket notation. In the interest of compressing the notation, we will often suppress the variables \( z, w \) in the computations, especially when using the \( \lambda \)-notation, where the presence of the multivariable \( w \) is assumed.

Lemma 5.1. \([10]\) Let \( i,j,k,l \in \mathbb{Z} \). Then for the generating functions \( a_{i,j}(w) \) and \( a_{i,j}^*(w) \), the following identities hold:

(a) \[ a_{i,j}(w)\lambda a_{i,j}^*(w) = \delta_{i,k}\delta_{j,l}, \]

(b) \[ a_{i,j}(w)a_{i,j}^*(w)\lambda a_{i,j}(w)a_{i,j}^*(w) = 0, \]

(c) \[ a_{i,j}(w)\lambda \cdot D a_{i,j}^*(w) = \delta_{i,k}\delta_{j,l} \sum_{p=0}^{N} \kappa_p \lambda_p = [\kappa \cdot D a_{i,j}^*(w)\lambda a_{i,j}(w)], \]

(d) \[ \sum_{j=r+2}^{s+1} \sum_{k=1}^{l+1} a_{k,s+1}(w)a_{l,j}(w)\lambda a_{j,r}(w)a_{r,j}^*(w) = -\delta_{s,r+1} a_{r,r+1}(w)a_{s,r+2}^*(w), \]

(e) \[ \sum_{j=r+2}^{s+1} \sum_{k=1}^{l+1} a_{k,s+1}(w)\lambda a_{j,r}(w)a_{r,j}^*(w) = 0, \]

(f) \[ \sum_{j=r+2}^{s+1} \sum_{k=1}^{l+1} a_{k,s+1}(w)a_{l,j}(w)\lambda a_{r+1,j}(w)a_{r+1,j}^*(w) = 0, \]

\[ \sum_{j=r+2}^{s+1} \sum_{k=1}^{l+1} a_{k,s+1}(w)a_{l,j}(w)\lambda a_{r+1,j}(w)a_{r+1,j}^*(w) = -2\delta_{r,s} \sum_{j=r+2}^{s+1} a_{r+1,j}(w)a_{r+1,j}^*(w) + \delta_{r,s+1} \sum_{j=r+3}^{s+1} a_{r+1,j}(w)a_{r-1,j}^*(w) \] + \delta_{r,s+1} \sum_{j=r+3}^{s+1} a_{r+2,k}(w)a_{r+1,j}^*(w), \]
\[
\sum_{j=1}^{r+1} a^*_j(z) \cdot \sum_{j=1}^{r+1} a_j(z) = a^*_r(z) \cdot a_r(z) + a^*_s(z) \cdot a_s(z) - a^*_r(z) \cdot a_s(z) - a^*_s(z) \cdot a_r(z).
\]

**Proof.** Only statement (c) is new, as shown. Therefore, we have completed the proof.

In addition, the following consequences of Lemma 5.1 are useful:

**Lemma 5.2.** The following identities hold:

\[
[a_{mn}(w) a^*_{mn}(w)] \cdot \kappa D^*_{mn}(w) = \delta_{mn} \delta_{j,s+1} a_j(z) a_s(z) \quad \text{if } n = s + 1,
\]

\[
[a_{ij}(w) a^*_{ij}(w)] \cdot \kappa D^*_{mn}(w) = \delta_{mn} \delta_{j,s+1} a^*_{ij}(w) \quad \text{if } n = s + 1,
\]

\[
[a_{ij}(w) a^*_{ij}(w)] \cdot \kappa D^*_{mn}(w) = \delta_{mn} \delta_{j,s+1} a^*_{ij}(w) \quad \text{if } n = s + 1,
\]

\[
[a_{ij}(w) a^*_{ij}(w)] \cdot \kappa D^*_{mn}(w) = \delta_{mn} \delta_{j,s+1} a^*_{ij}(w) \quad \text{if } n = s + 1,
\]

Note that the formal multivariable in the series \(a_{ij}(z)\) is not affected by the operator \(\kappa(w) D_w\) which acts on series in \(w\). Now applying the transform \(F^\Lambda_{z,w}\) gives the result.
5.2 Relations involving $H(z)$. The relations (T1) and (T2) involving the $H_i(z)$ are simpler to verify than those of type (T3) and (T4), so we begin with them. A reader familiar with other free field representations or vertex algebras can verify relation (T1) as an exercise (see also [9]). Because of the definition of $a_{ij}$ and $a^*_{ij}$ given in (7), there are no multiple contractions when computing out the operator product expansion for these terms. To further compress the notation, we sometimes omit the multivariable $w$ in our computations when the variable is clear from the context.

Lemma 5.3. (T2) $[\rho(H_r)(w)\lambda\rho(E_s)(w)] = A_{rs}\rho(E_s)(w)$.

Proof. First assume $r, s \neq 0$. If $|r - s| > 1$, observe that the indices of $a_{ij}$ and $a^*_{ij}$ that appear in $\rho(H_r)(z)$ and $\rho(E_s)(w)$ are disjoint and thus by Lemma 5.1(a) contribute nothing to the $\lambda$-bracket $[\rho(H_r)(w)\lambda\rho(E_s)(w)]$ (or equivalently to the commutator $[\rho(H_r)(z), \rho(E_s)(w)]$). The remaining terms coming from the $b_j$ have trivial commutator and thus $[\rho(H_r)(w)\lambda\rho(E_s)(w)] = 0$.

Now assume $r = s$ (with $r, s \neq 0$). In this case, $\rho(E_s)(w)$ is equal to

$$a_{r,r+1}a^*_{r,r+1} + \sum_{j=r+2}^{n+1} (a_{r,j}a^*_{r,j} - a_{r+1,j}a^*_{r+1,j})a^*_r$$

$$+ \sum_{j=r+2}^{n+1} a_{r,j}a^*_{r,j} - \sum_{j=r+2}^{n+1} a_{r+1,j}a^*_{r+1,j} + a_{r,r+1}\Phi(b_r) + \kappa \cdot Da^*_{r,r+1},$$

and $\rho(H_r)(w)$ expands to

$$2a_{r,r+1}a^*_{r,r+1} + \sum_{i=1}^{r-1} (a_{i,r+1}a^*_{i,r+1} - a_{i,r}a^*_{i,r}) + \sum_{j=r+2}^{n+1} (a_{r,j}a^*_{r,j} - a_{r+1,j}a^*_{r+1,j}) + \Phi(b_r)$$

(where we have suppressed the variable $w$). Now

$$2[a_{r,r+1}a^*_{r,r+1}\lambda\rho(E_r)]$$

$$= 2a_{r,r+1}a^*_{r,r+1}a^*_r + 2\sum_{j=r+2}^{n+1} (a_{r,j}a^*_{r,j} - a_{r+1,j}a^*_{r+1,j})a^*_r$$

$$+ 2a^*_{r,r+1}\Phi(b_r) + 2\kappa \cdot Da^*_{r,r+1} + 2a^*_{r,r+1}\sum_{l=0}^{N} \kappa_l \lambda_l. \quad (10)$$

