QUANTUM DEFORMATIONS OF IMAGINARY VERMA MODULES

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[Received 9 December 1994—Revised 20 November 1995]

Introduction

The root system of an affine Kac–Moody algebra has a standard partition into positive and negative roots. Corresponding to this partition is a standard Borel subalgebra, from which one may induce the standard Verma modules. However, it is quite possible to take other closed partitions of the root system, and form the corresponding non-standard Borel subalgebras. For finite-dimensional simple Lie algebras, we discover nothing new, but for affine Kac–Moody algebras, the induced Verma-type modules typically contain both finite and infinite-dimensional weight spaces. The classification of closed subsets of the root system for affine Kac–Moody algebras was obtained by Jakobsen and Kac [**10, 11**], and independently by Futorny [**6, 7**]. A categorical setting for these modules was introduced in [**2**], with certain restrictions, and generalized in [**3**]. For the algebra $A_1^{(1)}$, the only non-standard modules of Verma-type are the *imaginary Verma modules* [**8**].

Quantum groups were introduced independently by Drinfeld [**5**] and Jimbo [**12**] in their study of the quantum Yang–Baxter equation and two-dimensional solvable lattice models. They have since proved to be very important algebraic objects, providing many new insights into modules over symmetrizable Kac–Moody algebras and a diverse array of connected areas such as invariants of links and 3-manifolds, solvable lattice models, and quantum field theory. For generic q, Lusztig [**15**] showed that integrable highest-weight modules of symmetrizable Kac–Moody algebras can be deformed to those over the corresponding quantum groups in such a way that the dimensions of the weight spaces are invariant under the deformation.

In this paper, following the framework of [**15**] and [**14**], we construct *quantum imaginary Verma modules* for the quantum group $U_q(A_1^{(1)})$ and show that these modules are deformations of those over the universal enveloping algebra of $A_1^{(1)}$ in such a way that the weight multiplicities, both finite and infinite-dimensional, are preserved. Our work depends heavily on the PBW theorem for $U_q(A_1^{(1)})$ with respect to the triangular decomposition induced from the root partition corresponding to the imaginary Verma modules. The main ingredient of the proof of the PBW theorem is the Diamond Lemma obtained in [**1**].

Section 1 of the paper recalls the construction of imaginary Verma modules for $A_1^{(1)}$ and their standard properties [**8**]. Section 2 gives the two realizations of the

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Research of the third author supported in part by Basic Science Research Institute Program, Ministry of Education of Korea, BSRI-94-1414 and GARC-KOSEF at Seoul National University, Korea.
1991 *Mathematics Subject Classification*: 17B67, 17B65, 17B10.

Proc. London Math. Soc. (3) 74 (1997) 52–80.

quantum group $U_q(A_1^{(1)})$ that are needed for the subsequent constructions, and includes the PBW theorem for the triangular decomposition of the quantum group $U_q(A_1^{(1)})$ with respect to the non-standard root partition which yields the imaginary Verma modules. In § 3, we construct the quantum imaginary Verma modules and prove their standard properties. Section 4 introduces the $\mathbb{A}$-forms of the algebras and modules and proves a series of technical results of their structure. Finally, in § 5, we show that the constructions of § 4 reduce in the classical limit to the classical, or non-quantized, case. That is, we show that the quantum imaginary Verma modules over $U_q(A_1^{(1)})$ are the deformations of the imaginary Verma modules over $U(A_1^{(1)})$.

1. *Imaginary Verma modules for* $A_1^{(1)}$

We begin by recalling some basic facts and constructions for the affine Kac–Moody algebra $A_1^{(1)}$ and its imaginary Verma modules. See **[13]** for Kac–Moody algebra terminology and standard notation.

1.1. The algebra $A_1^{(1)}$ is the affine Kac–Moody algebra with generalized Cartan matrix

$$A = (a_{ij})_{0\leqslant i,j\leqslant 1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The algebra $A_1^{(1)}$ has a Chevalley–Serre presentation with generators e_0, e_1, f_0, f_1, h_0, h_1, d, and relations

$$[h_i, h_j] = 0, \quad [h_i, d] = 0,$$
$$[e_i, f_j] = \delta_{ij} h_i,$$
$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j,$$
$$[d, e_j] = \delta_{0,j} e_j, \quad [d, f_j] = -\delta_{0,j} f_j,$$
$$(\operatorname{ad} e_i)^3 e_j = (\operatorname{ad} f_i)^3 f_j = 0, \quad \text{for } i \neq j.$$

Alternatively, we may realize $A_1^{(1)}$ through the loop algebra construction

$$A_1^{(1)} \cong \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

with Lie bracket relations

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n\delta_{n+m,0}(x, y)c,$$
$$[x, c] = 0, \quad [d, x \otimes t^n] = nx \otimes t^n,$$

for $x, y \in \mathfrak{sl}_2$, $n, m \in \mathbb{Z}$, where $(\ , \ )$ denotes the Killing form on $\mathfrak{sl}_2$. For $x \in \mathfrak{sl}_2$ and $n \in \mathbb{Z}$, we write $x(n)$ for $x \otimes t^n$.

Let Δ denote the root system of $A_1^{(1)}$, and let $\{\alpha_0, \alpha_1\}$ be a basis for Δ. Let $\delta = \alpha_0 + \alpha_1$, the minimal imaginary root. Then

$$\Delta = \{\pm\alpha_1 + n\delta \mid n \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}\setminus\{0\}\}.$$

1.2. Let $\mathbb{F}$ be a field of characteristic 0. The universal enveloping algebra $U(A_1^{(1)})$ of $A_1^{(1)}$ is the associative algebra over $\mathbb{F}$ with 1 generated by the elements

$h_0, h_1, d, e_0, e_1, f_0, f_1$ with defining relations

$$[h_0, h_1] = [h_0, d] = [h_1, d] = 0,$$
$$h_i e_j - e_j h_i = a_{ij} e_j, \quad h_i f_j - f_j h_i = -a_{ij} f_j,$$
$$d e_j - e_j d = \delta_{0,j} e_j, \quad d f_j - f_j d = -\delta_{0,j} f_j,$$
$$e_i f_j - f_j e_i = \delta_{ij} h_i,$$
$$e_j e_i^3 - 3 e_i e_j e_i^2 + 3 e_i^2 e_j e_i - e_i^3 e_j = 0 \quad \text{for } i \neq j,$$
$$f_j f_i^3 - 3 f_i f_j f_i^2 + 3 f_i^2 f_j f_i - f_i^3 f_j = 0 \quad \text{for } i \neq j.$$

Corresponding to the loop algebra formulation of $A_1^{(1)}$ is an alternative description of $U(A_1^{(1)})$ as the associative algebra over $\mathbb{F}$ with 1 generated by the elements $e(k), f(k)$ $(k \in \mathbb{Z})$, $h(l)$ $(l \in \mathbb{Z}\setminus\{0\})$, $h, c, d,$ with relations

$$[c, u] = 0 \quad \text{for all } u \in U(A_1^{(1)}),$$
$$[h(k), h(l)] = 2k\delta_{k+l,0} c,$$
$$[h, d] = 0, \quad [h, h(k)] = 0,$$
$$[d, h(l)] = l h(l), \quad [d, e(k)] = k e(k), \quad [d, f(k)] = k f(k),$$
$$[h, e(k)] = 2e(k), \quad [h, f(k)] = -2f(k),$$
$$[h(k), e(l)] = 2e(k+l), \quad [h(k), f(l)] = -2f(k+l),$$
$$[e(k), f(l)] = h(k+l) + k\delta_{k+l,0} c.$$

1.3. A subset S of the root system Δ is called *closed* if $\alpha, \beta \in S$ and $\alpha + \beta \in \Delta$ implies $\alpha + \beta \in S$. The subset S is called a *closed partition* of the roots if S is closed, $S \cap (-S) = \emptyset$, and $S \cup -S = \Delta$ [**10**, **11**, **6**, **7**]. The set

$$S = \{\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{l\delta \mid l \in \mathbb{Z}_{>0}\}$$

is a closed partition of Δ and is $W \times \{\pm 1\}$-inequivalent to the standard partition of the root system into positive and negative roots [**8**].

For $\mathfrak{g} = A_1^{(1)}$, let $\mathfrak{g}_\pm^{(S)} = \sum_{\alpha \in S} \mathfrak{g}_{\pm\alpha}$. In the loop algebra formulation of $\mathfrak{g}$, we have that $\mathfrak{g}_+^{(S)}$ is the subalgebra generated by $e(k)$ $(k \in \mathbb{Z})$, and $h(l)$ $(l \in \mathbb{Z}_{>0})$ and $\mathfrak{g}_-^{(S)}$ is the subalgebra generated by $f(k)$ $(k \in \mathbb{Z})$, and $h(-l)$ $(l \in \mathbb{Z}_{>0})$. Since S is a partition of the root system, the algebra has a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_-^{(S)} \oplus \mathfrak{h} \oplus \mathfrak{g}_+^{(S)}.$$

Let $U(\mathfrak{g}_\pm^{(S)})$ be the universal enveloping algebra of $\mathfrak{g}_\pm^{(S)}$. Then, by the PBW theorem, we have

$$U(\mathfrak{g}) \cong U(\mathfrak{g}_-^{(S)}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}_+^{(S)}),$$

where $U(\mathfrak{g}_+^{(S)})$ is generated by $e(k)$ $(k \in \mathbb{Z})$, and $h(l)$ $(l \in \mathbb{Z}_{>0})$, $U(\mathfrak{g}_-^{(S)})$ is generated by $f(k)$ $(k \in \mathbb{Z})$ and $h(-l)$ $(l \in \mathbb{Z}_{>0})$, and $U(\mathfrak{h})$, the universal enveloping algebra of $\mathfrak{h}$, is generated by h, c, and d.

Let $\lambda \in P$, the weight lattice of $\mathfrak{g} = A_1^{(1)}$. A $U(\mathfrak{g})$-module V is called a *weight* module if $V = \bigoplus_{\mu \in P} V_\mu$, where

$$V_\mu = \{v \in V \mid h \cdot v = \mu(h)v,\ c \cdot v = \mu(c)v,\ d \cdot v = \mu(d)v\}.$$

Any submodule of a weight module is a weight module. A $U(\mathfrak{g})$-module V is called an *S-highest weight module* with highest weight λ if there is a non-zero

$v_\lambda \in V$ such that

(i) $u^+ \cdot v_\lambda = 0$ for all $u^+ \in U(\mathfrak{g}_+^{(S)}) \setminus \mathbb{F}^*$,

(ii) $h \cdot v_\lambda = \lambda(h) v_\lambda$, $c \cdot v_\lambda = \lambda(c) v_\lambda$, $d \cdot v_\lambda = \lambda(d) v_\lambda$,

(iii) $V = U(\mathfrak{g}) \cdot v_\lambda = U(\mathfrak{g}_-^{(S)}) \cdot v_\lambda$.

An S-highest weight module is a weight module.

For $\lambda \in P$, let $I_S(\lambda)$ denote the ideal of $U(A_1^{(1)})$ generated by $e(k)$ $(k \in \mathbb{Z})$, $h(l)$ $(l > 0)$, $h - \lambda(h)1$, $c - \lambda(c)1$, and $d - \lambda(d)1$. Then we define $M(\lambda) = U(A_1^{(1)})/I_S(\lambda)$ to be the *imaginary Verma module* of $A_1^{(1)}$ with highest weight λ. Imaginary Verma modules have many structural features similar to those of standard Verma modules, with the exception of the infinite-dimensional weight spaces. Their properties were investigated in [**8**], from which we recall the following proposition [**8**, Proposition 1, Theorem 1].

Proposition 1.4. (i) *The module $M(\lambda)$ is a $U(\mathfrak{g}_-^{(S)})$-free module of rank* 1 *generated by the S-highest weight vector $1 \otimes 1$ of weight λ.*

(ii) *The dimension* $\dim M(\lambda)_\lambda = 1$ *and* $0 < \dim M(\lambda)_{\lambda - k\delta} < \infty$ *for any integer $k > 0$; if $\mu \neq \lambda - k\delta$ for any integer $k \geqslant 0$ and $M^q(\lambda)_\mu \neq 0$, then* $\dim M(\lambda)_\mu = \infty$.

(iii) *Let V be a $U(A_1^{(1)})$-module generated by some S-highest weight vector v of weight λ. Then there exists a unique surjective homomorphism φ: $M(\lambda) \to V$ such that $\varphi(1 \otimes 1) = v$.*

(iv) *The module $M(\lambda)$ has a unique maximal submodule.*

(v) *Let $\lambda, \mu \in P$. Any non-zero element of* $\mathrm{Hom}_{U(A_1^{(1)})}(M(\lambda), M(\mu))$ *is injective.*

(vi) *The module $M(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.*

2. *The quantum group $U_q(A_1^{(1)})$*

2.1. The *quantum group $U_q(A_1^{(1)})$* is the $\mathbb{F}(q^{\frac{1}{2}})$-algebra with 1 generated by

$$e_0,\ e_1,\ f_0,\ f_1,\ K_0^{\pm 1},\ K_1^{\pm 1},\ D^{\pm 1}$$

with defining relations:

$$DD^{-1} = D^{-1}D = K_i K_i^{-1} = K_i^{-1} K_i = 1,$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$
$$K_i e_i K_i^{-1} = q^2 e_i, \quad K_i f_i K_i^{-1} = q^{-2} f_i,$$
$$K_i e_j K_i^{-1} = q^{-2} e_j, \quad K_i f_j K_i^{-1} = q^2 f_j, \quad \text{for } i \neq j,$$
$$K_i K_j - K_j K_i = 0, \quad K_i D - D K_i = 0,$$
$$D e_i D^{-1} = q^{\delta_{i,0}} e_i, \quad D f_i D^{-1} = q^{-\delta_{i,0}} f_i,$$
$$e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0, \quad \text{for } i \neq j,$$
$$f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 = 0, \quad \text{for } i \neq j,$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$.

