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## GENERALIZATIONS OF DEODHAR'S $\alpha$ -LOCALIZATION FUNCTOR

BEN COX

(Communicated by Roe Goodman)

**ABSTRACT.** In this paper we generalize the result of Deodhar (see *Invent. Math.* 57 (1980), 101–118) on  $\alpha$ -localization functors. Namely, we show that localization with respect to a larger family of left denominator sets “intertwines” with tensoring by finite-dimensional representations. In the language of the author’s previous work, localization with respect to such a left denominator set produces a new example of an  $\mathfrak{F}$ -functor and an  $\mathfrak{F}$ -category.

### 1. INTRODUCTION

Let  $\mathfrak{g}$  be a semisimple finite-dimensional Lie algebra with triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ . In [D], Deodhar discovered that Enright’s completion functor (see [E1, E2]) is actually a subfunctor of an  $\alpha$ -localization functor. This latter functor is obtained by noncommutative localization with respect to the left denominator set  $S_\alpha = \{y_\alpha^n | n \in \mathbb{N}\}$  where  $y_\alpha$  is a nilpotent element in  $\mathfrak{n}_-$ . Localization with respect to this set “intertwines” with tensoring by finite-dimensional  $\mathfrak{g}$ -modules. Below we find new examples of noncommutative localization which also intertwine with tensoring by finite-dimensional representations.

### 2. PRELIMINARIES AND NOTATION

2.1. Recall that an *additive category*  $\mathfrak{A}$  is a category satisfying the following three axioms:

- (i)  $\mathfrak{A}$  has a zero object;
- (ii) any two objects in  $\mathfrak{A}$  have a product; and
- (iii) for all objects  $A, B \in \text{Ob } \mathfrak{A}$  the set of morphisms  $\text{Hom}_{\mathfrak{A}}(A, B)$  forms an abelian group such that the composition

$$\text{Hom}_{\mathfrak{A}}(A, B) \times \text{Hom}_{\mathfrak{A}}(B, C) \rightarrow \text{Hom}_{\mathfrak{A}}(A, C)$$

is bilinear. One also has the following proposition.

**Proposition** [HS, Chapter 2]. *Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two additive categories and  $F: \mathfrak{A} \rightarrow \mathfrak{B}$  a functor. Then the following are equivalent:*

- (i)  $F$  preserves sums (of two objects).
- (ii)  $F$  preserves products (of two objects).

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(iii) For each  $A, A' \in \text{Ob } \mathfrak{A}$  one has that

$$F: \text{Hom}_{\mathfrak{A}}(A, A') \rightarrow \text{Hom}_{\mathfrak{B}}(FA, FA')$$

is a group homomorphism.

A functor satisfying the above equivalent conditions is called an *additive functor*.

For a vector space  $V$  over a field  $k$ , let  $T^n(V)$  denote the  $n$ -fold tensor product of  $V$  with itself, and let  $T^0 = k$ . Then  $T(V) := \bigoplus_{n=0}^{\infty} T^n(V)$  is the *tensor algebra* of  $V$ . Elements in  $T^n(V)$  are called *homogeneous of degree  $n$* . If  $\mathfrak{g}$  is a Lie algebra, let  $U(\mathfrak{g})$  denote the universal enveloping algebra of  $\mathfrak{g}$ .

### 3. $\mathfrak{F}$ -CATEGORIES AND $\mathfrak{F}$ -FUNCTORS

3.1. For any Lie algebra  $\mathfrak{g}$  (possibly infinite dimensional) defined over a field  $k$  of characteristic zero, let  $M_{\mathfrak{g}}$  denote the category of all  $\mathfrak{g}$ -modules. Throughout we will assume that  $\mathfrak{F}$  denotes an additive subcategory of  $M_{\mathfrak{g}}$  satisfying the following two conditions:

- (1)  $\mathfrak{F}$  is closed under tensoring; i.e., if  $E_j, F_j \in \mathfrak{F}$  and  $f_j \in \text{Hom}_{\mathfrak{F}}(E_j, F_j)$  for  $j = 1, 2$ , then  $E_1 \otimes E_2, F_1 \otimes F_2 \in \text{Ob } \mathfrak{F}$  and  $f_1 \otimes f_2 \in \text{Hom}_{\mathfrak{F}}(E_1 \otimes E_2, F_1 \otimes F_2)$ .
- (2)  $\mathfrak{g} \in \text{Ob } \mathfrak{F}$  as a  $\mathfrak{g}$ -module under the adjoint action.