The second summation in $\rho(H_r)(w)$ contributes

$$\sum_{i=1}^{r-1} [(a_{i,r+1}a^*_{i,r+1} - a_{i,r}a^*_{i,r})\lambda\rho(E_r)]$$

$$= \sum_{i=1}^{r-1} [(a_{i,r+1}a^*_{i,r+1} - a_{i,r}a^*_{i,r})a_{ir}a^*_{ir,r+1}] = 2\sum_{i=1}^{r-1} a_{ir}a^*_{ir,r+1}. \quad (11)$$

Now in the third summation in $\rho(H_r)(w)$, the index $j$ is greater than or equal to $r + 2$, and so commutes with all but the second and fourth terms of $\rho(E_r)(w)$ above,
thus

\[ \sum_{j=r+2}^{n+1} \left( (a_{r,j}a_{r,j}^* - a_{r+1,j}a_{r+1,j}^*) \chi(\rho(E_r)) \right) = \sum_{j=r+2}^{n+1} \left( [a_{r,j}a_{r,j}^* \lambda a_{r,j}^*] + [a_{r+1,j}a_{r+1,j}^* \lambda a_{r+1,j}^*] \right) - \sum_{i=r+2}^{n+1} \sum_{j=r+2}^{n+1} \left( (a_{r,i}a_{r,i}^* - a_{r+1,i}a_{r+1,i}^*) \chi a_{r+1,i}a_{r+1,i}^* \right) = -2 \sum_{j=r+2}^{n+1} a_{r+1,j}a_{r+1,j}^* \]

by Lemma 5.1(b) and (g). The last term in \( \rho(H_r)(w) \) contributes

\[ [\Phi(b_r) \chi(\rho(E_r))] = [\Phi(b_r) \lambda a_{r+1,r+1}^* \Phi(b_r)] = -2a_{r,r+1}^* \sum_{r=0}^{N} k_r \lambda_r. \]

The previous four calculations (10), (11), (12) and (13) sum up to give us the desired result \( [\rho(H_r) \chi(\rho(E_r))] = 2\rho(E_r). \)

Now suppose \( s = r + 1 \) so that \( \rho(E_{r+1})(w) \) is equal to

\[ a_{r+1,r+2}(a_{r+1,r+2}^*)^2 + \sum_{r+3}^{n+1} (a_{r+1,j}a_{r+1,j}^* - a_{r+2,j}a_{r+2,j}^*) a_{r+1,r+2}^* \]

\[ + \sum_{i=r+1}^{r} a_{i,r+1}a_{i,r+2}^* - \sum_{j=r+3}^{n+1} a_{r+2,j}a_{r+1,j}^* \]

Then the first summand in \( \rho(H_r)(w) \) contributes

\[ 2[a_{r,r+1}a_{r+1,r+1}^* \chi(\rho(E_{r+1}))] = -2a_{r,r+1}a_{r+1,r+2}^*. \]

The second summand in \( H_r(w) \) contributes

\[ \sum_{i=r+1}^{r-1} \frac{[a_{i,r+1}a_{i,r+1}^* - a_{i,r}a_{i,r}^*] \chi(\rho(E_{r+1}))}{\lambda} = \sum_{i=r+1}^{r-1} \frac{[a_{i,r+1}a_{i,r+1}^* - a_{i,r}a_{i,r}^*] \chi a_{i,r+1}a_{i,r+2}^*}{\lambda} = -\sum_{i=r+1}^{r-1} a_{i,r+1}a_{i,r+2}^*. \]

The third summand contributes by Lemma 5.1

\[ \sum_{j=r+2}^{n+1} \left( (a_{r,j}a_{r,j}^* - a_{r+1,j}a_{r+1,j}^*) \chi(\rho(E_{r+1})) \right) = -a_{r+1,r+2}a_{r+1,r+2}^* + \sum_{j=r+3}^{n+1} (a_{r+1,j}a_{r+1,j}^* - a_{r+2,j}a_{r+2,j}^*) a_{r+1,r+2}^* \]

\[ -a_{r+1,2}a_{r+1,2}^* = -\sum_{j=r+3}^{n+1} a_{r+2,j}a_{r+1,j}^* - a_{r+1,r+2}^* \Phi(b_{r+1}) \]

\[ -\kappa \cdot Da_{r+1,r+2}^* - a_{r+1,r+2}^* \sum_{i=0}^{N} k_i \lambda_i. \]
The last summation in \( \rho(H_r)(w) \) has \( \lambda \)-bracket with \( \rho(E_{r+1})(w) \) equal to
\[
[\Phi(b_r)\lambda \rho(E_{r+1})] = a_{r+1,r+2}^+ \rho(b_r) \lambda \rho(b_{r+1}) = a_{r+1,r+2}^+ \sum_{p=0}^N \kappa_p \lambda_p.
\]
Adding the previous four equations up, we get \( [\rho(H_r)\lambda \rho(E_{r+1})] = -\rho(E_{r+1}). \)

The final nontrivial case to consider is when \( s = r - 1 \) (and \( rs \neq 0 \)) so that
\( \rho(E_{r-1})(w) \) is equal to
\[
ar_{r-1,r}(a_{r-1,r}^+)^2 + \sum_{j=r+1}^{n+1} (ar_{r-1,j}a_{r-1,j}^+ - ar_{r,j}a_{r,j}^+) a_{r-1,r}^+ + \sum_{j=1}^{r-2} a_{j,r-1}a_{j,r} + a_{r-1,r}^+ \Phi(b_{r-1}) + \kappa \cdot Da_{r-1,r}^+.
\]
Then \( 2[a_{r+1,r+1}a_{r+1,r+1}^+ \lambda \rho(E_{r-1})] = -2a_{r-1,r+1}a_{r+1,a_{r+1,r}}. \)

The second summation in \( \rho(H_r)(w) \) contributes by Lemma 5.1
\[
\sum_{i=1}^{r-1} [(a_{i,r+1}a_{i,r+1}^+ - a_{ir}a_{ir}^+) \lambda \rho(E_{r-1})] = -a_{r-1,r}a_{r-1,r}^+ - \sum_{j=r+1}^{n+1} (ar_{r-1,j}a_{r-1,j}^+ - ar_{r,j}a_{r,j}^+) a_{r-1,r}^+ + \sum_{j=1}^{r-2} a_{j,r+1}a_{j,r} + a_{r-1,r}^+ \Phi(b_{r-1}) - \kappa \cdot Da_{r-1,r}^+ - \sum_{p=0}^N \kappa_p \lambda_p.
\]

The third summation contributes
\[
\sum_{j=r+2}^{n+1} [(ar_{r,j}a_{r,j}^+ - a_{r+1,j}a_{r+1,j}^+) \lambda \rho(E_{r-1})] = \sum_{j=r+2}^{n+1} a_{r,j}a_{r,j}^+.
\]