The quantum group $U_q(A_1^{(1)})$ can be given a Hopf algebra structure with a comultiplication given by

$$\Delta(K_i) = K_i \otimes K_i,$$
$$\Delta(D) = D \otimes D,$$
$$\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i,$$
$$\Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i,$$

and an antipode given by

$$s(e_i) = -e_i K_i^{-1},$$
$$s(f_i) = -K_i f_i,$$
$$s(K_i) = K_i^{-1},$$
$$s(D) = D^{-1}.$$

There is an alternative realization for $U_q(A_1^{(1)})$, due to Drinfeld [**5**], which we shall also need. Let U be the associative algebra with 1 over $\mathbb{F}(q^{\frac{1}{2}})$ generated by the elements $x^{\pm}(k)$ $(k \in \mathbb{Z})$, $a(l)$ $(l \in \mathbb{Z}\setminus\{0\})$, $K^{\pm 1}$, $D^{\pm 1}$, and $\gamma^{\pm\frac{1}{2}}$ with the following defining relations:

(0) $$DD^{-1} = D^{-1}D = KK^{-1} = K^{-1}K = 1,$$

(1) $$[\gamma^{\pm\frac{1}{2}}, u] = 0 \quad \text{for all } u \in U,$$

(2) $$[a(k), a(l)] = \delta_{k+l,0} \frac{[2k]}{k} \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}},$$

(3) $$[a(k), K] = 0, \quad [D, K] = 0,$$

(4) $$Da(k)D^{-1} = q^k a(k),$$

(5) $$Dx^{\pm}(k)D^{-1} = q^k x^{\pm}(k),$$

(6) $$Kx^{\pm}(k)K^{-1} = q^{\pm 2} x^{\pm}(k),$$

(7) $$[a(k), x^{\pm}(l)] = \pm \frac{[2k]}{k} \gamma^{\mp(|k|/2)} x^{\pm}(k+l),$$

(8) $$x^{\pm}(k+1)x^{\pm}(l) - q^{\pm 2}x^{\pm}(l)x^{\pm}(k+1) = q^{\pm 2}x^{\pm}(k)x^{\pm}(l+1) - x^{\pm}(l+1)x^{\pm}(k),$$

(9) $$[x^+(k), x^-(l)] = \frac{1}{q - q^{-1}} (\gamma^{(k-l)/2}\psi(k+l) - \gamma^{(l-k)/2}\phi(k+l)),$$

where

$$\sum_{k=0}^{\infty} \psi(k) z^{-k} = K \exp\Big((q - q^{-1}) \sum_{k=1}^{\infty} a(k) z^{-k}\Big),$$
$$\sum_{k=0}^{\infty} \phi(-k) z^{k} = K^{-1} \exp\Big(-(q - q^{-1}) \sum_{k=1}^{\infty} a(-k) z^{k}\Big).$$

The algebras $U_q(A_1^{(1)})$ and U are isomorphic [**5**]. The action of the isomorphism, which we shall call the *Drinfeld Isomorphism,* on the generators of $U_q(A_1^{(1)})$ is:

$$e_0 \mapsto x^-(1)K^{-1}, \quad f_0 \mapsto Kx^+(-1),$$
$$e_1 \mapsto x^+(0), \quad f_1 \mapsto x^-(0),$$
$$K_0 \mapsto \gamma K^{-1}, \quad K_1 \mapsto K, \quad D \mapsto D.$$

2.2. Using the root partition $S = \{\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{l\delta \mid l \in \mathbb{Z}_{>0}\}$ from § 1.3, we define:

$U^+(S)$ to be the subalgebra of U generated by $x^+(k)$ $(k \in \mathbb{Z})$ and $a(l)$ $(l > 0)$;

$U^-(S)$ to be the subalgebra of U generated by $x^-(k)$ $(k \in \mathbb{Z})$ and $a(-l)$ $(l > 0)$; and

$U^0(S)$ to be the subalgebra of U generated by $K^{\pm 1}$, $\gamma^{\pm \frac{1}{2}}$, and $D^{\pm 1}$.

Then we have the following PBW theorem.

PROPOSITION 2.2. *There exists a basis for U consisting of the set of monomials of the form*

$$x^- a^- K^\alpha D^\beta \gamma^{\mu/2} a^+ x^+$$

where

$$x^{\pm} = x^{\pm}(m_1)^{n_1} \dots x^{\pm}(m_k)^{n_k}, \quad \text{for } m_i < m_{i+1},\ m_i \in \mathbb{Z},$$
$$a^{\pm} = a(r_1)^{s_1} \dots a(r_l)^{s_l}, \quad \text{for } r_i < r_{i+1},\ \pm r_i \in \mathbb{N}^*,$$

and $\alpha, \beta, \mu \in \mathbb{Z}$, $n_i, s_i \in \mathbb{N}$. *In particular,* $U \cong U^-(S) \otimes U^0(S) \otimes U^+(S)$.

Proof. First, we recall some of the definitions and results of [**1**]. Let $\mathcal{K}$ be a commutative associative ring with 1 and $\mathcal{K}\langle X\rangle$ denote the free associative $\mathcal{K}$-algebra and $\langle X\rangle$ the free semigroup on a set X. We will consider sets T of the form $\{\sigma = (W_\sigma, f_\sigma)\}$ where $W_\sigma \in \langle X\rangle$ and $f_\sigma \in \mathcal{K}\langle X\rangle$. If $\sigma \in T$ and $A, B \in \langle X\rangle$ then let $r_{A\sigma B}$: $\mathcal{K}\langle X\rangle \to \mathcal{K}\langle X\rangle$ denote the $\mathcal{K}$-module endomorphism that fixes all elements of $\langle X\rangle$ other than $AW_\sigma B$ and $r_{A\sigma B}(AW_\sigma B) = Af_\sigma B$. Such a set T is called a *reduction system* and $r_{A\sigma B}$ is a *reduction.* An element $a \in \mathcal{K}\langle X\rangle$ is *irreducible* (with respect to T) if $r_{A\sigma B}(a) = a$ for all $\sigma \in T$.

A 5-tuple $\chi = (\sigma, \tau, A, B, C)$, with $\sigma, \tau \in T$, and $A, B, C \in \mathcal{K}\langle X\rangle \backslash \{1\}$ such that $W_\sigma = AB$, $W_\tau = BC$, is called an *overlap ambiguity* of T; χ is *resolvable* if there exist compositions of reductions r and r' such that $r(f_\sigma C) = r'(Af_\tau)$. A 5-tuple $\chi = (\sigma, \tau, A, B, C)$ is an *inclusion ambiguity* if $W_\sigma = B$ and $W_\tau = ABC$, and is said to be *resolvable* if there exist r and r' as above such that $r(Af_\sigma B) = r'(f_\tau)$. A *semigroup partial ordering* on $\langle X\rangle$ is a partial ordering $\leqslant$ such that if $B < B'$ then $ABC < AB'C$ for all $A, B, C, B' \in \langle X\rangle$, and is said to be *compatible* with T if for all $\sigma \in T$, f_σ is a linear combination of monomials $< W_\sigma$.

If $\leqslant$ is a partial ordering on $\langle X\rangle$ compatible with T and $A \in \langle X\rangle$ then let I_A denote the $\mathcal{K}$-submodule in $\mathcal{K}\langle X\rangle$ generated by all elements $B(W_\sigma - f_\sigma)C$ such that $BW_\sigma C < A$. An overlap ambiguity χ is resolvable *relative* to $\leqslant$ if $f_\sigma C - Af_\tau \in I_{ABC}$, and an inclusion ambiguity χ is resolvable *relative* to $\leqslant$ if $Af_\sigma B - f_\tau \in I_{ABC}$.

DIAMOND LEMMA [**1**]. *Let T be a reduction system for a free algebra* $\mathcal{K}\langle X\rangle$, *and* $\leqslant$ *a semigroup ordering on* $\langle X\rangle$ *compatible with T satisfying the descending chain condition. If all ambiguities of T are resolvable relative to* $\leqslant$, *then a set of representatives in* $\mathcal{K}\langle X\rangle$ *for the elements of the algebra* $R = \mathcal{K}\langle X\rangle / I$, *where I is the ideal generated by* $\{W_\sigma - f_\sigma \mid \sigma \in T\}$, *is given by the* $\mathcal{K}$-*submodule* $\mathcal{K}\langle X\rangle_{\text{irred}}$ *spanned by the T-irreducible monomials of* $\mathcal{K}\langle X\rangle$.

What we now need to do is to find a reduction system T for the free algebra

$$\mathcal{A} = \mathbb{F}(q^{\frac{1}{2}})\langle x^{\pm}(k), a(l), K^{\pm 1}, D^{\pm 1}, \gamma^{\pm \frac{1}{2}} \mid k \in \mathbb{Z},\ l \in \mathbb{Z} \backslash \{0\}\rangle$$

together with a partial ordering $\leqslant$ that satisfies the hypothesis of the Diamond Lemma.

As our reduction system we take $T = \{\sigma = (W_\sigma, f_\sigma)\}$, where the W_σ is the word listed in the first column and f_σ that in the second column below:

$$
(1)\quad
\begin{array}{lll}
\gamma^{\pm\frac{1}{2}}x^-(k), & x^-(k)\gamma^{\pm\frac{1}{2}}, & \text{for } k \in \mathbb{Z},\\
\gamma^{\pm\frac{1}{2}}a(-l), & a(-l)\gamma^{\pm\frac{1}{2}} & \\
a(l)\gamma^{\pm\frac{1}{2}}, & \gamma^{\pm\frac{1}{2}}a(l), & \text{for } l \in \mathbb{Z}_{>0},\\
\gamma^{\pm\frac{1}{2}}K^{\pm1}, & K^{\pm1}\gamma^{\pm\frac{1}{2}}, & \\
\gamma^{\pm\frac{1}{2}}D^{\pm1}, & D^{\pm1}\gamma^{\pm\frac{1}{2}}, & \\
\gamma^{\frac{1}{2}}\gamma^{-\frac{1}{2}}, & 1, & \\
\gamma^{\frac{1}{2}}\gamma^{-\frac{1}{2}}, & \gamma^{-\frac{1}{2}}\gamma^{\frac{1}{2}}, & \\
KK^{-1}, & 1, & \\
KK^{-1}, & K^{-1}K, & \\
DD^{-1}, & 1, & \\
DD^{-1}, & D^{-1}D, & \\
x^+(k)\gamma^{\pm\frac{1}{2}}, & \gamma^{\pm\frac{1}{2}}x^+(k), & \text{for } k \in \mathbb{Z};
\end{array}
$$

$$
(2)\quad
\begin{array}{lll}
a(l)K^{\pm1}, & K^{\pm1}a(l), & \text{for } -l \in \mathbb{Z}_{>0},\\
K^{\pm1}a(l), & a(l)K^{\pm1}, & \text{for } l \in \mathbb{Z}_{>0},\\
DK^{\pm1}, & K^{\pm1}D; &
\end{array}
$$

$$
(3)\quad a(k)a(l),\quad a(l)a(k) + \delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}},\quad \text{for } k, l \in \mathbb{Z},\ k > l;
$$

$$
(4)\quad
\begin{array}{lll}
a(l)D^{\pm1}, & q^{\mp l}D^{\pm1}a(l), & \text{for } l \in \mathbb{Z}_{>0},\\
D^{\pm1}a(l), & q^{\pm l}a(l)D^{\pm1}, & \text{for } -l \in \mathbb{Z}_{>0},\\
x^+(k)D^{\pm1}, & q^{\mp k}D^{\pm1}x^+(k) & \text{for } k \in \mathbb{Z},\\
D^{\pm1}x^-(k), & q^{\pm k}x^-(k)D^{\pm1}, & \text{for } k \in \mathbb{Z};
\end{array}
$$

$$
(5)\quad
\begin{array}{lll}
Kx^-(k), & q^{-2}x^-(k)K, & \\
x^+(k)K, & q^{-2}Kx^+(k), & \text{for } k \in \mathbb{Z};
\end{array}
$$

$$
(6)\quad
\begin{array}{l}
a(k)x^-(l),\quad x^-(l)a(k) - \dfrac{[2k]}{k}x^-(k+l)\gamma^{|k|/2},\\[2ex]
x^+(l)a(k),\quad a(k)x^+(l) - \dfrac{[2k]}{k}x^+(k+l)\gamma^{-|k|/2},\quad \text{for } k \in \mathbb{Z};
\end{array}
$$

$$
(7)\quad
\begin{array}{ll}
x^\pm(k+1)x^\pm(l), & q^{\pm2}(x^\pm(l)x^\pm(k+1) + x^\pm(k)x^\pm(l+1))\\
 & \quad -x^\pm(l+1)x^\pm(k),\quad \text{for } k > l,\\
x^\pm(k+1)x^\pm(k), & q^{\pm2}x^\pm(k)x^\pm(k+1);
\end{array}
$$

$$
(8)\quad x^+(k)x^-(l),\quad x^-(l)x^+(k) + \frac{1}{q - q^{-1}}(\gamma^{(k-l)/2}\psi(k+l) - \phi(k+l)\gamma^{(l-k)/2}).
$$

Let J be the ideal generated by the $W_\sigma - f_\sigma$, so that $U = \mathscr{A}/J$. Moreover, the images in $\mathscr{A}/J$ of the words irreducible under T give the desired PBW basis.