For  $F \in \text{Ob } \mathfrak{F}$ ,  $\mathfrak{g}$ -modules  $A$  and  $B$ , and  $h \in \text{Hom}_{\mathfrak{g}}(A, B)$ , let  $T_F$  denote the tensor product functor on  $M_{\mathfrak{g}}$  given by  $A \mapsto F \otimes A$  and  $h \mapsto 1_F \otimes h$ . If  $\mathfrak{n} \subset \mathfrak{g}$  is a Lie subalgebra and  $\mathfrak{C}$  is a subcategory of  $M_{\mathfrak{n}}$ , we shall use the symbol  $T_F$  to denote the tensor product functor on  $\mathfrak{C}$  when no confusion is likely to arise. We call the category  $\mathfrak{C}$  an  *$\mathfrak{F}$ -category* if it is additive and  $T_F$  carries  $\mathfrak{C}$  into itself for all  $F \in \text{Ob } \mathfrak{F}$ .

Now let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two Lie subalgebras of  $\mathfrak{g}$ , and let  $\mathfrak{A}$  (resp.  $\mathfrak{B}$ ) be an additive subcategory of  $M_{\mathfrak{a}}$  (resp.  $M_{\mathfrak{b}}$ ). Suppose further that both  $\mathfrak{A}$  and  $\mathfrak{B}$  are  $\mathfrak{F}$ -categories and  $\tau$  is a functor from  $\mathfrak{A}$  to  $\mathfrak{B}$ . We call  $\tau$  an *intertwining functor* (or  *$\mathfrak{F}$ -intertwining functor* when more precision is necessary) if  $\tau$  is additive and there exists a natural equivalence, for each  $F \in \text{Ob } \mathfrak{F}$ ,  $i_F: T_F \circ \tau \rightarrow \tau \circ T_F$ .

Suppose  $\tau$  is an intertwining functor, and let  $\mathcal{S} = \{i_F | F \in \text{Ob } \mathfrak{F}\}$  denote the family of natural equivalences above. Recall that a natural equivalence  $i_E: T_E \circ \tau \rightarrow \tau \circ T_E$  is a rule that assigns to each object  $A$  of  $\mathfrak{A}$  an isomorphism  $i_E(A): T_E \circ \tau(A) \rightarrow \tau \circ T_E(A)$  such that for every homomorphism  $f: A \rightarrow B$  in  $\mathfrak{A}$  one has  $i_E(B) \circ ((T_E \circ \tau)(f)) = (\tau \circ T_E)(f) \circ i_E(A)$ . For convenience we set  $i_{E,A} = i_E(A)$  for all  $A \in \text{Ob } \mathfrak{A}$ ,  $E \in \text{Ob } \mathfrak{F}$ .

Suppose now that for every  $E, F \in \text{Ob } \mathfrak{F}$  and  $h \in \text{Hom}_{\mathfrak{F}}(E, F)$  one has  $h \otimes 1_A \in \text{Hom}_{\mathfrak{A}}(E \otimes A, F \otimes A)$  for  $A \in \text{Ob } \mathfrak{A}$ . Assume the category  $\mathfrak{B}$  has this same property. Then we say that  $\mathcal{S}$  is *natural in the  $\mathfrak{F}$ -variable* (or *natural in  $\mathfrak{F}$* ) if the following diagram is commutative for all  $E, F \in \text{Ob } \mathfrak{F}$ ,  $A \in \text{Ob } \mathfrak{A}$ , and  $f \in \text{Hom}_{\mathfrak{F}}(E, F)$ :

$$(1) \quad \begin{array}{ccc} E \otimes \tau A & \xrightarrow{i_{E,A}} & \tau(E \otimes A) \\ f \otimes 1_A \downarrow & & \downarrow \tau(f \otimes 1_A) \\ F \otimes \tau A & \xrightarrow{i_{F,A}} & \tau(F \otimes A) \end{array}$$