The last summation in \( \rho(H_r)(w) \) has \( \lambda \)-bracket with \( \rho(E_{r-1})(w) \) that reduces to
\[
[\Phi(b_r)\lambda \rho(E_{r-1})] = -N \sum_{p=0}^N \kappa_p \lambda_p.
\]

Summing the previous four equations gives \( [\rho(H_r)(w), \rho(E_{r-1})(w)] = -\rho(E_{r-1})(w). \)
We now consider the case of \( s = 0 \) and \( r \neq 0 \). Then \( \rho(E_s)(w) = \rho(E_0)(w) = -a_{1,n+1} \) and hence \( 2[a_{r+1,a_{r+1}^+} \lambda \rho(E_0)] = 0. \)

The second summation in \( \rho(H_r)(w) \) contributes
\[
-\sum_{i=1}^{r-1} [(a_{i,r+1}a_{i,r+1}^+ - a_{ir}a_{ir}^+) \lambda a_{1,n+1}] = \delta_{r,n}a_{1,n+1}.
\]

The third summation contributes
\[
-\sum_{j=r+2}^{n+1} [(ar_{r,j}a_{r,j}^+ - a_{r+1,j}a_{r+1,j}^+) \lambda a_{1,n+1}] = \delta_{r,1}a_{1,n+1}.
\]

The last summation in \( \rho(H_r)(z) \) has commutator with \( \rho(E_0)(w) \) equal to 0, and hence does not contribute to the \( \lambda \)-bracket. Summing the previous three equations, we get \( [\rho(H_r)(w)\lambda \rho(E_0)(w) = -\rho(E_0)(w). \)
If \( r = 0 \), since \( \rho(H_0) = -\sum_{r=1}^{n} \rho(H_r) \), we get

\[
[r(H_0)](w) = -\sum_{r=1}^{n} [r(H_r)](w)
\]

which holds for any \( s \). This completes the proof of the lemma.

Since our expression for \( F_0(w) \) is quite different from that of \( F_i(w) \) if \( i \neq 0 \), we will prove that case separately. First we consider the case:

**Lemma 5.4.** (T2) For \( r, s \neq 0 \), \( [r(H_r)](w) \rho(F_s)(w)] = -A_{rs} \rho(F_s)(w) \).

**Proof.** We assume \( s, r \neq 0 \) in

\[
[H_r F_s] = \left[ \sum_{i=1}^{r} a_{i,r+1} a_{i,r+1}^* - \sum_{i=1}^{r-1} a_{i,r} a_{i,r}^* + \sum_{j=r+1}^{n} a_{r,j} a_{r,j}^* \right.
\]

\[
- \sum_{j=r+2}^{n+1} a_{r+1,j} a_{r+1,j}^* + \Phi(b_r) \lambda a_{s,s+1} - \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^*
\]

(omitting the multivariable \( w \) as before). Using Lemma 5.2 and the fact that \( \Phi(b_r) \) commutes with \( a_{ij,m} \) and \( a_{ij,m}^* \) gives

\[
\left[ \sum_{i=1}^{r} a_{i,r+1} a_{i,r+1}^* - \sum_{i=1}^{r-1} a_{i,r} a_{i,r}^* + \sum_{j=r+1}^{n} a_{r,j} a_{r,j}^* - \sum_{j=r+2}^{n+1} a_{r+1,j} a_{r+1,j}^* + \Phi(b_r) \lambda a_{s,s+1} \right.
\]

\[
- \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^*
\]

\( = -\delta_{r+1,s+1} a_{s,s+1} + \delta_{r,s+1} a_{s,s+1} - \delta_{r,s} a_{s,s+1} + \delta_{r+1,s} a_{s,s+1} \). For the remaining component, we must show

\[
\left[ \left( \sum_{i=1}^{r} a_{i,r+1} a_{i,r+1}^* - \sum_{i=1}^{r-1} a_{i,r} a_{i,r}^* + \sum_{j=r+1}^{n} a_{r,j} a_{r,j}^* - \sum_{j=r+2}^{n+1} a_{r+1,j} a_{r+1,j}^* \right) \lambda \right.
\]

\[
- \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^*
\]

\( = A_{rs} \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^* \). (14)

First note that by Lemma 5.1, \( \left[ \sum_{i=1}^{r} a_{i,r+1} a_{i,r+1}^* - \sum_{i=1}^{r-1} a_{i,r} a_{i,r}^* \lambda \right. \)

\( - \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^* \) = 0 unless \( r = s \). \( r = 1 = s \), or \( r = s + 1 \).

Equation (8) of Lemma 5.2 allows us to compute each case. Suppose \( r = s \), then

\[
\left[ \sum_{i=1}^{r} a_{i,r+1} a_{i,r+1}^* - \sum_{i=1}^{r-1} a_{i,r} a_{i,r}^* \lambda \right. \)

\( - \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^* \]

\( = \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^* + \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^* = 2 \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^* \).

Suppose \( r = s - 1 \), then

\[
\left[ - \sum_{i=1}^{r-1} a_{i,r} a_{i,r}^* \lambda \right. \)

\( - \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^* \]

\( = - \sum_{j=1}^{s-1} a_{j,s+1} a_{j,s+1}^* \.

\[
\sum_{i=1}^{r} a_{i,r+1} a_{i,r+1}^* - \sum_{i=1}^{r-1} a_{i,r} a_{i,r}^* \lambda \]
Similarly, if \( r = s + 1 \), then
\[
\sum_{i=1}^{r} a_{i,r+1} a_{i,r+1}^{*} - \sum_{j=1}^{s+1} a_{j,s+1} a_{j,s}^{*} = 0 \quad \text{and} \quad - \sum_{i=1}^{r} a_{ir} a_{ir}^{*} - \sum_{j=1}^{s+1} a_{j,s+1} a_{j,s}^{*} = - \sum_{j=1}^{s+1} a_{j,s+1} a_{j,s}^{*}.
\]

We have shown
\[
\left( \sum_{i=1}^{r} a_{i,r+1} a_{i,r+1}^{*} - \sum_{i=1}^{r} a_{i,r} a_{i,r}^{*} \right) - \sum_{j=1}^{s+1} a_{j,s+1} a_{j,s}^{*} = A_{rs} \sum_{j=1}^{s+1} a_{j,s+1} a_{j,s}^{*}. \tag{15}
\]

Applying Lemmas 5.1 and 5.2 and splitting into cases \( r > s - 1 \), \( r = s - 1 \) and \( 1 \leq r \leq s - 1 \), a straightforward computation shows the remaining component satisfies
\[
\sum_{i=r+1}^{n+1} a_{ri} a_{ri}^{*} - \sum_{i=r+1}^{n+1} a_{ri+1} a_{ri+1}^{*} - \sum_{j=1}^{s+1} a_{j,s+1} a_{j,s}^{*} = 0 \tag{16}
\]
for all \( r, s \neq 0 \). Equations (15) and (16) give (14) and the desired result. \( \square \)

The case of \([H_0(w)A_1(w)]\) is similar to the above and is left to the reader. Next we consider the case of \( F_0(w) \).