We must show that the system of reductions T terminates. Set

$$\mathscr{X} = \{\gamma^{\pm\frac{1}{2}}, K^{\pm 1}, D^{\pm 1}, a(l), x^{\pm}(k) \mid l, k \in \mathbb{Z}, l \neq 0\}$$

and order the elements in $\mathscr{X}$ by

$$v(l) < v(p) \quad \text{if } l < p \text{ and } v \in \{x^{\pm}, a\},$$

$$x^-(k) < a(-l) < K^{-1} < K < D^{-1} < D < \gamma^{-\frac{1}{2}} < \gamma^{\frac{1}{2}} < a(l) < x^+(k),$$

for $k \in \mathbb{Z}$, $l \in \mathbb{Z}_{>0}$. Define an ordering on $\langle \mathscr{X} \rangle$ by first defining the *weight* of an element $x_1 \dots x_n \in \langle \mathscr{X} \rangle$, where $x_i \in \mathscr{X}$, to be the number of occurrences of $x^{\pm}(k)$ ($k \in \mathbb{Z}$). For example,

$$x^+(3)x^-(2)a(2)x^+(0)^2K$$

has weight 4. Define the *misordering index* of an element $x_1 \dots x_n \in \langle \mathscr{X} \rangle$, with $x_i \in \mathscr{X}$, to be the number of pairs (i, j) such that $i < j$ but $x_i > x_j$. We can partially order elements in $\langle \mathscr{X} \rangle$ by setting $A < B$ if A is of less weight than B, or if A is the same weight as B but A has less length than B, or A is a permutation of B but has smaller misordering index. We leave it to the reader to check that this is a semigroup partial ordering of $\langle \mathscr{X} \rangle$ compatible with T which satisfies the descending chain condition.

We now need to check that all of the ambiguities of T are resolvable relative to $\leqslant$.

Any overlap ambiguities that arise from words in (1) together with other words in (1)–(8) are resolvable with respect to $\leqslant$, as the reader can check.

The ambiguities that arise from words in (2) together with words in (2), (3), or (4) are all resolvable as the reader can check. We first consider overlap ambiguities of words in (2) together with words in (5). Using the reductions in T we have

$$\begin{aligned} a(k)Kx^-(l) &\mapsto q^{-2}a(k)x^-(l)K \\ &\mapsto q^{-2}\left(x^-(l)a(k) - \frac{[2k]}{k} x^-(k+l)\gamma^{|k|/2}\right)K \\ &\mapsto q^{-2}\left(x^-(l)Ka(k) - \frac{[2k]}{k} x^-(k+l)K\gamma^{|k|/2}\right). \end{aligned}$$

On the other hand, if we reduce on the first two symbols, we have

$$\begin{aligned} a(k)Kx^-(l) &\mapsto Ka(k)x^-(l) \\ &\mapsto K\left(x^-(l)a(k) - \frac{[2k]}{k} x^-(k+l)\gamma^{|k|/2}\right) \\ &\mapsto q^{-2}\left(x^-(l)Ka(k) - \frac{[2k]}{k} x^-(k+l)K\gamma^{|k|/2}\right). \end{aligned}$$

One can then use the fact that both reduce to the same element to prove that the two possible reductions of $a(k)Kx^-(l)$ are resolvable relative to $\leqslant$. Indeed the above implies that

$$
\begin{aligned}
&a(k)x^-(l)K - Ka(k)x^-(l) \\
&= \Big(q^{-2}a(k)x^-(l)K - q^{-2}\Big(x^-(l)a(k) - \frac{[2k]}{k}x^-(k+l)\gamma^{|k|/2}\Big)K\Big) \\
&\quad + \Big(q^{-2}\Big(x^-(l)a(k) - \frac{[2k]}{k}x^-(k+l)\gamma^{|k|/2}\Big)K \\
&\quad - q^{-2}\Big(x^-(l)Ka(k) - \frac{[2k]}{k}x^-(k+l)K\gamma^{|k|/2}\Big)\Big) \\
&\quad - \Big(Ka(k)x^-(l) - K\Big(x^-(l)a(k) - \frac{[2k]}{k}x^-(k+l)\gamma^{|k|/2}\Big)\Big) \\
&\quad - \Big(K\Big(x^-(l)a(k) - \frac{[2k]}{k}x^-(k+l)\gamma^{|k|/2}\Big) \\
&\quad - q^{-2}\Big(x^-(l)Ka(k) - \frac{[2k]}{k}x^-(k+l)K\gamma^{|k|/2}\Big)\Big) \\
&\in I_{a(k)Kx^-(l)}.
\end{aligned}
$$

Similarly, the reductions of words in (2) together with words in (6) are resolvable relative to $\leqslant$. Words in (2) do not have any ambiguities with words in (7) and (8).

Next we consider ambiguities arising from words in (3) together with other words in (3). Suppose $k > l > m$. Using the reductions in T we have

$$
\begin{aligned}
a(k)a(l)a(m) &\mapsto a(k)\Big(a(m)a(l) + \delta_{m+l,0}\frac{[2l]}{l}\Big(\frac{\gamma^l - \gamma^{-l}}{q - q^{-1}}\Big)\Big) \\
&\mapsto \Big(a(m)a(k) + \delta_{m+k,0}\frac{[2k]}{k}\Big(\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}\Big)\Big)a(l) \\
&\quad + a(k)\delta_{m+l,0}\frac{[2l]}{l}\frac{\gamma^l - \gamma^{-l}}{q - q^{-1}} \\
&\mapsto a(m)a(l)a(k) + a(m)\delta_{k+l,0}\frac{[2k]}{k}\Big(\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}\Big) \\
&\quad + \delta_{m+k,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}a(l) + a(k)\delta_{m+l,0}\frac{[2l]}{l}\Big(\frac{\gamma^l - \gamma^{-l}}{q - q^{-1}}\Big).
\end{aligned}
$$

Alternatively, if we reduce on the first two symbols, we have

$$
\begin{aligned}
a(k)a(l)a(m) &\mapsto \Big(a(l)a(k) + \delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}\Big)a(m) \\
&\mapsto a(l)\Big(a(m)a(k) + \delta_{m+k,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}\Big) \\
&\quad + a(m)\delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}} \\
&\mapsto a(m)a(l)a(k) + a(m)\delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}
\end{aligned}
$$

$$+\delta_{m+k,0}\frac{[2k]}{k}\frac{\gamma^k-\gamma^{-k}}{q-q^{-1}}a(l)$$
$$+a(k)\delta_{m+l,0}\frac{[2l]}{l}\frac{\gamma^l-\gamma^{-l}}{q-q^{-1}}.$$

One can then use the fact that both reduce to the same element to prove that the two possible reductions of $a(k)a(l)a(m)$ are resolvable relative to $\leqslant$. The fact that any ambiguities that arise from (3) and (4) are resolvable relative to $\leqslant$ is straightforward and left to the reader. There are no ambiguities arising from words from (3) together with words from (5).

Next we have ambiguities that arise from words in (3) together with other words in (6). Suppose $k>l$. Using the reductions in T we have

$$a(k)a(l)x^-(m)\mapsto a(k)\Big(x^-(m)a(l)-\frac{[2l]}{l}x^-(m+l)\gamma^{|l|/2}\Big)$$
$$\mapsto x^-(m)a(k)a(l)-\frac{[2k]}{k}x^-(k+m)\gamma^{|k|/2}a(l)-\frac{[2l]}{l}a(k)x^-(m+l)\gamma^{|l|/2}$$
$$\mapsto x^-(m)\Big(a(l)a(k)+\delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k-\gamma^{-k}}{q-q^{-1}}\Big)$$
$$-\frac{[2k]}{k}x^-(k+m)\gamma^{|k|/2}a(l)-\frac{[2l]}{l}x^-(m+l)a(k)\gamma^{|l|/2}$$
$$+\frac{[2k]}{k}\frac{[2l]}{l}x^-(k+l+m)\gamma^{(|k|+|l|)/2}.$$

Reducing on the first two symbols we have

$$a(k)a(l)x^-(m)\mapsto\Big(a(l)a(k)+\delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k-\gamma^{-k}}{q-q^{-1}}\Big)x^-(m)$$
$$\mapsto a(l)\Big(x^-(m)a(k)-\frac{[2k]}{k}x^-(m+k)\gamma^{|k|/2}\Big)$$
$$+\delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k-\gamma^{-k}}{q-q^{-1}}x^-(m)$$
$$\mapsto x^-(m)a(l)a(k)-x^-(m+1)a(k)\frac{[2l]}{l}\gamma^{|l|/2}$$
$$-a(l)x^-(m+k)\frac{[2k]}{k}\gamma^{|k|/2}+\delta_{m+l,0}\frac{[2k]}{k}\frac{\gamma^k-\gamma^{-k}}{q-q^{-1}}x^-(m)$$
$$\mapsto x^-(m)\Big(a(l)a(k)+\delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k-\gamma^{-k}}{q-q^{-1}}\Big)$$
$$-\frac{[2k]}{k}x^-(k+m)\gamma^{|k|/2}a(l)-\frac{[2l]}{l}x^-(m+l)a(k)\gamma^{|l|/2}$$
$$+\frac{[2k]}{k}\frac{[2l]}{l}x^-(k+l+m)\gamma^{(|k|+|l|)/2}.$$

Now use the fact that both reduce to the same element to prove that the two possible reductions of $a(k)a(l)x^-(m)$ are resolvable relative to $\leqslant$.

There are no ambiguities arising from words from (3) together with words from (7) and (8).

It is straightforward to check that any ambiguities that arise from words in (4) together with words from (4), (5), or (6) are resolvable with respect to $\leqslant$. Words from (4) do not have ambiguities with words from (7) and (8).

It is straightforward to check that any ambiguities that arise from words in (5) together with words from (5) or (7) are resolvable with respect to $\leqslant$. Words from (5) do not have ambiguities with words from (6) and (8).

Next we consider ambiguities arising from words in (6) together with words in (6):

$$\begin{aligned}
x^+(m)a(k)x^-(l) &\mapsto x^+(m)\Big(x^-(l)a(k) - \frac{[2k]}{k}x^-(k+l)\gamma^{|k|/2}\Big)\\
&\mapsto x^-(l)x^+(m)a(k) + r(m,l)a(k) - \frac{[2k]}{k}x^-(k+l)\gamma^{|k|/2}x^+(m)\\
&\quad - \frac{[2k]}{k}\gamma^{|k|/2}r(k+l,m)\\
&\mapsto x^-(l)a(k)x^+(m) - \frac{[2k]}{k}x^-(l)\gamma^{-|k|/2}x^+(m+k) + r(m,l)a(k)\\
&\quad - \frac{[2k]}{k}x^-(k+l)\gamma^{|k|/2}x^+(m) - \frac{[2k]}{k}\gamma^{|k|/2}r(k+l,m),
\end{aligned}$$

where

$$r(m,l) = \frac{1}{q-q^{-1}}(\gamma^{(m-l)/2}\psi(m+l) - \phi(m+l)\gamma^{(l-m)/2}).$$

On the other hand, if we reduce the first two symbols, we have

$$\begin{aligned}
x^+(m)a(k)x^-(l) &\mapsto \Big(a(k)x^+(m) - \frac{[2k]}{k}x^+(k+m)\gamma^{-|k|/2}\Big)x^-(l)\\
&\mapsto a(k)x^-(l)x^+(m) + a(k)r(m,l) - \frac{[2k]}{k}\gamma^{-|k|/2}x^-(l)x^+(k+m)\\
&\quad - \frac{[2k]}{k}\gamma^{-|k|/2}r(k+m,l)\\
&\mapsto x^-(l)a(k)x^+(m) - \frac{[2k]}{k}x^-(l)\gamma^{-|k|/2}x^+(m+k) + a(k)r(m,l)\\
&\quad - \frac{[2k]}{k}x^-(k+l)\gamma^{|k|/2}x^+(m) - \frac{[2k]}{k}\gamma^{-|k|/2}r(m+k,l).
\end{aligned}$$

Hence, to show that the ambiguity arising from $x^+(m)a(k)x^-(l)$ is resolvable relative to $\leqslant$, we need to prove that

$$a(k)r(m,l) - \frac{[2k]}{k}\gamma^{-|k|/2}r(m+k,l) = r(m,l)a(k) - \frac{[2k]}{k}\gamma^{|k|/2}r(k+l,m).$$

This can in turn be derived from expanding out

$$a(m)K\exp\Big((q-q^{-1})\sum_{k=1}^{\infty}a(k)z^{-k}\Big),$$

$$K\exp\Big((q-q^{-1})\sum_{k=1}^{\infty}a(k)z^{-k}\Big)a(m),$$

$$a(m)K^{-1}\exp\Big(-(q-q^{-1})\sum_{k=1}^{\infty}a(-k)z^{k}\Big),$$

$$K^{-1}\exp\Big(-(q-q^{-1})\sum_{k=1}^{\infty}a(-k)z^{k}\Big)a(m).$$

Now words from (6) have overlap ambiguities with words from (7): suppose $k>l$. Then using reductions from T we have

$$\begin{aligned}
&a(m)x^-(k+1)x^-(l)\\
&\quad\mapsto\Big(x^-(k+1)a(m)-\frac{[2m]}{m}x^+(k+m+1)\gamma^{|m|/2}\Big)x^-(l)\\
&\quad\mapsto x^-(k+1)x^-(l)a(m)-\frac{[2m]}{m}\gamma^{|m|/2}x^-(k+1)x^-(m+l)\\
&\qquad-\frac{[2m]}{m}q^{-2}\gamma^{|m|/2}(x^-(l)x^-(m+k+1)-x^-(m+k)x^-(l+1))\\
&\qquad+\frac{[2m]}{m}\gamma^{|m|/2}x^-(l+1)x^-(m+k)\\
&\quad\mapsto q^{-2}(x^-(l)x^-(k+1)+x^-(k)x^-(l+1))a(m)+x^-(l+1)x^-(k)a(m)\\
&\qquad-\frac{[2m]}{m}\gamma^{|m|/2}(x^-(k+1)x^-(m+l)\\
&\qquad+q^{-2}(x^-(l)x^-(m+k+1)-x^-(m+k)x^-(l+1))\\
&\qquad+x^-(l+1)x^-(m+k)).
\end{aligned}$$

On the other hand, if we reduce the last two symbols, we have

$$\begin{aligned}
&a(m)x^-(k+1)x^-(l)\\
&\quad\mapsto a(m)q^{-2}(x^-(l)x^-(k+1)+x^-(k)x^-(l+1))-a(m)x^-(l+1)x^-(k)\\
&\quad\mapsto q^{-2}\Big(x^-(l)a(m)x^-(k+1)-x^-(m+l)\frac{[2m]}{m}\gamma^{|m|/2}x^-(k+1)\\
&\qquad+x^-(k)a(m)x^-(l+1)-x^-(m+k)\frac{[2m]}{m}\gamma^{|m|/2}x^-(l+1)\Big)\\
&\qquad+x^-(l+1)a(m)x^-(k)+\frac{[2m]}{m}\gamma^{|m|/2}x^-(m+l+1)x^-(k)\\
&\quad\mapsto q^{-2}(x^-(l)x^-(k+1)+x^-(k)x^-(l+1))a(m)+x^-(l+1)x^-(k)a(m)
\end{aligned}$$

$$-\frac{[2m]}{m}\gamma^{|m|/2}(x^-(k+1)x^-(m+l)$$
$$+q^{-2}(x^-(l)x^-(m+k+1)-x^-(m+k)x^-(l+1))+x^-(l+1)x^-(m+k)).$$

The calculations above tell us that the ambiguities from $a(m)x^-(k+1)x^-(l)$ are resolvable with respect to $\leqslant$. Similarly, one proves that the ambiguities arising from $x^-(k+1)x^-(l)a(m)$ with $k \geqslant l$ are resolvable relative to $\leqslant$.