We call  $\mathcal{S}$  *distributive* if the following diagram is commutative for  $E, F \in \text{Ob } \mathfrak{F}$  and  $A \in \text{Ob } \mathfrak{A}$ :

$$(2) \quad \begin{array}{ccc} (E \oplus F) \otimes \tau A & \xrightarrow{i_{E \oplus F, A}} & \tau((E \oplus F) \otimes A) \\ \downarrow & & \downarrow \\ (E \otimes \tau A) \oplus (F \otimes \tau A) & \xrightarrow{i_{E, A} \oplus i_{F, A}} & \tau(E \otimes A) \oplus \tau(F \otimes A) \end{array}$$

The left map (2) expresses the bilinearity of  $\otimes$ , and the right map expresses this bilinearity combined with additivity of  $\tau$ .

We say that  $\mathcal{S}$  is *associative* if the following diagram is commutative for all  $E, F \in \text{Ob } \mathfrak{F}$  and  $A \in \text{Ob } \mathfrak{A}$ :

$$(3) \quad \begin{array}{ccc} E \otimes F \otimes \tau A & \xrightarrow{1 \otimes i_{F, A}} & E \otimes \tau(F \otimes A) \\ & \searrow i_{E \otimes F, A} & \downarrow i_{E, F \otimes A} \\ & & \tau(E \otimes F \otimes A) \end{array}$$

**3.2 Lemma [C].** *Suppose  $\mathfrak{A}$  and  $\mathfrak{B}$  are  $\mathfrak{F}$ -categories and  $\tau: \mathfrak{A} \rightarrow \mathfrak{B}$  is an intertwining functor. If the family  $\mathcal{S} = \{i_{F, A} | F \in \text{Ob } \mathfrak{F}, A \in \text{Ob } \mathfrak{A}\}$  is natural in the  $\mathfrak{F}$ -variable then  $\mathcal{S}$  is distributive.*

Suppose  $\tau$  is an intertwining functor with the family of natural equivalences  $\mathcal{S} = \{i_F | F \in \text{Ob } \mathfrak{F}\}$ . We call the pair  $(\tau, \mathcal{S})$  an  *$\mathfrak{F}$ -functor* whenever  $\mathcal{S}$  is both distributive and associative. When  $\mathcal{S}$  is understood to be fixed we call  $\tau$  an  *$\mathfrak{F}$ -functor*. (See [C] and the references listed there for several examples of  $\mathfrak{F}$ -functors in the representation theory of Lie algebras.)

**4. DEODHAR'S  $\alpha$ -LOCALIZATION FUNCTOR AND NEW EXAMPLES OF  $\mathfrak{F}$ -FUNCTORS**

4.1. Let  $\mathfrak{g}$  be a Kac-Moody algebra, and let  $\mathfrak{F}$  be the category of integrable  $\mathfrak{g}$ -modules. In this section we review Deodhar's  $\alpha$ -localization functor  $D_\alpha$  for  $\alpha$  a real root of  $\mathfrak{g}$ . Our main result, Theorem 4.18, is a substantial generalization of Deodhar's results on localization and will provide us with new examples of  $\mathfrak{F}$ -functors. Let us begin by reviewing some noncommutative ring theory (our general reference will be by [GW]).