**Lemma 5.5.** (T2) For all \( 0 \leq k \leq n \), \([\rho(H_k)(w)A_1(w)] = -A_{k0}F_0(w)\).

**Proof.** To simplify the computation for \( \rho(F_0)(w) \), one should note that for all positive integers \( s, t, i \), and fixed \((i + 1)\)-tuple \( q = (q_1, q_2, \ldots, q_{i+1}) \in \mathbb{Z}^{i+1} \) with \( 1 = q_1 < q_2 < \cdots < q_i \), it follows immediately from Lemma 5.1 that
\[
[a_{st} a_{st}^{*} \prod_{l=1}^{i} a_{q_{l} q_{l+1}}^{*}] = \prod_{l=1}^{i} \delta_{q_{l} q_{l+1}} a_{q_{l} q_{l+1}}^{*}. \tag{17}
\]

In other words, the expression \([a_{st} a_{st}^{*} \prod_{l=1}^{i} a_{q_{l} q_{l+1}}^{*}]\) is zero unless \( s \) and \( t \) appear as consecutive integers in the increasing sequence \( q_1 < q_2 < \cdots < q_i \) in which case \([a_{st} a_{st}^{*} \prod_{l=1}^{i} a_{q_{l} q_{l+1}}^{*}]\) acts as an identity operator. From this observation and noting that the expression \( \rho(F_0)(w) \) contains sums of strings of such products, one is motivated to arrange terms of \( \rho(H_k)(w) \) to promote cancellation, writing (where we suppress the multivariable):
\[
\rho(H_k) = \sum_{i=1}^{k} a_{ik+1} a_{ik+1}^{*} - \sum_{j=k+2}^{n+1} a_{k+1,j} a_{k+1,j}^{*} - \sum_{i=1}^{k-1} a_{ik} a_{ik}^{*} + \sum_{j=k+1}^{n+1} a_{kj} a_{kj}^{*} + \Phi(b_k).
\]

Now consider the \( \lambda \) (or equivalently the commutators) of components of \( \rho(H_k)(w) \) and \( \rho(F_0)(w) \), and will show that \([\rho(H_k)(w)A_1(w)]\) is zero except in cases of \( k = 0, 1, n \). Let
\[
A := - \sum_{1 \leq r < n+1} \sum_{m=r+1}^{n+1} a_{rm} \prod_{m=q_i > q_{i-1} > \cdots > q_1 \downarrow l=1}^{i} a_{q_{l} q_{l+1}}^{*} a_{r,n+1}^{*},
\]
\[ B := - \sum_{1 \leq r < n + 1} \Phi(b_r) \sum_{q} \prod_{i=1}^{i} a^*_{r,q,i+1}, \]
\[ C := - \sum_{r=q_i, q_{i-1}, \ldots, q_1} \prod_{i=1}^{i} a^*_{r,q,i+1} k \cdot Da^*_{r,n+1}, \]

so \( \rho(F_0)(w) = A + B + C. \) Because it is simpler, we first consider the second component \( B. \) Fix \( r \) with \( 1 \leq r < n + 1, \) recall \( q_{i+1} = n + 1, \) and fix \( k \) with \( 1 \leq k < n. \) In a fixed \( q \in \mathbb{Z}^{n+1} \) as above, if none of the \( q_j = k + 1 \) for \( 1 \leq j \leq i, \) then by (17), \( \left[ \left( \sum_{i=1}^{i} a_{i,k+1}a^*_{i,k+1} - \sum_{j=k+2}^{n+1} a_{k+1,j}a^*_{k+1,j} \right) \prod_{i=1}^{i} a^*_{r,q,i+1} \right] = 0. \)

On the other hand, if \( q_t = k + 1 \) for some \( 1 < t < i + 1 \) (unique because of the conditions on \( q \)), then (17) shows
\[
\left[ \left( \sum_{i=1}^{i} a_{i,k+1}a^*_{i,k+1} - \sum_{j=k+2}^{n+1} a_{k+1,j}a^*_{k+1,j} \right) \prod_{i=1}^{i} a^*_{r,q,i+1} \right] = \prod_{i=1}^{i} a^*_{r,q,i+1} - \prod_{i=1}^{i} a^*_{q_i,q_{i+1},i} = 0.
\]

Since \( \Phi(b_r)(z) \) commutes with the operators \( a_{i,j}(z) \) and \( a^*_{i,j}(z), \) summing over \( r, q \) with \( r \geq q, \) we have shown: For \( 1 \leq k < n, \)
\[
\left[ \sum_{i=1}^{i} a_{i,k+1}a^*_{i,k+1} - \sum_{j=k+2}^{n+1} a_{k+1,j}a^*_{k+1,j} \right] = 0.
\]

A similar argument shows that for \( 1 < k \leq n, \)
\[
\left[ \sum_{j=k+1}^{n} a_{j,k}a^*_{j,k} - \sum_{i=1}^{i} a_{i,k}a^*_{i,k} \right] = 0.
\]

Adding equations (18) and (19) shows \( \rho(F_0)(w)_A B = 0 \) for \( 1 < k < n. \)

Now consider the first component \( A \) of our realization of \( F_0(w). \) Note that the last term \( a^*_{r,n+1} \) may also appear in the product \( \prod_{i=1}^{i} a^*_{r,q,i+1}. \) Assume \( k \neq n, \) apply (17) and (2) for simplifying the brackets:

\[
\sum_{l,j} a_{i,k+1}a^*_{i,k+1} - a_{k+1,j}a^*_{k+1,j}A
\]

= \[
\sum_{l,j} \sum_{r<m} a_{i,k+1} \delta_{i,r} \delta_{k+1,m} \sum_{q} \prod_{i=1}^{i} a^*_{r,q,i+1} a^*_{r,n+1} - a_{k+1,j} \delta_{k+1,r} \delta_{j,m} \sum_{q} \prod_{i=1}^{i} a^*_{r,q,i+1} a^*_{r,n+1}
\]

+ \[
\sum_{r<m} \left( - a_{r,m} \delta_{k+1,m} \sum_{q} \prod_{i=1}^{i} a^*_{r,q,i+1} a^*_{r,n+1}
\right)
\]

+ \[
a_{r,m} \sum_{m=q_i, q_{i-1}, \ldots, q_1} \prod_{i=1}^{i} a^*_{r,q,i+1} [a_{k+1,j}a^*_{k+1,j}a^*_{r,n+1} + a_{k+1,j}a^*_{r,n+1}]
\]

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\[= \sum_{r < m} \left( a_{r,k+1} \delta_{n,k+1} + \sum_{q_i \geq q_1} \prod_{l=1}^{i-1} a_{q_l,q_{l+1}}^* a_{r,n+1}^{*} \right) \\
- \sum_{r \leq m} \left( a_{r,k+1} \delta_{k+1,r} \right) \sum_{m \geq q_i} \prod_{l=1}^{i-1} a_{q_l,q_{l+1}}^* a_{r,n+1}^{*} \\
+ \sum_{r < m} \left( -a_{r,m} \delta_{k+1,m} \right) \sum_{q_i \geq q_1} \prod_{l=1}^{i-1} a_{q_l,q_{l+1}}^* a_{r,n+1}^{*} \\
+ \delta_{r,k+1} a_{rm} \sum_{m \geq q_i} \prod_{l=1}^{i-1} a_{q_l,q_{l+1}}^* a_{r,n+1}^{*} \right) = 0. \]