Next we turn to words from (7). First we set

$$\phi(u)=\sum_p \phi(p)u^{-p}, \quad \psi(u)=\sum_p \psi(p)u^{-p}, \quad x^\pm(u)=\sum_p x^\pm(p)u^{-p}.$$

Then from [**5**] we have

$$(6') \qquad [\phi(u), \phi(v)] = 0 = [\psi(u), \psi(v)],$$

$$(6'') \qquad \phi(u)\psi(v)\phi(u)^{-1}\psi(v)^{-1} = g(uv^{-1}\gamma^{-1})/g(uv^{-1}\gamma),$$

$$(7') \qquad \phi(u)x^\pm(v)\phi(u)^{-1} = g(uv^{-1}\gamma^{\mp\frac{1}{2}})^{\pm 1}x^\pm(v),$$

$$(7'') \qquad \psi(u)x^\pm(v)\psi(u)^{-1} = g(u^{-1}v\gamma^{\mp\frac{1}{2}})^{\mp 1}x^\pm(v),$$

$$(8') \qquad (u-q^{\pm 2}v)x^\pm(u)x^\pm(v) = (q^{\pm 2}u - v)x^\pm(v)x^\pm(u),$$

$$(9') \qquad [x^+(u), x^-(v)] = (q-q^{-1})^{-1}(\delta(u/v\gamma)\psi(v\gamma^{\frac{1}{2}}) - \delta(u\gamma/v)\phi(u\gamma^{\frac{1}{2}})),$$

where $g(t)$ is the Taylor series at $t=0$ of the functions $(q^2t-1)/(t-q^2)$ and $\delta(t)$ is the Dirac delta function. These formulas are derived directly from the defining relations (6), (7), (8), and (9). Now consider the calculations below:

$$(u-q^2v)x^+(u)x^+(v)x^-(w)$$
$$= (q-q^{-1})^{-1}(u-q^2v)x^+(u)(\delta(v/w\gamma)\psi(w\gamma^{\frac{1}{2}}) - \delta(v\gamma/w)\phi(v\gamma^{\frac{1}{2}}))$$
$$+ (u-q^2v)x^+(u)x^-(w)x^+(v)$$
$$(10) \qquad = (q-q^{-1})^{-1}(\delta(v/w\gamma)g(uw^{-1}\gamma^{-1})\psi(w\gamma^{\frac{1}{2}})$$
$$- (\delta(v\gamma/w)g(u^{-1}v)^{-1}\phi(v\gamma^{\frac{1}{2}}))x^+(u)(u-q^2v)$$
$$+ (q-q^{-1})^{-1}(u-q^2v)(\delta(u/w\gamma)\psi(w\gamma^{\frac{1}{2}}) - \delta(u\gamma/w)\phi(u\gamma^{\frac{1}{2}}))x^+(v)$$
$$+ (u-q^2v)x^-(w)x^+(u)x^+(v).$$

If we expand $x^+(u)x^-(w)$ first then we have

$$(u-q^2v)x^+(u)x^+(v)x^-(w)$$
$$= (uq^2-v)x^+(v)x^+(u)x^-(w)$$
$$= (uq^2-v)(q-q^{-1})^{-1}(\delta(u/w\gamma)g(vw^{-1}\gamma^{-1})\psi(w\gamma^{\frac{1}{2}})$$
$$- \delta(u\gamma/w)g(uv^{-1})^{-1}\phi(u\gamma^{\frac{1}{2}}))x^+(v) + (q^2u-v)x^+(v)x^-(w)x^+(u)$$
$$(11) \qquad = (q^2u-v)(q-q^{-1})^{-1}(\delta(u/w\gamma)g(vw^{-1}\gamma^{-1})\psi(w\gamma^{\frac{1}{2}})$$
$$- \delta(u\gamma/w)g(uv^{-1})\phi(u\gamma^{\frac{1}{2}}))x^+(v)$$
$$+ (q-q^{-1})^{-1}(q^2u-v)(\delta(v/w\gamma)\psi(w\gamma^{\frac{1}{2}})$$
$$- \delta(v\gamma/w)\phi(v\gamma^{\frac{1}{2}}))x^+(u) + (uq^2-v)x^-(w)x^+(v)x^+(u).$$

A straightforward computation shows that the last two expressions in both (10) and (11) are equal.

One can use (10) and (11) as guides to show that ambiguities of words in (7) with words in (8) are resolvable with respect to $\leqslant$. We will explain how one uses (10) to obtain this. The first equality can be expanded as

$$\sum_{p,r,s}(x^+(p-1)x^+(r)x^-(s)-q^2x^+(p)x^+(r-1)x^-(s))u^{-p}v^{-r}w^{-s}$$
$$=(q^2u-v)(q-q^{-1})^{-1}$$
$$\times\Bigg(\Big(\sum_p \gamma^{-p}u^pw^{-p}\Big)\Big(q^{-2}+\sum_{r\geqslant 1}(q^{-2}-q^2)q^{-2}\gamma^{-r}v^rw^{-r}\Big)\Big(\sum_s \psi(s)\gamma^{-s/2}w^{-s}\Big)$$
$$-\Big(\sum_p \gamma^pu^pw^{-p}\Big)\Big(q^2+\sum_{r\geqslant 1}(q^2-q^{-2})q^2u^rv^{-r}\Big)\Big(\sum_s \phi(s)\gamma^{-s/2}u^{-s}\Big)\Bigg)x^+(v)$$
$$+\sum_{p,r,s}(q^2x^+(r)x^-(s)x^+(p-1)-x^+(r-1)x^-(s)x^+(p))u^{-p}v^{-r}w^{-s}$$
$$=(q^2u-v)(q-q^{-1})^{-1}\Big(\sum_{p,s,t}q^{-2}\gamma^{(2p-s)/2}\psi(s)x^+(t)u^{-p}v^{-t}w^{p-s}$$
$$+\sum_{r\geqslant 1,p,s,t}q^{-2}\gamma^{(2(p+r)-s)/2}(q^{-2}-q^2)\psi(s)x^+(t)u^{-p}v^{-r-t}w^{p+r-s}$$
$$-\sum_{p,s,t}q^2\gamma^{(-2p-s)/2}\phi(s)x^+(t)u^{-p-s}v^{-t}w^p$$
$$-\sum_{r\geqslant 1,p,s,t}q^2\gamma^{(-2p-s)/2}(q^2-q^{-2})\phi(s)x^+(t)u^{-p-s+t}v^{-r-t}w^p\Big)$$
$$+\sum_{p,r,s}(q^2x^+(r)x^-(s)x^+(p-1)-x^+(r-1)x^-(s)x^+(p))u^{-p}v^{-r}w^{-s} \tag{12}$$
$$=\sum_{p,r,s}\Bigg((q-q^{-1})^{-1}\Big(\gamma^{(p-1-s)/2}\psi(p+s-1)x^+(r)-q^{-2}\gamma^{(p-s)/2}\psi(p+s)x^+(r-1)$$
$$+\sum_{t\leqslant r-1}\Big(\gamma^{(p+r-s-t-1)/2}(q^{-2}-q^2)\psi(p+r+s-t-1)x^+(t)$$
$$-\sum_{1\leqslant t\leqslant r}q^{-2}\gamma^{(p+r-s-t)/2}(q^{-2}-q^2)\psi(p+r+s-t)x^+(t+1)\Big)$$
$$-\gamma^{(-p+s+1)/2}q^4\phi(p+s-1)x^+(r)+q^2\gamma^{(-p+s)/2}\phi(p+s)x^+(r-1)$$
$$-\sum_{t\leqslant r-1}q^4\gamma^{(s+t-r-p+1)/2}(q^2-q^{-2})\phi(p+s+r-t-1)x^+(t)$$
$$-\sum_{1\leqslant t\leqslant r}q^2\gamma^{(s+t-r-p)/2}(q^2-q^{-2})\phi(p+s+r-t)x^+(t+1)\Big)$$
$$+q^2x^+(r)x^-(s)x^+(p-1)-x^+(r-1)x^-(s)x^+(p)\Bigg)u^{-p}v^{-r}w^{-s}.$$

This leads to the following reduction:

$$
\begin{aligned}
&x^+(p)x^+(r-1)x^-(s)\\
&\quad\mapsto -q^{-2}x^+(p-1)x^+(r)x^-(s)\\
&\qquad -q^{-2}(q-q^{-1})^{-1}\Big(\gamma^{(p-1-s)/2}\psi(p+s-1)x^+(r)\\
&\qquad -q^{-2}\gamma^{(p-s)/2}\psi(p+s)x^+(r-1)\\
&\qquad +\sum_{t\leqslant r-1}\Big(\gamma^{(p+r-s-t-1)/2}(q^{-2}-q^2)\psi(p+r+s-t-1)x^+(t)\\
&\qquad -\sum_{1\leqslant t\leqslant r}q^{-2}\gamma^{(p+r-s-t)/2}(q^{-2}-q^2)\psi(p+r+s-t)x^+(t-1)\Big)\\
&\qquad -\gamma^{(-p+s+1)/2}q^4\phi(p+s-1)x^+(r)+q^2\gamma^{(-p+s)/2}\phi(p+s)x^+(r-1)\\
&\qquad -\sum_{t\leqslant r-1}\Big(q^4\gamma^{(s+t-r-p+1)/2}(q^2-q^{-2})\phi(p+s+r-t-1)x^+(t)\\
&\qquad -\sum_{1\leqslant t\leqslant r}q^2\gamma^{(s+t-r-p)/2}(q^2-q^{-2})\phi(p+s+r-t)x^+(t+1)\Big)\\
&\qquad +q^2x^+(r)x^-(s)x^+(p-1)-x^+(r-1)x^-(s)x^+(p)\Big)\\
&\quad= -q^{-2}x^+(p-1)x^+(r)x^-(s)\\
&\qquad -q^{-2}(q-q^{-1})^{-1}\Big((\gamma^{(p-1-s)/2}\psi(p+s-1)-\gamma^{(-p+s+1)/2}q^4\phi(p+s-1))x^+(r)\\
&\qquad +(q^2\gamma^{(-p+s)/2}\phi(p+s)-q^{-2}\gamma^{(p-s)/2}\psi(p+s))x^+(r-1)\\
&\qquad +\sum_{0\leqslant t\leqslant r-1}(\gamma^{(p+r-s-t-1)/2}(q^{-2}-q^2)\psi(p+r+s-t-1)\\
&\qquad -q^4\gamma^{(s+t-r-p+1)/2}(q^2-q^{-2})\phi(p+s+r-t-1))x^+(t)\\
&\qquad -\sum_{1\leqslant t\leqslant r}(q^{-2}\gamma^{(p+r-s-t)/2}(q^{-2}-q^2)\psi(p+r+s-t)\\
&\qquad -q^2\gamma^{(s+t-r-p)/2}(q^2-q^{-2})\phi(p+s+r-t))x^+(t+1)\\
&\qquad +q^2x^+(r)x^-(s)x^+(p-1)-x^+(r-1)x^-(s)x^+(p)\Big).
\end{aligned}
\tag{13}
$$

Now using the second equality in (11) one can obtain a further reduction of the terms after $q^{-2}x^+(p-1)x^+(r)x^-(s)$ in (13). Let us call this last reduced expression R_{11}. Recall that the last equation in (10) agrees with the last equation in (11). Thus the end reduction R_{11} will have to agree with the reduction R_{10} obtained by using (10). Hence the ambiguities that arise from words in (7) with words in (8) are all resolvable relative to $\leqslant$.