4.2. All rings in this article are assumed to have an identity. If  $X$  is a multiplicative subset then a *left ring of fractions for  $R$  with respect to  $X$*  is a ring homomorphism  $\phi: R \rightarrow S$  such that

- (a)  $\phi(x)$  is a unit in  $S$  for all  $x \in X$ ,
- (b) each element of  $S$  can be written in the form  $\phi(x)^{-1}\phi(r)$  for some  $x \in X$  and  $r \in R$ , and
- (c)  $\ker(\phi) = \{r \in R | xr = 0 \text{ for some } x \in X\}$ .

A multiplicative set  $X$  in a ring  $R$  that satisfies the following two conditions is called a *left denominator set*:

- (Da)  $Xr \cap Rx \neq \emptyset$  for all  $r \in R$  and  $x \in X$ ;
- (Db) if  $r \in R$  and  $x \in X$  are such that  $xr = 0$  then there exists  $x' \in X$  such that  $rx' = 0$ .

A well-known result due to Goldie is

**4.3. Theorem** [GW, Proposition 9.7]. *Let  $X$  be a multiplicative set in a ring  $R$ . Then there exists a left ring of fractions for  $R$  with respect to  $X$  if and only if  $X$  is a left denominator set.*

Moreover, if a left ring of fractions exists then it is unique up to isomorphism (see [GW, Corollary 9.5]). If  $X$  is a left denominator set then we let  $X^{-1}R$  denote its unique ring of fractions. This (as is well known) can be constructed as follows: Define on  $X \times R$  an equivalence relation  $\sim$  where  $(x, r) \sim (x', r')$  if there exists  $s \in R$  and  $y \in X$  such that  $yr = sr'$  and  $yx = sx'$ . The set of equivalence classes of  $X \times R$  has an obvious ring structure which gives us  $X^{-1}R$ . We let  $x \setminus r$  denote the equivalence class of  $(x, r)$ .

**4.4.** Let  $X$  be a left denominator set, and let  $A$  be a left  $R$ -module. A *module of fractions for  $A$  with respect to  $X$*  is an  $R$ -module homomorphism  $f: A \rightarrow B$  where  $B$  is a left  $X^{-1}R$ -module such that

- (a) every element of  $B$  can be written in the form  $x^{-1}f(a)$  for some  $x \in X$  and  $a \in A$ , and
- (b)  $\ker f = \{a \in A \mid ax = 0 \text{ for some } x \in X\}$ .

Another basic fact is

**Theorem** [GW, Corollary 9.11 and Theorem 9.13]. *If  $X$  is a left denominator set in a ring  $R$  then there exists a unique (up to isomorphism) module of fractions for any left  $R$ -module  $A$  with respect to  $X$ .*

**4.5.** When  $X$  is a left denominator set and  $A$  is a left  $R$ -module then we let  $X^{-1}A$  denote its unique module of fractions with respect to  $X$ . In addition there is the following useful construction of  $X^{-1}A$ .

**4.6. Theorem** [GW, Proposition 9.14]. *For  $X$  a left denominator set in  $R$  and  $A$  a left  $R$ -module we have*

$$X^{-1}R \otimes_R A \cong X^{-1}A$$

where the map is given by  $s \otimes a \mapsto sa$  for  $a \in A$  and  $s \in X^{-1}R$ .

One has the following universal mapping property of localizations.

**Proposition** [GW, Proposition 9.10]. *Let  $X$  be a left denominator set in a ring  $R$ , let  $A$  be a left  $R$ -module, and suppose  $C$  is a left  $X^{-1}R$ -module with  $g: A \rightarrow C$  an  $R$ -module homomorphism. If  $f: A \rightarrow X^{-1}A$  is the module of fractions for  $A$  with respect to  $X$ , then there exists a unique  $X^{-1}R$ -module homomorphism  $h: X^{-1}A \rightarrow C$  such that  $g = h \circ f$ .*

**4.7.** Next we define Deodhar’s  $\alpha$ -localization functor and record the well-known fact that it is just “localization” with respect to the multiplicative set  $S_\alpha = \{y_\alpha^n \mid n \in \mathbb{N}\}$  where  $y_\alpha \in \mathfrak{g}_{-\alpha}$  is nonzero and  $\alpha$  is a positive root.