Thus, for \(1 \leq k < n\) we have shown
\[
\left[ \sum_{i=1}^{k} a_{i,k+1} a_{i,k+1}^* - \sum_{j=k+2}^{n+1} a_{k+1,j} a_{k+1,j}^* \right] A = 0. \tag{20}
\]

Using a similar argument, one can also show for \(1 < k \leq n\),
\[
\left[ \sum_{i=1}^{k-1} -a_{ik} a_{ik}^* + \sum_{j=k+1}^{n+1} a_{kj} a_{kj}^* A \right] = 0. \tag{21}
\]

We have shown \([H_k(w)A] = 0\) for \(1 < k < n\).

Using Lemma 5.2 and equation (17) as above, for \(1 \leq k < n\), one obtains
\[
\sum_{r=1}^{n} \sum_{i=1}^{k} a_{i,k+1} a_{i,k+1}^* - \sum_{j=k+2}^{n+1} a_{k+1,j} a_{k+1,j}^* C \right] \\
= \sum_{k=1}^{n} \prod_{l=1}^{i-1} a_{q_l,q_{l+1}}^* a_{k+1,n+1}^{*} \cdot \lambda, \tag{22}
\]

and for \(k\) with \(1 < k \leq n\),
\[
\sum_{r=1}^{n} \left[ \sum_{i=1}^{k-1} -a_{ik} a_{ik}^* + \sum_{j=k+1}^{n+1} a_{kj} a_{kj}^* C \right] = - \sum_{q_i \geq q_1} \prod_{l=1}^{i-1} a_{q_l,q_{l+1}}^* a_{k,n+1}^{*} \cdot \lambda. \tag{23}
\]

The last term \(\Phi(b_k)(z)\) appearing in \(H_k(z)\) commutes with all of the operators \(a_{ij}(w), a_{ij}^*(w)\), so all that remains is to compute for \(1 \leq k < n\):
\[
\left[ \Phi(b_k)(w) \right] A - \sum_{1 \leq r < n+1} \Phi(b_r)(w) \sum_{r \geq q_i} \prod_{l=1}^{i} a_{q_l,q_{l+1}}^* \cdot \lambda \\
= \sum_{1 \leq r < n+1} A_{kr} \sum_{r \geq q_i} \prod_{l=1}^{i} a_{q_l,q_{l+1}}^* \cdot \lambda \\
= (-1)^{k-1} \sum_{k \geq q_i} \prod_{l=1}^{i} a_{q_l,q_{l+1}}^* \cdot \lambda + \sum_{k \geq q_i} \prod_{l=1}^{i} a_{q_l,q_{l+1}}^* \cdot \lambda \\
+ (-1)^{k+1} \sum_{k \geq q_i} \prod_{l=1}^{i} a_{q_l,q_{l+1}}^* \cdot \lambda 
\]
where we collect partitions in the last equality.

Now for $k \neq 0, 1, n$, we have shown
\[ [\rho(H_k)(w)\lambda\rho(F_0)(w)] = (20) + (21) + (18) + (19) + (22) + (23) + (24) = 0. \]

Now we consider the case $k = 1$, where
\[
H_1 = a_{1,2}a_{1,2}^* - \sum_{j=3}^{n+1} a_{2,j}a_{2,j}^* + \sum_{j=2}^{n+1} a_{1,j}a_{1,j}^* + \Phi(b_1).
\]

Equations (18) and (20) hold for $k = 1$ as does equation (22), so
\[
\left[ a_{1,2}a_{1,2}^* - \sum_{j=3}^{n+1} a_{2,j}a_{2,j}^* F_0 \right] = - \sum_{q} \prod_{i=1}^{r} a_{q_i,q_i+1}^* \kappa \cdot \lambda = a_{1,2}a_{2,n+1}^* \kappa \cdot \lambda. \]  

(25)

Furthermore, since in all of our $q, q_1 = 1$, equations (17) and (9) give
\[
\left[ \sum_{j=2}^{n+1} a_{1,j}a_{1,j}^* F_0 \right] = A + B + \left[ \sum_{j=2}^{n+1} a_{1,j}a_{1,j}^* \lambda - \sum_{r=1}^{r-1} \prod_{i=1}^{r} a_{q_i,q_i+1}^* \kappa \cdot Da_{n+1}^* \right]
= F_0 - a_{1,n+1}^* \kappa \cdot \lambda
\]

(26)

Finally,
\[
[\Phi(b_1)F_0] = 2a_{1,n+1}^* \kappa \cdot \lambda - 1(a_{1,2}a_{2,n+1}^* + a_{1,n+1}^* \kappa \cdot \lambda)
= -a_{1,2}a_{2,n+1}^* \kappa \cdot \lambda + a_{1,n+1}^* \kappa \cdot \lambda.
\]

(27)

Summing equations (27), (25) and (26) yields $[\rho(H_1)(w)\lambda\rho(F_0)(w)] = \rho(F_0)(w)$ as desired.

Now consider $H_n = \sum_{i=1}^{n} a_{i,n+1}a_{i,n+1}^* - \sum_{i=1}^{n-1} a_{i,n}a_{i,n}^* + a_{n,n+1}a_{n,n+1}^* + \Phi(b_n)$, writing
\[
F_0(w) = A + B + C as above, and recalling our assumption that in all the $(i+1)$-tuples $q_i$, the term $q_{i+1} = n + 1$, a straightforward computation using (17) shows
\[
\left[ \sum_{i=1}^{n} a_{i,n+1}a_{i,n+1}^* A \right] = A,
\]
\[
\left[ \sum_{i=1}^{n} a_{i,n+1}a_{i,n+1}^* B \right] = B.
\]

(28)

(29)

Furthermore, using Lemmas 5.1, 5.2 and collecting partitions, one obtains
\[
\left[ \sum_{i=1}^{n} a_{i,n+1}a_{i,n+1}^* C \right] = C - \sum_{q \geq q_1 > q_1 - 1 \ldots > q_1 = 1} \prod_{i=1}^{r} a_{q_i,q_i+1}^* \kappa \cdot \lambda.
\]
Equations (19), (21) and (23) hold for \( k = n \) and show

\[
- \sum_{i=1}^{n-1} a_{in} a_{in}^* + a_{n,n+1} a_{n,n+1}^* A = 0, \quad (30)
\]

\[
- \sum_{i=1}^{n-1} a_{in} a_{in}^* + a_{n,n+1} a_{n,n+1}^* B = 0, \quad (31)
\]

\[
- \sum_{i=1}^{n-1} a_{in} a_{in}^* + a_{n,n+1} a_{n,n+1}^* C = - \sum_{n=q_1 > q_2 > \cdots > q_{i+1} = 1} \prod_{i=1}^{q_{i+1}} a_{q_{i+1}}^* \cdot \lambda, \quad (32)
\]