Now we study ambiguities of words from (7) together with words from (7). Consider the computation below where we first switch $x^\pm(u)$ and $x^\pm(v)$:

$$
\begin{aligned}
&(u-q^{\pm2}w)(u-q^{\pm2}v)(v-q^{\pm2}w)x^\pm(u)x^\pm(v)x^\pm(w)\\
&\quad=(u-q^{\pm2}w)(q^{\pm2}u-v)(v-q^{\pm2}w)x^\pm(v)x^\pm(u)x^\pm(w)\\
&\quad=(q^{\pm2}u-w)(q^{\pm2}u-v)(v-q^{\pm2}w)x^\pm(v)x^\pm(w)x^\pm(u)\\
&\quad=(q^{\pm2}u-w)(q^{\pm2}u-v)(q^{\pm2}v-w)x^\pm(w)x^\pm(v)x^\pm(u).
\end{aligned}
\tag{14}
$$

Similarly, first switching $x^{\pm}(v)$ and $x^{\pm}(w)$, one can reduce to

$$(q^{\pm 2}u - w)(q^{\pm 2}u - v)(q^{\pm 2}v - w)x^{\pm}(w)x^{\pm}(v)x^{\pm}(u). \tag{15}$$

As in the case of ambiguities of (7) and (8), one can use (14) and (15) to show that all ambiguities of words in (7) with words in (7) are resolvable relative to $\leqslant$.

3. *Quantum imaginary Verma modules*

3.1. Let $\lambda \in P$, the weight lattice of $A_1^{(1)}$. A U-module V^q is called a *weight* module if $V^q = \bigoplus_{\mu \in P} V^p_\mu$, where

$$V^q_\mu = \{v \in V \mid K^{\pm 1} \cdot v = q^{\pm \mu(h)} v,\ D^{\pm 1} \cdot v = q^{\pm \mu(d)} v,\ \gamma^{\pm \frac{1}{2}} \cdot v = q^{\pm \frac{1}{2}\mu(c)} v\}.$$

Denote $P(V^q) = \{\mu \in P \mid V^q_\mu \neq 0\}$. Using the standard Vandermonde determinant argument, one can easily show that any submodule of a weight module is a weight module, since q is not a root of unity. A U-module V^q is called an *S-highest weight module* with highest weight λ if there is a non-zero $v_\lambda \in V^q$ such that:

(i) $u^+ \cdot v_\lambda = 0$ for all $u^+ \in U^+(S)\backslash\{1\}$;

(ii) $K^{\pm 1} \cdot v_\lambda = q^{\pm \lambda(h)} v_\lambda$, $\gamma^{\pm \frac{1}{2}} \cdot v_\lambda = q^{\pm \frac{1}{2}\lambda(c)} v_\lambda$, $D^{\pm 1} \cdot v_\lambda = q^{\pm \lambda(d)} v_\lambda$;

(iii) $V^q = U \cdot v_\lambda = U^-(S) \cdot v_\lambda$.

An S-highest weight module is a weight module.

3.2. For $\lambda \in P$, let $I^q(\lambda)$ denote the ideal of U generated by $x^+(k)$ $(k \in \mathbb{Z})$, $a(l)$ $(l > 0)$, $K^{\pm 1} - q^{\pm \lambda(h)}1$, $\gamma^{\pm \frac{1}{2}} - q^{\pm \frac{1}{2}\lambda(c)}1$, and $D^{\pm 1} - q^{\pm \lambda(d)}1$. We define the *imaginary Verma module* over U with highest weight λ to be $M^q(\lambda) = U/I^q(\lambda)$. As in the classical case, we have the following standard proposition.

PROPOSITION 3.2. (i) *The module $M^q(\lambda)$ is a $U^-(S)$-free module generated by the S-highest weight vector $1 \otimes 1$ of weight λ.*

(ii) *The dimension* $\dim M^q(\lambda)_\lambda = 1$, *and* $0 < \dim M^q(\lambda)_{\lambda - k\delta} < \infty$ *for any integer* $k > 0$; *if* $\mu \neq \lambda - k\delta$ *for any integer* $k \geqslant 0$ *and* $M^q(\lambda)_\mu \neq 0$, *then* $\dim M^q(\lambda)_\mu = \infty$.

(iii) *Let V be a U-module generated by some S-highest weight vector v of weight λ. Then there exists a unique surjective homomorphism φ: $M^q(\lambda) \to V$ such that $\varphi(1 \otimes 1) = v$.*

(iv) *The module $M^q(\lambda)$ has a unique maximal submodule.*

(v) *Let λ, $\mu \in P$. Any non-zero element of* $\operatorname{Hom}_U(M^q(\lambda), M^q(\mu))$ *is injective.*

Proof. It follows from Proposition 2.2 and the definitions in § 3.1 that $M^q(\lambda)$ is $U^-(S)$-free. Using the defining relation (7) of U, we also obtain that any element of $M^q(\lambda)$ is a linear combination of elements of the type

$$u(k_1, \ldots, k_l, n_1, \ldots, n_m) = x^-(k_1) \ldots x^-(k_l) a(-n_1) \ldots a(-n_m) \cdot v_\lambda,$$

where $k_i \in \mathbb{Z}$, for $1 \leqslant i \leqslant l$ and $n_j \in \mathbb{N}$ for $1 \leqslant j \leqslant m$. Since

$$u(k_1, \ldots, k_l, n_1, \ldots, n_m) \in M^q(\lambda)_{\lambda - l\alpha_1 - (k_1 + \ldots + k_l + n_1 + \ldots + n_m)\delta},$$

we conclude that $M^q(\lambda)$ is a weight module and

$$P(M^q(\lambda)) = \{\lambda - l\alpha_1 + n\delta \mid l \in \mathbb{Z}\backslash\{0\},\ n \in \mathbb{Z}\} \cup \{\lambda - m\delta \mid m \in \mathbb{Z}_{\geqslant 0}\}.$$

This implies (i). Statements (ii)–(iv) now follow from (i) and § 3.1. Statement (v) follows from (i) and the fact that U has no zero-divisors [**4**].

Denote by G the subalgebra of U generated by $a(l)$ $(l \in \mathbb{Z}\setminus\{0\})$, $K^{\pm 1}$, $D^{\pm 1}$, and $\gamma^{\pm\frac{1}{2}}$. Consider an imaginary Verma module $M^q(\lambda)$ and its subspace $M = \sum_{m\in\mathbb{Z}_+} M^q(\lambda)_{\lambda-m\delta}$. Then M can be viewed as a G-module.

LEMMA 3.3. *The module M is an irreducible G-module if and only if $\lambda(c)\neq 0$.*

Proof. Let $0\neq v\in M^q(\lambda)_\lambda$. By Proposition 2.2, the subspace $M^q(\lambda)_{\lambda-m\delta}$ has a basis consisting of the elements

$$(*) \qquad a^{k_1}(-l_1)\dots a^{k_n}(-l_n)\cdot v,$$

with $k_i,\ l_i\in\mathbb{N},\ l_1<\dots<l_n$ and $\sum_{i=1}^n k_il_i=m$. Suppose that $\lambda(c)\neq 0$ and $0\neq u\in M^q(\lambda)_{\lambda-m\delta}$. Then u can be written as a linear combination of monomials of the type $(*)$. Choose among them the one with the largest l_n and denote $l^*=l_n$. Since, for any monomial $a^{k_1}(-l_1)\dots a^{k_p}(-l_p)\cdot v$ in u we have

$$a(l^*)a^{k_1}(-l_1)\dots a^{k_m}(-l_p)\cdot v = k_p\frac{[2l^*]}{l^*}\frac{q^{l^*\lambda(c)}-q^{-l^*\lambda(c)}}{q-q^{-1}}a^{k_1}(-l_1)\dots a^{k_p-1}(-l^*)\cdot v$$

if $l_p=l^*$, and

$$a(l^*)a^{k_1}(-l_1)\dots a^{k_p}(-l_p)\cdot v=0$$

if $l_p<l^*$, we conclude that $a(l^*)u\neq 0$. But $a(l^*)u\in M^q(\lambda)_{\lambda-(m-l^*)\delta}$. Hence, by induction on m, we obtain that there exists an element $x\in G$ such that $0\neq xa(l^*)u\in M^q(\lambda)_\lambda$, which implies the irreducibility of $M^q(\lambda)$.

Conversely, suppose now that $\lambda(c)=0$. Then $a(l)a^{k_1}(-l_1)\dots a^{k_n}(-l_n)\cdot v=0$ for any $l,\ k_1,\dots,k_n,\ l_1,\dots,l_n\in\mathbb{N}$ and hence

$$Ga^{k_1}(-l_1)\dots a^{k_n}(-l_n)\cdot v\subset M^q(\lambda)$$

(where $\subset$ is used in its strict sense). That is, for each m, monomials of the form $(*)$ generate a proper G-submodule of $M^q(\lambda)$. The lemma is now proved.

LEMMA 3.4. *Let $0\neq u\in M^q(\lambda)_{\lambda-n\alpha_1+m\delta}$ and $n>0$. Then there exists $x\in U$ such that $0\neq x\cdot u\in M$.*

Proof. Let $0\neq v\in M^q(\lambda)_\lambda$. By the PBW theorem (Proposition 2.2), the element u can be written as the following linear combination:

$$u=\sum_i a_iu(s_1^{(i)},\dots,s_n^{(i)};l_1^{(i)},\dots,l_{k_i}^{(i)})\cdot v,$$

where the a_i are non-zero elements of $\mathbb{F}$,

$$u(s_1^{(i)},\dots,s_n^{(i)};l_1^{(i)},\dots,l_{k_i}^{(i)})=x^-(s_1^{(i)})\dots x^-(s_n^{(i)})a(-l_1^{(i)})\dots a(-l_{k_i}^{(i)}),$$

$s_1^{(i)}\geqslant\dots\geqslant s_n^{(i)}$ and $s_n^{(i)}\leqslant s_n^{(i+1)}$ for each i, $l_j^{(i)}\in\mathbb{N}$, $s_j^{(i)}\in\mathbb{Z}$, and $-\sum_{j=1}^n s_j^{(i)}-\sum_{j=1}^{k_i}l_j^{(i)}=m$ for all i. Denote $\mathrm{ht}(u(s_1^{(i)},\dots,s_n^{(i)};l_1^{(i)},\dots,l_{k_i}^{(i)}))=\min_{1\leqslant j\leqslant k_i}l_j^{(i)}$. Consider an $s\in\mathbb{N}$ such that $-s+s_n^{(1)}<-l_j^{(i)}$ for all i,j and $m-s<0$. Then $x^+(-s)u\in M^q(\lambda)_{\lambda-(n-1)\alpha_1+(m-s)\delta}$. From the defining relation (9) of U, we have

$$[x^+(-s),x^-(s_n^{(1)})]=\frac{1}{q-q^{-1}}(-\gamma^{(s_n^{(1)}+s)/2})\phi(s_n^{(1)}-s).$$

Note that $\phi(s_n^{(1)} - s) = -K^{-1}(q - q^{-1})a(s_n^{(1)} - s) +$ (terms involving $a(p)$ with $s_n^{(1)} - s < p$). Thus

$$x^+(-s)u = \frac{q^{(\lambda(c)(s+s_n^{(1)})/2-\lambda(h))}}{q^2-1} \sum (q^{2m_j}-1)a_j u(s_1^{(j)}, \dots, s_{n-1}^{(j)}; -s + s_n^{(1)}, l_1^{(j)}, \dots, l_{k_j}^{(j)}) \cdot v$$
$$+ (\text{terms of greater height}) \cdot v,$$

where m_j is the number of values of i such that $s_i^{(j)} = s_n^{(1)}$ and the sum runs over all j such that $s_n^{(j)} = s_n^{(1)}$. Hence, we conclude that $x^+(-s) \cdot u \neq 0$. The proof of the lemma now follows by induction on n.

COROLLARY 3.5. *Let $0 \neq N \subset M^q(\lambda)$ be a U-submodule. Then $N \cap M \neq 0$.*

Proof. The result follows immediately from Lemma 3.4 and Lemma 3.1.

Now we can prove the criterion for irreducibility for the imaginary Verma modules $M^q(\lambda)$.

THEOREM 3.6. *The imaginary Verma module $M^q(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.*

Proof. Assume that $\lambda(c) \neq 0$ and $0 \neq N \subset M^q(\lambda)$. Then, by Corollary 3.5, $N \cap M \neq 0$, and thus by Lemma 3.3, $N = M^q(\lambda)$. Conversely, suppose that $\lambda(c) = 0$. Then for any non-zero weight vector $u \in M$, using Lemma 3.3, we obtain that $Uu \subset M^q(\lambda)$ is a proper submodule. This completes the proof.

4. $\mathbb{A}$-*forms*

4.1. In $U_q(A_1^{(1)})$, let

$$\begin{bmatrix} K_0; t \\ N \end{bmatrix} = \prod_{r=1}^{N} \frac{K_0 q^{(t-r+1)} - K_0^{-1} q^{-(t-r+1)}}{q^r - q^{-r}},$$
$$\begin{bmatrix} K_1; t \\ N \end{bmatrix} = \prod_{r=1}^{N} \frac{K_1 q^{(t-r+1)} - K_1^{-1} q^{-(t-r+1)}}{q^r - q^{-r}},$$
$$\begin{bmatrix} D; t \\ N \end{bmatrix} = \prod_{r=1}^{N} \frac{D q^{(t-r+1)} - D^{-1} q^{-(t-r+1)}}{q^r - q^{-r}},$$

for $t \in \mathbb{Z}$ and $N \in \mathbb{Z}_{\geqslant 0}$.

4.2. Let $\mathbb{A} = F[q^{\frac{1}{2}}, q^{-\frac{1}{2}}, 1/[2]]$, and define the $\mathbb{A}$-*form* $U_{\mathbb{A}}(A_1^{(1)})$ of the quantum group $U_q(A_1^{(1)})$ to be the $\mathbb{A}$-subalgebra of $U_q(A_1^{(1)})$ with 1 generated by the elements

$$e_0^{(n)},\ e_1^{(n)},\ f_0^{(n)},\ f_1^{(n)}\ (n > 0),\ K_0^{\pm 1},\ K_1^{\pm 1},\ D^{\pm 1},$$
$$\begin{bmatrix} K_0; t \\ N \end{bmatrix}, \begin{bmatrix} K_1; t \\ N \end{bmatrix}, \begin{bmatrix} D; t \\ N \end{bmatrix}\ (t \in \mathbb{Z}, N \in \mathbb{Z}_{\geqslant 0}),$$

where $u^{(n)} = u^n/[n]!$ for $u \in U_q(A_1^{(1)})$.

Now we define the $\mathbb{A}$-*form* $U_{\mathbb{A}}$ of the quantum group U to be the image of $U_{\mathbb{A}}(A_1^{(1)})$ under the Drinfeld isomorphism. Explicitly, this makes $U_{\mathbb{A}}$ the $\mathbb{A}$-subalgebra of U with 1 generated by the elements

$$x^{\pm}(0)^{(n)},\quad x^{+}(-1)^{(n)},\ x^{-}(1)^{(n)}\ (n>0),\ K^{\pm 1},\ \gamma^{\pm\frac{1}{2}},\ D^{\pm 1},$$
$$\begin{bmatrix} K\,;t \\ N \end{bmatrix},\ \begin{bmatrix} \gamma K^{-1}\,;t \\ N \end{bmatrix},\ \begin{bmatrix} D\,;t \\ N \end{bmatrix}.$$

4.3. We make the following useful observation about $U_{\mathbb{A}}$.

LEMMA 4.3. *As generators of* $U_{\mathbb{A}}$*, the elements* $\begin{bmatrix} \gamma K^{-1}\,;t \\ N \end{bmatrix}$ *can be replaced by* $\begin{bmatrix} \gamma\,;t \\ N \end{bmatrix}$ $(t\in\mathbb{Z},\ N\in\mathbb{Z}_{\geqslant 0})$.

Proof. First, note that, for $t\in\mathbb{Z},\ r\in\mathbb{Z}_{\geqslant 0}$,

$$\left(\frac{\gamma K^{-1}q^{(t-r+1)}-\gamma^{-1}Kq^{-(t-r+1)}}{q^{r}-q^{-r}}\right)K+\left(\frac{Kq^{(t-r+1)}-K^{-1}q^{-(t-r+1)}}{q^{r}-q^{-r}}\right)K\gamma^{-1}$$
$$-\left(\frac{q^{(t-r+1)}-q^{-(t-r+1)}}{q^{r}-q^{-r}}\right)K^{2}\gamma^{-1}=\frac{\gamma q^{(t-r+1)}-\gamma^{-1}q^{-(t-r+1)}}{q^{r}-q^{-r}}.$$

Hence, for $t\in\mathbb{Z},\ N\in\mathbb{Z}_{\geqslant 0}$,

$$\begin{bmatrix} \gamma K^{-1}\,;t \\ N \end{bmatrix}K^{N}+\begin{bmatrix} K\,;t \\ N \end{bmatrix}(K\gamma^{-1})^{N}-(K^{2}\gamma^{-1})^{N}\prod_{r=1}^{N}\frac{[t-r+1]}{[r]}=\begin{bmatrix} \gamma\,;t \\ N \end{bmatrix}.$$

While an individual term $[t-r+1]/[r]$ is not necessarily in $\mathbb{A}$, the product

$$\prod_{r=1}^{N}\frac{[t-r+1]}{[r]}=\begin{pmatrix} [t] \\ [N] \end{pmatrix},$$

a quantum binomial, is in $\mathbb{A}$, and so the $\begin{bmatrix} \gamma\,;t \\ N \end{bmatrix}$ are in $U_{\mathbb{A}}$.

Solving this formula for $\begin{bmatrix} \gamma K^{-1}\,;t \\ N \end{bmatrix}$, we see that we can replace the $\begin{bmatrix} \gamma K^{-1}\,;t \\ N \end{bmatrix}$ as generators of $U_{\mathbb{A}}$ by the $\begin{bmatrix} \gamma\,;t \\ N \end{bmatrix}$.

4.4. Additionally, we have the following convenient lemma.

LEMMA 3.3. *The* $\mathbb{A}$-*form* $U_{\mathbb{A}}$ *contains* $x^{\pm}(k)$ $(k\in\mathbb{Z})$ *and* $a(l)$ $(l\in\mathbb{Z}\setminus\{0\})$.

Proof. Notice that $x^{+}(0)$ and $x^{-}(1)$ are generators of $U_{\mathbb{A}}$. From the defining relation (9) of U, we have

$$[x^{+}(0),x^{-}(1)]=\frac{1}{q-q^{-1}}(\gamma^{\frac{1}{2}}\psi(1)),$$

where

$$\sum_{k=0}^{\infty} \psi(k) z^{-k} = K \exp\Big((q-q^{-1}) \sum_{k=1}^{\infty} a(k) z^{-k}\Big).$$

That is, $\psi(1) = K(q-q^{-1})a(1)$, and so $[x^+(0), x^-(1)] = K\gamma^{\frac{1}{2}} a(1)$. Since K^{-1} and $\gamma^{-\frac{1}{2}}$ are generators of $U_{\mathbb{A}}$, we have $a(1) \in U_{\mathbb{A}}$.

From relation (7), we have

$$[a(1), x^{\pm}(k)] = \pm [2] \gamma^{\mp\frac{1}{2}} x^{\pm}(k+1).$$

But $1/[2] \in \mathbb{A}$, and $\gamma^{\pm\frac{1}{2}}$ and $x^{\pm}(0)$ are in $U_{\mathbb{A}}$, so, by induction $x^{\pm}(k) \in U_{\mathbb{A}}$ for all $k > 0$.

Next, we need to show that $a(l) \in U_{\mathbb{A}}$ for $l \in \mathbb{Z} \setminus \{0\}$. For $l > 0$, consider the relations

$$[x^+(l), x^-(0)] = \frac{1}{q-q^{-1}} \gamma^{l/2} \psi(l) \in U_{\mathbb{A}}.$$

Now note that

$$\psi(l) = K(q-q^{-1})a(l) + (\text{terms involving } a(j) \text{ for } j < l \text{ and higher powers of } (q-q^{-1})).$$

Since $a(1) \in U_{\mathbb{A}}$, by induction $a(l) \in U_{\mathbb{A}}$ for all $l > 0$. Similarly, from the relation

$$[x^+(-1), x^-(0)] = \frac{1}{q-q^{-1}} (-\gamma^{\frac{1}{2}} \phi(-1)),$$

we see that $a(-1)$ is an element of $U_{\mathbb{A}}$, and hence, by similar inductive arguments, $x^{\pm}(l)$ and $a(l)$ are in $U_{\mathbb{A}}$ for all $l < 0$.

4.5. Now define $U_{\mathbb{A}}^{\pm}(S) = U_{\mathbb{A}} \cap U^{\pm}(S)$ and $U_{\mathbb{A}}^{0}(S) = U_{\mathbb{A}} \cap U^{0}(S)$.

PROPOSITION 4.5.

$$U_{\mathbb{A}}^{0}(S) = \left\langle K^{\pm 1}, \gamma^{\pm\frac{1}{2}}, D^{\pm 1}, \begin{bmatrix} K\,;\,t \\ N \end{bmatrix}, \begin{bmatrix} \gamma\,;\,t \\ N \end{bmatrix}, \begin{bmatrix} D\,;\,t \\ N \end{bmatrix} \right\rangle.$$

Proof. Certainly the elements $K^{\pm 1}$, $\gamma^{\pm\frac{1}{2}}$, $D^{\pm 1}$, $\begin{bmatrix} K\,;\,t \\ N \end{bmatrix}$, $\begin{bmatrix} \gamma\,;\,t \\ N \end{bmatrix}$ and $\begin{bmatrix} D\,;\,t \\ N \end{bmatrix}$ are in $U_{\mathbb{A}}^{0}(S)$. Looking at the defining relations of U, we see that the only ones with which we need to be concerned are those of the form

$$[a(k), a(-k)] = \frac{[2k]}{k} \frac{\gamma^k - \gamma^{-k}}{q-q^{-1}} \in U^0(S).$$

But $[2k]$ is in $\mathbb{A} = \mathbb{F}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}, 1/[2]]$ and

$$\frac{\gamma^k - \gamma^{-k}}{q-q^{-1}} = \frac{\gamma - \gamma^{-1}}{q-q^{-1}} \sum_{i=1}^{2k-1} \gamma^{k-i} = \begin{bmatrix} \gamma\,;\,0 \\ 1 \end{bmatrix} \sum_{i=1}^{2k-1} \gamma^{k-i}.$$

Hence, $U_{\mathbb{A}}^{0}(S)$ is generated by the above elements.

4.6. The $\mathbb{A}$-form $U_{\mathbb{A}}$ inherits a triangular decomposition from U.

PROPOSITION 4.6.

$$U_{\mathbb{A}} \cong U_{\mathbb{A}}^{-}(S) \otimes U_{\mathbb{A}}^{0}(S) \otimes U_{\mathbb{A}}^{+}(S).$$

Proof. Let $u \in U_{\mathbb{A}}$. Then u can be written as a sum of monomials in the generators of $U_{\mathbb{A}}$. Suppose one of these monomials contains the expression xy where x and y are misordered generators of $U_{\mathbb{A}}$. Using the commutation relations of U, and Proposition 2.2, we may reduce this expression to $xy = yx + a$ where a is a sum of monomials in U. By virtue of the reduction process of Proposition 2.2, the resulting monomials are better ordered than the original. Since x and y are generators of $U_{\mathbb{A}}$, we find that $a \in U_{\mathbb{A}}$ and may be written in terms of generators of $U_{\mathbb{A}}$. Repeating this process as necessary, we ultimately obtain an expression of u as a sum of monomials each of the form $u^{-}u^{0}u^{+}$ where $u^{\pm} \in U_{\mathbb{A}}^{\pm}(S)$ and $u^{0} \in U_{\mathbb{A}}^{0}(S)$.

4.7. Let $\lambda \in P$, the weight lattice of $A_1^{(1)}$, and let V^q be an S-highest weight module over U. We define the $\mathbb{A}$-*form* $V^{\mathbb{A}}$ of V^q to be the $U_{\mathbb{A}}$-submodule of V^q generated by v_λ. That is, $V^{\mathbb{A}} = U_{\mathbb{A}} \cdot v_\lambda$.

PROPOSITION 4.7.

$$V^{\mathbb{A}} = U_{\mathbb{A}}^{-}(S) \cdot v_\lambda.$$

Proof. Let $u \in U_{\mathbb{A}}$. Then, by Proposition 4.6, u can be written as a sum of monomials of the form $u^{-}u^{0}u^{+}$, with $u^{\pm} \in U_{\mathbb{A}}^{\pm}(S)$ and $u^{0} \in U_{\mathbb{A}}^{0}(S)$. By definition, $u^{+} \cdot v_\lambda = 0$ if $u^{+} \notin \mathbb{F}^{*}$.

Next we need to show that the generators of $U_{\mathbb{A}}^{0}(S)$ act as scalars in $\mathbb{A}$ on v_λ. By Proposition 4.5, the generators of $U_{\mathbb{A}}^{0}(S)$ are $K^{\pm 1}$, $\gamma^{\pm\frac{1}{2}}$, $D^{\pm 1}$, $\begin{bmatrix} K ; t \\ N \end{bmatrix}$, $\begin{bmatrix} \gamma ; t \\ N \end{bmatrix}$, and $\begin{bmatrix} D ; t \\ N \end{bmatrix}$. Now $K^{\pm 1} \cdot v_\lambda = q^{\pm\lambda(h)} v_\lambda$, $\gamma^{\pm\frac{1}{2}} \cdot v_\lambda = q^{\pm\lambda(c)/2} v_\lambda$, and $D^{\pm 1} \cdot v_\lambda = q^{\pm\lambda(d)} v_\lambda$. Since $\lambda \in P$, the numbers $\lambda(h)$, $\lambda(c)$, and $\lambda(d)$ are integers, and so $q^{\pm\lambda(h)}$, $q^{\pm\lambda(c)/2}$, and $q^{\pm\lambda(d)}$ are all in $\mathbb{A}$. Also,

$$\begin{aligned}
\begin{bmatrix} K ; t \\ N \end{bmatrix} \cdot v_\lambda &= \prod_{r=1}^{N} \frac{Kq^{(t-r+1)} - K^{-1}q^{-(t-r+1)}}{q^r - q^{-r}} \cdot v_\lambda \\
&= \prod_{r=1}^{N} \frac{q^{(\lambda(h)+t-r+1)} - q^{-(\lambda(h)+t-r+1)}}{q^r - q^{-r}} v_\lambda \\
&= \prod_{r=1}^{N} \frac{q^{(\lambda(h)+t-r+1)} - q^{-(\lambda(h)+t-r+1)}}{q - q^{-1}} \frac{q - q^{-1}}{q^r - q^{-r}} v_\lambda \\
&= \prod_{r=1}^{N} \frac{[\lambda(h) + t - r + 1]}{[r]} v_\lambda \\
&= \binom{[\lambda(h) + t]}{[N]} v_\lambda.
\end{aligned}$$

Similarly,

$$\begin{bmatrix} \gamma\,;t \\ N \end{bmatrix} \cdot v_\lambda = \binom{[\lambda(c)+t]}{[N]} v_\lambda$$

and

$$\begin{bmatrix} D\,;t \\ N \end{bmatrix} \cdot v_\lambda = \binom{[\lambda(d)+t]}{[N]} v_\lambda.$$

The quantum binomials are all in $\mathbb{A}$, and so the generators of $U^0_{\mathbb{A}}(S)$ act as scalars in $\mathbb{A}$ on v_λ. Hence, u^0 acts as a scalar in $\mathbb{A}$, and the result is proved.