Summing equations (28)–(33), we have

\[
[\rho(H_n)(w)\lambda \rho(F_0)(w)] = A + B + C - \sum_{n \geq q_1 > q_2 > \cdots > q_{i+1} = 1} \prod_{n=q_1 > q_2 > \cdots > q_{i+1} = 1} \prod_{i=1}^{q_{i+1}} a_{q_{i+1}}^* \cdot \lambda + 2 \sum_{n \geq q_1 > q_2 > \cdots > q_{i+1} = 1} \prod_{i=1}^{q_{i+1}} a_{q_{i+1}}^* \cdot \lambda
\]

\[
- \left( \sum_{n \geq q_1 > q_2 > \cdots > q_{i+1} = 1} \prod_{n=q_1 > q_2 > \cdots > q_{i+1} = 1} \prod_{i=1}^{q_{i+1}} a_{q_{i+1}}^* \cdot \lambda + \sum_{n \geq q_1 > q_2 > \cdots > q_{i+1} = 1} \prod_{i=1}^{q_{i+1}} a_{q_{i+1}}^* \cdot \lambda \right)
\]

\[
= F_0(w).
\]

The case of \( k = 0 \) follows from the above, and is left to the reader.

5.3 Relations involving \( E_k(w), F_r(w), \) and Serre relations. We will prove a selection of the relations involving the elements \( E_k(w), F_r(w) \) including the Serre type relations.

Lemma 5.6. (T3) \[\rho(E_s)(w)\lambda \rho(F_r)(w) = -\delta_{r,s} \left( \rho(H_r)(w) + \frac{2}{\lambda_{i}} \sum_{i=0}^{N} \rho(K_i)(w) \lambda_{i}\right).\]

Proof. For \( r \neq 0 \) and \( s \neq 0 \), the proof is similar to that in [9, Lemma 3.4], where \(-\gamma b_{r}(z) - \frac{1}{2} (b_{r-1}^+(z) + b_{r+1}^+(z)) \) is replaced by \( \Phi(b_r) \) and \(-\frac{1}{2} \delta_{s,r+1} a_{s}^* r+1 \) is replace by \( \kappa \cdot Da_{s}^* \cdot \). We refer the interested reader to that paper for the proof.

It is also straightforward to check that

\[
[\rho(E_0)\lambda \rho(F_r)] = -\delta_{0,r} \left( \rho(H_r) + \frac{2}{\lambda_{i}} \sum_{i=0}^{N} \rho(K_i) \lambda_{i}\right).
\]

The case \[\rho(E_s)(w) \lambda \rho(F_0)\] with \( s > 0 \) is shown via lengthy calculation made available in [8].

We are now left with the Serre relations:
Lemma 5.7. (T4) Let $\rho(E_r)(w)$, $\rho(F_r)(w)$, $\rho(H_r)(w)$ be defined as in Theorem 4.1, then the following relations hold:

\[
[p(\rho(F_r))(w)|\rho(\rho(E_r))(w)] = [p(\rho(E_r))(w)|\rho(\rho(E_r))(w)] = 0 \quad \text{if} \quad A_{rs} \neq -1,
\]

\[
[p(\rho(F_r))(w)|\rho(\rho(F_r))(w)] = [p(\rho(E_r))(w)|\rho(\rho(E_r))(w)] = 0 \quad \text{if} \quad A_{rs} = -1.
\]

Proof. As in the previous lemmas, we first assume $rs \neq 0$. In this case, the proof is exactly the same as in [9, Lemma 3.5] with the exception of a sign change in the formulation of $\rho(F_r)$.

Now suppose $r = 0$, by expanding all $\lambda$-brackets and collecting over partitions along with other simplifications, one obtains for $1 < s \leq n$:

\[
[p(\rho(F_0))|\rho(\rho(F_0))] = \delta_{s,n}a_{n,n+1} \sum_{q_i \geq q_{i+1}} \frac{1}{q_i!q_{i+1}!} \frac{(D + \lambda)}{D + \lambda - 1} \cdot \frac{(D + \lambda - 2)}{D + \lambda - 2} = \delta_{s,n}a_{n,n+1} \sum_{q_i \geq q_{i+1}} \frac{1}{q_i!q_{i+1}!} \frac{(D + \lambda)}{D + \lambda - 1} \cdot \frac{(D + \lambda - 2)}{D + \lambda - 2}.
\]

This proves the Serre relation for $s \neq 0, 1, n$. Consider the case $s = n$. We want to show $[[\rho(F_0)|\rho(F_0)]|\rho(F_0)] = 0$.

To prove this, first recall $\rho(F_n) = a_{n,n+1} - \sum_{p=1}^{n-1} a_{p,n+1}a_{p,n}^*$. Now using (34),

\[
[[\rho(F_0)|\rho(F_0)]|\rho(F_0)]] = [a_{n,n+1} \sum_{q_i \geq q_{i+1}} \frac{1}{q_i!q_{i+1}!} \frac{(D + \lambda)}{D + \lambda - 1} \cdot \frac{(D + \lambda - 2)}{D + \lambda - 2} = \delta_{s,n}a_{n,n+1} \sum_{q_i \geq q_{i+1}} \frac{1}{q_i!q_{i+1}!} \frac{(D + \lambda)}{D + \lambda - 1} \cdot \frac{(D + \lambda - 2)}{D + \lambda - 2}.
\]

whereas

\[
- \sum_{p=1}^{n-1} [[\rho(F_0)|\rho(F_0)]|\rho(F_0))] = [a_{n,n+1} \sum_{q_i \geq q_{i+1}} \frac{1}{q_i!q_{i+1}!} \frac{(D + \lambda)}{D + \lambda - 1} \cdot \frac{(D + \lambda - 2)}{D + \lambda - 2} = \delta_{s,n}a_{n,n+1} \sum_{q_i \geq q_{i+1}} \frac{1}{q_i!q_{i+1}!} \frac{(D + \lambda)}{D + \lambda - 1} \cdot \frac{(D + \lambda - 2)}{D + \lambda - 2}.
\]
For a partition

\[ \sum_{1 \leq p} a_{n,p+1}^* \sum_{1 \leq q} \prod_{k=1}^{i-2} a_{q,k+1}^* a_{p,n}^* + \sum_{l=1}^{n-1} a_{l,n+1} \sum_{q: q_1 \leq q \leq q_{l-1}} \prod_{m=1}^{i-1} a_{q,m,q_{m+1}}^* a_{l,n}^* \]

\[ - \sum_{l=1}^{n-1} a_{l,n+1} \sum_{q: q_1 \leq q \leq q_{l-1}} \prod_{m=1}^{i-2} a_{q,m,q_{m+1}}^* a_{p,n}^* a_{l,n}^* . \]

Hence, \([\rho(F_0)\lambda \rho(F_n)]_{\mu} \rho(F_n) = 0\).