PROPOSITION 4.8. *Let* ϕ: $\mathbb{F}(q^{\frac{1}{2}})\otimes_{\mathbb{A}} V^{\mathbb{A}} \to V^q$ *be defined by* $\phi(f\otimes v) = fv$ *for any* $f \in \mathbb{F}(q^{\frac{1}{2}})$ *and* $v \in V^{\mathbb{A}}$. *Then* ϕ *is an* $\mathbb{F}(q^{\frac{1}{2}})$*-linear isomorphism.*

Proof. Let $v \in V^q$. By the proof of the PBW theorem (Proposition 2.2), $v = u^- \cdot v_\lambda$ for some $u^- \in U^-(S)$. Recall that $U^-(S)$ is the $\mathbb{F}(q^{\frac{1}{2}})$-subalgebra of U generated by the $x^-(k)$ $(k \in \mathbb{Z})$ and $a(-l)$ $(l \in \mathbb{Z}_{>0})$. By the PBW theorem, u^- can be written uniquely as

$$u^- = \sum_{\text{finite}} cm(x, a),$$

where the scalars c are in $\mathbb{F}(q^{\frac{1}{2}})$ and $m(x, a)$ denotes a monomial in $x^-(k)$ and $a(-l)$. By Lemma 4.4, $m(x, a) \in U^-_{\mathbb{A}}(S)$.

Define ψ: $V^q \to \mathbb{F}(q^{\frac{1}{2}})\otimes_{\mathbb{A}} V^{\mathbb{A}}$ by

$$v = u^- \cdot v_\lambda \mapsto \sum_{\text{finite}} c \otimes m(x, a) \cdot v_\lambda.$$

Since $V^{\mathbb{A}} = U^-_{\mathbb{A}}(S) \cdot v_\lambda$ and the expression for u^- is unique, ψ is well-defined. It is straightforward to verify that ϕ and ψ are inverses to each other.

4.9. For $\mu \in P$, define $V^{\mathbb{A}}_\mu = V^{\mathbb{A}} \cap V^q_\mu$.

PROPOSITION 4.9.

$$V^{\mathbb{A}} = \bigoplus_{\mu \in P} V^{\mathbb{A}}_\mu.$$

Proof. Let $v \in V^{\mathbb{A}}$. Then $v = v_1 + v_2 + \ldots + v_p$, where $v_j \in V^q_{\mu_j}$ and $\mu_j \in P$, for $1 \leqslant j \leqslant p$. We need to show that each v_j is in $V^{\mathbb{A}}$, for $1 \leqslant j \leqslant p$. We will show that $v_1 \in V^{\mathbb{A}}$. The remaining cases are similar.

For $j = 1, 2, \ldots, p$, write $\mu_j(h) = R_j$, $\mu_j(c) = S_j$ and $\mu_j(d) = T_j$. For each $j = 2, \ldots, p$, since $\mu_j \neq \mu_1$, at least one of the following is true: $R_1 \neq R_j$, $S_1 \neq S_j$, $T_1 \neq T_j$. Let $s = \max\{|R_1 - R_j|, |S_1 - S_j|, |T_1 - T_j| \mid 2 \leqslant j \leqslant p\}$. Define an element u of $U_{\mathbb{A}}$ as

$$u = \begin{bmatrix} K\,;-R_1+s \\ s \end{bmatrix} \begin{bmatrix} K\,;-R_1-1 \\ s \end{bmatrix} \begin{bmatrix} \gamma\,;-S_1+s \\ s \end{bmatrix} \begin{bmatrix} \gamma\,;-S_1-1 \\ s \end{bmatrix}$$
$$\times \begin{bmatrix} D\,;-T_1+s \\ s \end{bmatrix} \begin{bmatrix} D\,;-T_1-1 \\ s \end{bmatrix}.$$

We want to show that u acts on v_1 as a scalar:

$$\begin{bmatrix} D\,;-T_1-1 \\ s \end{bmatrix}\cdot v_1=\prod_{r=1}^{s}\frac{Dq^{-T_1-r}-D^{-1}q^{T_1+r}}{q^r-q^{-r}}\cdot v_1$$
$$=\prod_{r=1}^{s}\frac{q^{-r}-q^{r}}{q^r-q^{-r}}v_1=(-1)^s v_1$$

and

$$\begin{bmatrix} D\,;-T_1+s \\ s \end{bmatrix}\cdot v_1=\prod_{r=1}^{s}\frac{Dq^{-T_1+s-r+1}-D^{-1}q^{T_1-s+r-1}}{q^r-q^{-r}}\cdot v_1$$
$$=\prod_{r=1}^{s}\frac{q^{s-r+1}-q^{-s+r-1}}{q^r-q^{-r}}v_1$$
$$=v_1.$$

Similarly,

$$\begin{bmatrix} \gamma\,;-S_1-1 \\ s \end{bmatrix}\cdot v_1=(-1)^s v_1,\quad \begin{bmatrix} \gamma\,;-S_1+s \\ s \end{bmatrix}\cdot v_1=v_1,$$
$$\begin{bmatrix} K;-R_1-1 \\ s \end{bmatrix}\cdot v_1=(-1)^s v_1,\quad \begin{bmatrix} K\,;-R_1+s \\ s \end{bmatrix}\cdot v_1=v_1.$$

Hence,

$$u\cdot v_1=(-1)^{3s}v_1=(-1)^s v_1.$$

Next, we need to prove that $u\cdot v_j=0$ for any $2\leqslant j\leqslant p$. Now

$$\begin{bmatrix} D\,;-T_1-1 \\ s \end{bmatrix}\cdot v_j=\prod_{r=1}^{s}\frac{Dq^{-T_1-r}-D^{-1}q^{T_1+r}}{q^r-q^{-r}}\cdot v_j$$
$$=\prod_{r=1}^{s}\frac{q^{T_j-T_1-r}-q^{T_1-T_j+r}}{q^r-q^{-r}}v_j$$

and

$$\begin{bmatrix} D\,;-T_1+s \\ s \end{bmatrix}\cdot v_j=\prod_{r=1}^{s}\left(\frac{q^{T_j-T_1+s-r+1}-q^{T_1-T_j-s+r-1}}{q^r-q^{-r}}\right)v_j.$$

Thus,

$$\begin{bmatrix} D\,;-T_1+s \\ s \end{bmatrix}\begin{bmatrix} D\,;-T_1-1 \\ s \end{bmatrix}\cdot v_j$$
$$=\prod_{r,t=1}^{s}\frac{(q^{T_j-T_1-r}-q^{T_1-T_j+r})(q^{T_j-T_1+s-t+1}-q^{T_1-T_j-s+t-1})}{(q^r-q^{-r})(q^t-q^{-t})}v_j.$$

Similarly,

$$\begin{bmatrix} \gamma\,;-S_1+s \\ s \end{bmatrix}\begin{bmatrix} \gamma\,;-S_1-1 \\ s \end{bmatrix}\cdot v_j$$
$$=\prod_{r,t=1}^{s}\frac{(q^{S_j-S_1-r}-q^{S_1-S_j+r})(q^{S_j-S_1+s-t+1}-q^{S_1-S_j-s+t-1})}{(q^r-q^{-r})(q^t-q^{-t})}v_j$$

and

$$\begin{bmatrix} K\,;\,-R_1+s \\ s \end{bmatrix}\begin{bmatrix} K\,;\,-R_1-1 \\ s \end{bmatrix}\cdot v_j$$
$$= \prod_{r,t=1}^{s} \frac{(q^{R_j-R_1-r}-q^{R_1-R_j+r})(q^{R_j-R_1+s-t+1}-q^{R_1-R_j-s+t-1})}{(q^r-q^{-r})(q^t-q^{-t})} v_j.$$

Consider the product

$$\prod_{r,t=1}^{s} (q^{T_j-T_1-r}-q^{T_1-T_j+r})(q^{T_j-T_1+s-t+1}-q^{T_1-T_j-s+t-1}).$$

The $r+t=s+1$ factor in this product is

$$q^{2(T_j-T_1)}-q^{2r}-q^{-2r}+q^{-2(T_j-T_1)}.$$

Similarly, we get

$$q^{2(S_j-S_1)}-q^{2r}-q^{-2r}+q^{-2(S_j-S_1)}$$

and

$$q^{2(R_j-R_1)}-q^{2r}-q^{-2r}+q^{-2(R_j-R_1)}.$$

As we noted before, at least one of T_j-T_1, S_j-S_1 and R_j-R_1 is non-zero. Suppose without loss of generality that $S_j-S_1\neq 0$. The index r takes values from 1 to s and $s\geqslant S_j-S_1$, so let $r=S_j-S_1$. Then

$$q^{2(S_j-S_1)}-q^{2r}-q^{-2r}+q^{-2(S_j-S_1)}=0.$$

Hence the whole product is zero, $u\cdot v_j=0$ for any $2\leqslant j\leqslant p$.

Therefore, $u\cdot v=u\cdot v_1=(-1)^s v_1\in V^{\mathbb{A}}$, which implies that $v_1\in V^{\mathbb{A}}$. The proof for the remaining v_j is similar.

PROPOSITION 4.10. *Let $\mu\in P$. Then $V_\mu^{\mathbb{A}}$ is a free $\mathbb{A}$-module and*

$$\operatorname{rank}_{\mathbb{A}} V_\mu^{\mathbb{A}} = \dim_{\mathbb{F}(q)} V_\mu^q.$$

Proof. The proof is straightforward and follows from Propositions 4.8 and 4.9.

5. *Classical limits*

5.1. Recall that $\mathbb{A}=\mathbb{F}[q^{\frac12}, q^{-\frac12}, 1/[2]]$. Let $\mathbb{J}$ be the ideal of $\mathbb{A}$ generated by $q^{\frac12}-1$. Then we have an isomorphism of fields $\mathbb{A}/\mathbb{J}\cong\mathbb{F}$ given by $f+\mathbb{J}\mapsto f(1)$ for $f\in\mathbb{A}$. Define $\bar{U}=\mathbb{F}\otimes_{\mathbb{A}} U_{\mathbb{A}}$ and $\bar{V}=\mathbb{F}\otimes_{\mathbb{A}} V^{\mathbb{A}}$. Then $\bar{U}\cong U_{\mathbb{A}}/\mathbb{J}U_{\mathbb{A}}$ and $\bar{V}\cong V^{\mathbb{A}}/\mathbb{J}V^{\mathbb{A}}$. For each $\mu\in P$, define $\bar{V}_\mu=\mathbb{F}\otimes_{\mathbb{A}} V_\mu^{\mathbb{A}}$. Since $V^{\mathbb{A}}=\bigoplus_{\mu\in P} V_\mu^{\mathbb{A}}$, we have $\bar{V}=\bigoplus_{\mu\in P}\bar{V}_\mu$. Moreover, we have

PROPOSITION 5.1.

$$\dim_{\mathbb{F}} \bar{V}_\mu = \operatorname{rank}_{\mathbb{A}} V_\mu^{\mathbb{A}}.$$

Proof. If $\{v_j \mid j\in\Omega\}$ is a basis of the free $\mathbb{A}$-module $V_\mu^{\mathbb{A}}$, then every element v of $\bar{V}_\mu=\mathbb{F}\otimes_{\mathbb{A}} V_\mu^{\mathbb{A}}$ can be written uniquely as $v=\sum_{j\in\Omega} a_j\otimes v_j$ where $a_j\in\mathbb{F}$ [**9**, 5.11]. It follows that $\{\bar{v}_j=1\otimes v_j \mid j\in\Omega\}$ is a basis of the $\mathbb{F}$-vector space $\bar{V}_\mu$.

5.2. Consider the natural maps $U_{\mathbb{A}} \to U_{\mathbb{A}}/\mathbb{J}U_{\mathbb{A}} \cong \bar{U}$ and $V^{\mathbb{A}} \to V^{\mathbb{A}}/\mathbb{J}V^{\mathbb{A}} \cong \bar{V}$. Note that $q \to 1$ under the action of these maps. The passage from $U_{\mathbb{A}}$ to $\bar{U}$ and $V^{\mathbb{A}}$ to $\bar{V}$ under these natural maps is referred to as taking the *classical limit.* We will denote the images of general elements $u \in U_{\mathbb{A}}$ and $v \in V^{\mathbb{A}}$ by $\bar{u}$ and $\bar{v}$ respectively. For convenience, we denote the images of

$$\frac{K - K^{-1}}{q - q^{-1}}, \quad \frac{\gamma - \gamma^{-1}}{q - q^{-1}} \text{ and } \frac{D - D^{-1}}{q - q^{-1}}$$

by h, c, and d, respectively. Furthermore, we denote $\overline{x^+(k)} = e(k)$, $\overline{x^-(k)} = f(k)$, and $\overline{a(l)} = h(l)$.

LEMMA 5.2. *As endomorphisms of $\bar{V}$, $\bar{K} = \bar{\gamma} = \bar{D} = 1$.*

Proof. Since $K \cdot v_\mu = q^{\mu(h)} v_\mu$, we have $\bar{K} \cdot \bar{v}_\mu = \overline{q^{\mu(h)}} \bar{v}_\mu = \bar{v}_\mu$. This is true for all $\mu \in P$; hence $\bar{K} = 1$ on $\bar{V}$. Similarly, $\bar{\gamma} = 1$ and $\bar{D} = 1$.