For \(s = 1\) we get \(\rho(F_1) = a_{1,2}\) and hence

\[ [\rho(F_0)\lambda \rho(F_1)] = \sum_{1 \leq r < l \leq n+1} a_{r,l} a_{r,l+1} \sum_{q: j=q_1, r \leq q_j \leq q_{2}, q_2=2} \prod_{i=2}^{i-1} a_{q_i,q_{i+1}}^* a_{r,n+1}^* \]

\[ + \sum_{r<n+1} \sum_{l=1}^{l(q)-1} a_{r,l} \lambda \prod_{j=2}^{l(q)-1} a_{q_j,q_{j+1}}^* \Phi(b_r) + \sum_{l=r}^{l(q)-1} \sum_{q: q_1 \leq q \leq q_{l-1}} \prod_{j=2}^{l(q)-1} a_{q_j,q_{j+1}} \kappa \cdot Da_{r,n+1}^* . \]

Thus, \([\rho(F_0)\lambda \rho(F_1)]_{\mu} \rho(F_1) = 0\).

Next up is the calculation for \([\rho(F_0)\lambda \rho(F_0)]_{\mu} \rho(F_0)\):

For a partition \(q = (1 = q_1, q_2, \ldots, q_i, n+1)\), recall that we set \(l(q) = i\). We now write \([\rho(F_0)\lambda \rho(F_1)] = A_{01} + B_{01} + C_{01}\), where

\[ A_{01} = \sum_{1 \leq r < l \leq n+1} a_{r,l} \sum_{q: j=q_1, r \leq q_j \leq q_{2}, q_2=2} \prod_{i=2}^{l(q)-1} a_{q_i,q_{i+1}}^* a_{r,n+1}^* \]

\[ B_{01} = \sum_{r<n} \sum_{l=1}^{l(q) \lambda} \prod_{j=2}^{l(q)-1} a_{q_j,q_{j+1}}^* \Phi(b_r) \]

\[ C_{01} = \sum_{1 \leq r < l \leq n+1} a_{r,l} \sum_{q: q_1 \leq q \leq q_{l-1}} \prod_{j=2}^{l(q)-1} a_{q_j,q_{j+1}} \kappa \cdot Da_{r,n+1}^* , \]

and \(F_0 = A + B + C\), where

\[ A = \sum_{1 \leq r < l \leq n+1} -a_{r,l} \sum_{q: j=q_1, r \leq q_j \leq q_{2}, q_2=2} \prod_{i=2}^{l(q)-1} a_{q_i,q_{i+1}}^* a_{r,n+1}^* \]

\[ B = - \sum_{r<n+1} \sum_{l=1}^{l(q)} \prod_{j=2}^{l(q)-1} a_{q_j,q_{j+1}}^* \Phi(b_r) \]

\[ C = - \sum_{1 \leq r < l \leq n+1} \sum_{q: q_1 \leq q \leq q_{l-1}} \prod_{j=2}^{l(q)-1} a_{q_j,q_{j+1}} \kappa \cdot Da_{r,n+1}^* . \]

Then \([F_0\lambda[F_0\mu F_1]] = [A\lambda A_{01}] + [B\lambda B_{01}] + [A\lambda B_{01}] + [B\lambda A_{01}] + [A\lambda C_{01}] + [C\lambda A_{01}].

Now we calculate each summand above, and simplify

\[ [A\lambda A_{01}] \]

\[ = - \sum_{1 \leq r < l \leq n+1} \sum_{q: j=q_1, r \leq q_j \leq q_{2}, q_2=2} \prod_{i=2}^{l(q)-1} a_{q_i,q_{i+1}}^* a_{r,n+1}^* a_{s,n+1}^* \prod_{i=2}^{l(p)-1} a_{p_i,p_{i+1}}^* \prod_{i=2}^{l(s)-1} a_{s_i,s_{i+1}}^* . \]
= - \sum_{1 \leq r \leq n+1} \sum_{k=1}^{\bar{r}} a_{sk} \left( \left[ a_{rj} \prod_{\xi=2}^{l(p)-1} a_{p_{\xi} p_{\xi+1}} \right] a_{s,n+1} \right)
+ l(p)-1 \prod_{\xi=2}^{l(p)-1} a_{p_{\xi} p_{\xi+1}} \left[ a_{rj} \lambda a_{s,n+1} \right] a_{s,n+1}
- \sum_{1 \leq r \leq n+1} \sum_{k=1}^{\bar{r}} a_{sk} \left( \left[ \prod_{l=1}^{l(q)-1} a_{q_{r} q_{r+1}} \lambda a_{s,n+1} \right] a_{s,n+1} \right)
+ l(q)-1 \prod_{l=1}^{l(q)-1} a_{q_{r} q_{r+1}} a_{s,n+1}.

Re-indexing the above gives

= - \sum_{1 \leq s \leq k \leq n+1} a_{sk} \left( \left[ a_{rj} \prod_{\xi=2}^{l(p)-1} a_{p_{\xi} p_{\xi+1}} \right] a_{s,n+1} \right)
+ l(p)-1 \prod_{\xi=2}^{l(p)-1} a_{p_{\xi} p_{\xi+1}} \left[ a_{rj} \lambda a_{s,n+1} \right] a_{s,n+1}
- \sum_{1 \leq s \leq k \leq n+1} a_{sk} \left( \left[ \prod_{l=1}^{l(q)-1} a_{q_{r} q_{r+1}} \lambda a_{s,n+1} \right] a_{s,n+1} \right)
+ l(q)-1 \prod_{l=1}^{l(q)-1} a_{q_{r} q_{r+1}} a_{s,n+1}.

To show that the above summation is zero reduces to showing the following are zero:

I_{z} := - \sum_{1 \leq \xi \leq 2} l(p)-1 \prod_{\xi=2}^{l(p)-1} a_{p_{\xi} p_{\xi+1}} \prod_{l=1}^{l(q)-1} a_{q_{r} q_{r+1}}
+ \sum_{1 \leq \xi \leq 2} \sum_{1 \leq \xi \leq 2} \sum_{1 \leq \xi \leq 2} a_{p_{\xi} p_{\xi+1}} a_{s,n+1} a_{s,n+1}.
Note that the first summation is over all partitions $p$ and $q$ and has summands of the form $a^*_p a^*_q \cdots a^*_t a^*_q a^*_q \cdots a^*_u a^*_v \cdots a^*_w a^*_x a^*_y a^*_z a^*_w$ with $p(l(p)-1) \leq s$ and $q(l(q)-1) \leq s$. The second summation is over all partitions $p$ and $q$ and has summands of the form

$$a^*_1 a^*_2 \cdots a^*_q, a^*_q a^*_1, \cdots a^*_q a^*_q a^*_q a^*_q a^*_q a^*_q a^*_q a^*_q a^*_q a^*_q a^*_q,$$

where $s \geq q(l(q)-1), s \geq r \geq p(l(p)-1)$ and $r < j < k$. But these two sets of partitions are the same, so $I_1 = 0$.