PROPOSITION 5.2. *As an $\mathbb{F}$-algebra of endomorphisms of $\bar{V}$, $\bar{U}$ is generated by $e(k)$, $f(k)$ $(k \in \mathbb{Z})$, $h(l)$ $(l \in \mathbb{Z}\setminus\{0\})$, c, d, and h.*

Proof. In Lemma 4.4, we showed that the elements $x^\pm(k)$ and $a(l)$ were in $U_{\mathbb{A}}$, and so their images $e(k)$, $f(k)$, and $h(l)$ are in $\bar{U}$. In Lemma 4.3, we also showed that $(\gamma - \gamma^{-1})/(q - q^{-1}) \in U_{\mathbb{A}}$ so that $c \in \bar{U}$, while

$$\frac{K - K^{-1}}{q - q^{-1}} = \begin{bmatrix} K\,;0 \\ 1 \end{bmatrix} \in U_{\mathbb{A}}$$

and

$$\frac{D - D^{-1}}{q - q^{-1}} = \begin{bmatrix} D\,;0 \\ 1 \end{bmatrix} \in U_{\mathbb{A}},$$

so h, $d \in \bar{U}$.

Now consider the images of the generators of $U_{\mathbb{A}}$. Note that $\overline{x^+(0)^{(n)}} = e(0)^{(n)}$. In particular, $\overline{x^+(0)^{(1)}} = e(0) = e \in \bar{U}$. Similarly, $\overline{x^-(0)^{(n)}} = f^{(n)}$ implies $f = f(0) \in \bar{U}$, $\overline{x^+(-1)^{(n)}} = e(-1)^{(n)}$ implies $e(-1) \in \bar{U}$, and $\overline{x^-(1)^{(n)}} = f(1)^{(n)}$ implies $f(1) \in \bar{U}$. Lemma 5.2 showed that $\bar{K} = \bar{\gamma} = \bar{D} = 1$.

Recall that

$$\begin{bmatrix} K\,;t \\ N \end{bmatrix} = \prod_{r=1}^{N} \frac{Kq^{t-r+1} - K^{-1}q^{-(t-r+1)}}{q^r - q^{-r}}.$$

But

$$\frac{Kq^{t-r+1} - K^{-1}q^{-(t-r+1)}}{q^r - q^{-r}} = \frac{q^{t-r+1}(q - q^{-1})}{q^r - q^{-r}} \frac{K - K^{-1}}{q - q^{-1}} + K^{-1} \frac{q^{t-r+1} - q^{-(t-r+1)}}{q - q^{-1}} \frac{q - q^{-1}}{q^r - q^{-r}},$$

Letting $q \to 1$, we get

$$\overline{\begin{bmatrix} K\,;t \\ N \end{bmatrix}} = \prod_{r=1}^{N} \frac{1}{r}(h + (t - r + 1)1).$$

Therefore all images of $\begin{bmatrix} K\,;t \\ N \end{bmatrix}$ are generated by h. Similarly,

$$\overline{\begin{bmatrix} \gamma\,;t \\ N \end{bmatrix}} = \prod_{r=1}^{N} \frac{1}{r}(c + (t-r+1)1)$$

and

$$\overline{\begin{bmatrix} D\,;t \\ N \end{bmatrix}} = \prod_{r=1}^{N} \frac{1}{r}(d + (t-r+1)1).$$

Hence, as an $\mathbb{F}$-algebra of endomorphisms of $\bar{V}$, $\bar{U}$ is generated by $e(k)$, $f(k)$, $h(l)$, c, d, and h.

THEOREM 5.3. (1) *The endomorphisms* $e(k)$, $f(k)$, $h(l)$, c, d, *and* h *satisfy the relations for* $U(A_1^{(1)})$. *Hence* $\bar{V}$ *has a* $U(A_1^{(1)})$*-module structure.*

(2) *As a* $\bar{U}$*-module,* $\bar{V}$ *is an S-highest weight module with highest weight* λ *and highest weight vector* $\bar{v}_\lambda = 1 \otimes v_\lambda$.

(3) *The elements* h, c, *and* d *act on* $\bar{V}_\mu$ *as scalar multiplication by* $\mu(h)$, $\mu(c)$, *and* $\mu(d)$, *respectively. Hence* $\bar{V}_\mu$ *is the* μ*-weight space of the* $U(A_1^{(1)})$*-module* $\bar{V}$.

Proof. (1) Since γ is in the centre of U, we have

$$\left[\frac{\gamma - \gamma^{-1}}{q - q^{-1}}, u\right] = 0 \quad \text{for all } u \in U.$$

By taking the classical limit $q \to 1$, we obtain $[c, \bar{u}] = 0$ for all $\bar{u} \in \bar{U}$. Similarly, $[h, d] = 0$ and $[h, h(k)] = 0$ for $k \neq 0$.

Let $v \in V_\mu^{\mathbb{A}}$ be a weight vector. From defining relation (2) of U, we have

$$[a(k), a(l)] \cdot v = \delta_{k+l,0} \frac{[2k]}{k} \frac{q^{k\mu(c)} - q^{-k\mu(c)}}{q - q^{-1}} v.$$

Letting $q \to 1$, we get

$$[h(k), h(l)] \cdot \bar{v} = \delta_{k+l,0} \frac{2k}{k} k\mu(c)\bar{v} = 2k\delta_{k+l,0} c \cdot \bar{v}.$$

Hence we have $[h(k), h(l)] = 2k\delta_{k+l,0}c$.

From defining relation (4) of U, we have

$$\begin{aligned}\left[\frac{D - D^{-1}}{q - q^{-1}}, a(l)\right] \cdot v &= \frac{(q^l - 1)a(l)D - (q^{-l} - 1)a(l)D^{-1}}{q - q^{-1}} \cdot v \\ &= \frac{q^{\mu(d)}(q^l - 1) - q^{-\mu(d)}(q^{-l} - l)}{q - q^{-1}} a(l) \cdot v \\ &= \frac{(q^{\mu(d)+l} - q^{-\mu(d)-l}) - (q^{\mu(d)} - q^{-\mu(d)})}{q - q^{-1}} a(l) \cdot v.\end{aligned}$$

Thus letting $q \to 1$ yields

$$[d, h(l)] \cdot \bar{v} = (\mu(d) + l - \mu(d))h(l) \cdot \bar{v} = lh(l) \cdot \bar{v},$$

and hence $[d, h(l)] = lh(l)$.

Similarly, we obtain

$$[d, e(k)] = ke(k), \quad [d, f(k)] = kf(k),$$
$$[h, e(k)] = 2e(k), \quad [h, f(k)] = -2f(k).$$

We now consider

$$[a(k), x^+(l)] \cdot v = \frac{[2k]}{k}\gamma^{-[k]/2}x^+(k+l) \cdot v = \frac{[2k]}{k}q^{-(|k|/2)\mu(c)}x^+(k+l) \cdot v.$$

Letting $q \to 1$, we obtain

$$[h(k), e(l)] \cdot \bar{v} = 2e(k+l) \cdot \bar{v}.$$

Hence $[h(k), e(l)] = 2e(k+l)$, and similarly, $[h(k), f(l)] = -2f(k+l)$.

Finally, consider the defining relation (9) of U. If $k + l > 0$, we have

$$\begin{aligned}[x^+(k), x^-(l)] &= \frac{1}{q-q^{-1}}\gamma^{(k-l)/2}\psi(k+l)\\ &= \gamma^{(k-l)/2}Ka(k+l) + K(q-q^{-1})(*),\end{aligned}$$

where the expression $(*)$ has no terms involving $1/(q-q^{-1})$. Thus, taking the classical limit $q \to 1$, we get

$$[e(k), f(l)] = h(k+l).$$

Similarly, we obtain $[e(k), f(l)] = h(k+l)$ for $k + l < 0$.

If $k + l = 0$, then we have

$$\begin{aligned}[x^+(k), x^-(l)] \cdot v &= \frac{1}{q-q^{-1}}(\gamma^k\psi(0) - \gamma^{-k}\phi(0)) \cdot v\\ &= \frac{1}{q-q^{-1}}(\gamma^k K - \gamma^{-k}K^{-1}) \cdot v\\ &= \frac{q^{k\mu(c)+\mu(h)} - q^{-k\mu(c)-\mu(h)}}{q-q^{-1}}v.\end{aligned}$$

By taking the limit $q \to 1$, we obtain

$$[e(k), f(-k)] \cdot \bar{v} = (\mu(h) + k\mu(c))\bar{v} = (h + kc) \cdot \bar{v}.$$

Hence we have

$$[e(k), f(l)] = h(k+l) + k\delta_{k+l,0}c,$$

which proves (1).

(2) By definition, $x^+(k)\cdot v_\lambda = 0$, which implies that $e(k)\cdot \bar{v}_\lambda = 0$. Similarly, since $a(l)\cdot v_\lambda = 0$ for $l>0$, we have $h(l)\cdot \bar{v}_\lambda = 0$ for $l>0$. Now recall that

$$\frac{K-K^{-1}}{q-q^{-1}}\cdot v_\lambda = \frac{q^{\lambda(h)}-q^{-\lambda(h)}}{q-q^{-1}} v_\lambda.$$

Letting $q\to 1$, we obtain $h\cdot \bar{v}_\lambda = \lambda(h)\bar{v}_\lambda$. Similarly, $c\cdot \bar{v}_\lambda = \lambda(c)\bar{v}_\lambda$ and $d\cdot \bar{v}_\lambda = \lambda(d)\bar{v}_\lambda$.

Recall from Proposition 4.7 that $V^{\mathbb{A}} = U^-_{\mathbb{A}}(S)\cdot v_\lambda$. Hence, $\bar{V} = \overline{U^-_{\mathbb{A}}(S)}\cdot \bar{v}_\lambda$. By Lemma 4.4 and Proposition 4.6, the images of the generators of $U^-_{\mathbb{A}}(S)$ under the classical limit are the monomials in $f(k) = \overline{x^-(k)}$ $(k\in\mathbb{Z})$ and $h(-l) = \overline{a(-l)}$ $(l>0)$ with coefficients in $\mathbb{F}$. Hence, as an $\mathbb{F}$-algebra, $\overline{U^-_{\mathbb{A}}(S)}$ is generated by the elements $f(k)$ $(k\in\mathbb{Z})$ and $h(-l)$ $(l>0)$, which implies that $\overline{U^-_A(S)}\cong U(\mathfrak{g}^{(S)}_-)$. Therefore, $\bar{V}$ is an S-highest weight module with highest weight λ and highest weight vector $\bar{v}_\lambda$.

(3) For $\mu\in P$ and $v_\mu\in V^A_\mu$, we have

$$\frac{K-K^{-1}}{q-q^{-1}}\cdot v_\mu = \frac{q^{\mu(h)}-q^{-\mu(h)}}{q-q^{-1}} v_\mu,$$

and so $h\cdot \bar{v}_\mu = \mu(h)\bar{v}_\mu$. We treat c and d similarly.

5.4. Now we are ready to prove our main result.

THEOREM 5.4. *If V^q is the imaginary Verma module $M^q(\lambda)$ over U with highest weight λ, then $\bar{V}$ is isomorphic to the imaginary Verma module $M(\lambda)$ over $U(A_1^{(1)})$ with highest weight λ.*

Proof. Let v_λ be a highest weight vector of V^q. We need to prove that $\bar{V}$ is a free $\overline{U^-_{\mathbb{A}}(S)}$ module of rank 1 generated by $\bar{v}_\lambda$. Since $V^q = M^q(\lambda)$, V^q is a free $U^-(S)$-module of rank 1 generated by the highest weight vector v_λ. Thus, by Propositions 4.7 and 4.8, the $\mathbb{A}$-form $V^{\mathbb{A}}$ of V^q is a free $U^-_{\mathbb{A}}(S)$-module generated by v_λ. Since $V^{\mathbb{A}} = U^-_{\mathbb{A}}(S)\cdot v_\lambda$, on passing to the classical limit, we have $\bar{V} = \overline{U^-_{\mathbb{A}}(S)}\cdot \bar{v}_\lambda$, where $\overline{U^-_{\mathbb{A}}(S)} = \mathbb{F}\otimes_{\mathbb{A}} U^-_{\mathbb{A}}(S)\cong U^-_{\mathbb{A}}(S)/\mathbb{J}U^-_{\mathbb{A}}(S)$ is the $\mathbb{F}$-subalgebra of $U(A_1^{(1)})$ with 1 generated by the elements $f(k)$ $(k\in\mathbb{Z})$ and $h(-l)$ $(l>0)$.

Suppose $\bar{u}\cdot\bar{v}_\lambda = 0$ for some $\bar{u}\in\overline{U^-_{\mathbb{A}}(S)}$. Write $\bar{u} = u + \mathbb{J}U^-_{\mathbb{A}}(S)$ for some $u\in U^-_{\mathbb{A}}(S)$. Since $\bar{u}\cdot\bar{v}_\lambda = 0$, we have $u\cdot v_\lambda\in\mathbb{J}V^{\mathbb{A}} = \mathbb{J}U^-_{\mathbb{A}}(S)\cdot v_\lambda$, so $u\cdot v_\lambda = u'\cdot v_\lambda$ for some $u'\in\mathbb{J}U^-_{\mathbb{A}}(S)\subseteq U^-_{\mathbb{A}}(S)$. Since $V^{\mathbb{A}}$ is a free $U^-_{\mathbb{A}}(S)$-module generated by v_λ, we must have $u = u'$. But this implies that $\bar{u} = \bar{u}' = 0$ in $\overline{U^-_{\mathbb{A}}(S)}$. Therefore, $\bar{V}$ is a free $\overline{U^-_{\mathbb{A}}(S)}$-module generated by $\bar{v}_\lambda$ and hence $\bar{V}\cong M(\lambda)$.

We have shown that any imaginary Verma module $M(\lambda)$ over $U(A_1^{(1)})$ with highest weight $\lambda\in P$ admits a quantum deformation to the imaginary Verma module $M^q(\lambda)$ over U with highest weight $\lambda\in P$ in such a way that the dimensions of the weight spaces are invariant under the deformation.

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