Similarly, if we look at the partitions for the summands of

$$I_2 := \sum_{l=q(l(q)-1)+2}^{l(p)-1} \prod_{i=1}^{l(q)-1} a^*_q a^*_q \prod_{\ell=2}^{l(p)-1} a^*_p p_{l-1} a^*_p \prod_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{q(l(q)-1)+1}^{l(q)-1} \sum_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{p(l(p)-1)+1}^{l(p)-1} a^*_p p_{l-1} a^*_p \prod_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{q(l(q)-1)+1}^{l(q)-1} \sum_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{p(l(p)-1)+1}^{l(p)-1} a^*_p p_{l-1} a^*_p \prod_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{q(l(q)-1)+1}^{l(q)-1} \sum_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{p(l(p)-1)+1}^{l(p)-1} a^*_p$$

collecting partitions shows that $I_2 = 0$. Hence, $[A_{\lambda} A_{01}] = 0$.

One can also obtain:

$$[B_{\lambda} B_{01}]$$

and the following identities

$$[A_{\lambda} B_{01}] = \sum_{l=1}^{l(q)-1} \prod_{i=1}^{l(q)-1} a^*_q a^*_q \prod_{\ell=2}^{l(p)-1} a^*_p p_{l-1} a^*_p \prod_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{q(l(q)-1)+1}^{l(q)-1} \sum_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{p(l(p)-1)+1}^{l(p)-1} a^*_p p_{l-1} a^*_p \prod_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{q(l(q)-1)+1}^{l(q)-1} \sum_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{p(l(p)-1)+1}^{l(p)-1} a^*_p$$

and

$$[B_{\lambda} A_{01}] = \sum_{l=1}^{l(q)-1} \prod_{i=1}^{l(q)-1} a^*_q a^*_q \prod_{\ell=2}^{l(p)-1} a^*_p p_{l-1} a^*_p \prod_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{q(l(q)-1)+1}^{l(q)-1} \sum_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{p(l(p)-1)+1}^{l(p)-1} a^*_p p_{l-1} a^*_p \prod_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{q(l(q)-1)+1}^{l(q)-1} \sum_{2 \leq r \leq l+n+1}^{l(q)-1} \sum_{p(l(p)-1)+1}^{l(p)-1} a^*_p$$
applies the following type of calculation:

\[ \text{While on the other hand, we have} \]

\[ \text{Computing the } \lambda \text{-brackets above, then collecting and re-labelling partitions yields} \]

\[ [A_\lambda B_{A01}] + [B_\lambda A_{01}] = 0. \]

Next we calculate

\[ [A_\lambda C_{01}] \]

\[ = - \sum_{s=1}^{n} \sum_{2 \leq r < j \leq n+1} \sum_{q \preceq \lambda(r-1), q_2 \geq 2} \sum_{p}^f \sum_{l(q)-1}^{l(p)} \prod_{k=2}^{l(p)-1} a_{q; r, j+1} \sum_{l=1}^{\lambda(r-1)+1} a_{s,r,n+1} \sum_{p; s=p_i(p), p_2=2} \prod_{k=2}^{l(p)-1} a_{p; r, t} a_{r,n+1} \Phi(b_k). \]

While on the other hand, we have

\[ [C_\lambda A_{01}] \]

\[ = - \left[ \sum_{s=1}^{n} \sum_{q \preceq \lambda(r-1), q_2 \geq 2} \sum_{p}^f \sum_{l(q)-1}^{l(p)-1} a_{p; r, k+1} \sum_{q_{\lambda(r-1)+1}} a_{s,r,n+1} \sum_{l=1}^{\lambda(r-1)+1} a_{r,j} \sum_{p; s=p_i(p), p_2=2} \prod_{k=2}^{l(p)-1} a_{p; r, t} a_{r,n+1} \Phi(b_k) \right] \]

\[ = \sum_{s=1}^{n} \sum_{q \preceq \lambda(r-1), q_2 \geq 2} \sum_{p; s=p_i(p), p_2=2} \sum_{l(q)-1}^{l(p)-1} a_{p; r, k+1} \sum_{q_{\lambda(r-1)+1}} a_{s,r,n+1} \sum_{l=1}^{\lambda(r-1)+1} a_{r,j} \sum_{p; s=p_i(p), p_2=2} \prod_{k=2}^{l(p)-1} a_{p; r, t} a_{r,n+1} \Phi(b_k). \]
This completes the proof that
\[
\text{(the identity above holds for any elements)}
\]
then by induction one can show that
\[
\sum_{q=1}^{n} \left( \sum_{q=1}^{n} B(q) \right) C(s) + \sum_{q=1}^{n} B(s) \left( \sum_{q=1}^{n} C(q) \right) = 2 \left( \sum_{q=1}^{n} B(q) \right) \left( \sum_{q=1}^{n} C(q) \right) - \sum_{s=1}^{n-1} \left( \left( \sum_{q=1}^{s} B(q) \right) C(s+1) \right) - \sum_{s=1}^{n-1} \left( \left( \sum_{q=1}^{s} B(q) \right) C(s) \right)
\]
(the identity above holds for any elements $B(q)$ and $C(s)$ in an algebra with coefficients in $\mathbb{Z}$). Applying this identity to the sum of (35) and (36), we show
\[
[B_{\lambda} B_{\mu}] + [A_{\lambda} C_{\mu}] + [C_{\lambda} A_{\mu}] = 0.
\]
This completes the proof that $[\rho(F_{0})][\rho(F_{0})\mu\rho(F_{0})] = 0$.

The remaining relations $[\rho(F_{0})\lambda][\rho(F_{0})\mu\rho(F_{0})] = 0$ and $[\rho(F_{0})\lambda\rho(F_{0})] = 0$ are proven in a similar manner, where one applies to the following formal identities:
\[
\sum_{r=1}^{n} \left( \sum_{s=1}^{r} \left( -\delta_{r,s-1} + 2\delta_{r,s} - \delta_{r,s+1} \right) \left( \sum_{v=1}^{r} \left( \sum_{t=1}^{r} B(t) C(v) \right) \right) \right)
\]
and
\[
\left( \sum_{r=1}^{n} \left( \sum_{q=1}^{r} A(q) \right) A(r+1) \right) - \left( \sum_{q=1}^{n} A(q) \right)^{2} + \sum_{s=1}^{n} \left( \left( \sum_{q=1}^{s} A(q) \right) A(s) \right) = 0.
\]
For the Serre type relations for the $E_r$, the calculations are the same as those in [9, Lemma 3.5], where $-\gamma b_r(z) - \frac{1}{2} (b_{r-1}^r(z) + b_{r+1}^r(z))$ is replaced by $\Phi(b_r)$ and $-\frac{1}{2} \delta^*_{r,r+1}(z)$ is replaced by $\kappa : Da^*_{r,r+1}$. We refer the interested reader to that paper for the proof. □

References