Let  $\mathcal{A}$  denote the category of  $\mathfrak{h}$ -semisimple  $U(\mathfrak{n}_-)$ -torsionfree  $\mathfrak{g}$ -modules. If  $A \in \text{Ob } \mathcal{A}$ , let  $A' = \{y_\alpha^{-n} \mid n \in \mathbb{N}\} \times A$ , and define an equivalence relation on  $A'$  by  $(y_\alpha^{-n}, a) \sim (y_\alpha^{-m}, a')$  if and only if  $y_\alpha^n a' = y_\alpha^m a$ . Set  $D_\alpha(A) = A' / \sim$ .  $D_\alpha(A)$  is given a  $\mathfrak{g}$ -module structure as follows.

**4.8. Lemma** [D, Lemma 2.1]. *If  $z \in \mathfrak{g}$  and  $0 \leq r \in \mathbb{Z}$  then there exists  $0 \leq s \in \mathbb{Z}$  such that  $y_\alpha^s z = uy_\alpha^r$  for some  $u \in U(\mathfrak{g})$ .*

*Remark.* Deodhar’s proof is based on a straightforward  $\mathfrak{sl}(2, \mathbb{C})$  computation, and from the proof it is easy to see that this lemma is also true for  $z \in U(\mathfrak{g})$

so that  $S_\alpha$  is a left Ore set, i.e., it satisfies (Da). It then follows from a result of Goldie (see [GW, Proposition 9.9]) that  $S_\alpha$  is a left denominator set.

Now if  $(y_\alpha^{-r}, a) \in D_\alpha(A)$  and  $z \in \mathfrak{g}$  then  $z(y_\alpha^{-r}, a)$  is defined to be  $(y_\alpha^{-s}, ua)$  where  $u$  is in the lemma above.

**4.9. Proposition** [D, Proposition 2.2]. *Under the action of  $\mathfrak{g}$  on  $D_\alpha$  given above,  $D_\alpha(A)$  becomes a well-defined  $\mathfrak{g}$ -module.*

*Remark.* By definition  $D_\alpha$  is Deodhar's  $\alpha$ -localization functor.

Next for  $A \in \mathcal{A}$ ,  $A \hookrightarrow D_\alpha(A)$  as a  $\mathfrak{g}$ -module so that by the universal mapping property of the module of fractions there exists a unique  $S_\alpha^{-1}U(\mathfrak{g})$ -module map  $S_\alpha^{-1}A \rightarrow D_\alpha A$ . In fact this map is given by  $y_\alpha^{-n}a \mapsto (y_\alpha^{-n}, a)$ . This map is clearly a bijection. Hence  $S_\alpha^{-1}|_{\mathcal{A}} \cong D_\alpha$ .

**4.10.** We now turn to our generalization of these results of Deodhar. Let  $X_0 \subset \mathfrak{g} \oplus \mathbb{C} = U(\mathfrak{g})_1$ , and let  $X \subset U(\mathfrak{g})$  be a left denominator set generated by 1 and  $X_0$  with the following property: for  $x \in X_0$  and  $E$  any finite-dimensional  $\mathfrak{g}$ -module

- (1) if  $\lambda$  is an eigenvalue of the action of  $x$  on  $E$  then  $x + \lambda \cdot 1 \in X$ .

We shall suppose throughout this section that  $X$  satisfies (1).

For convenience we set  $R = U(\mathfrak{g})$  and  $S = X^{-1}R$ .

**4.11. Lemma.** *Let  $x \in X$ . Then  $x$  acts by an isomorphism on  $S \otimes E$ .*

*Proof.* It is sufficient to check the lemma for  $x \in X_0$ . Introduce an  $x$ -stable filtration  $0 = E_0 \subset \dots \subset E_d$  on  $E$  with  $x$  acting by  $\lambda_i \cdot 1$  on  $E_i/E_{i-1}$ . Then suppose  $x$  acts by isomorphism on  $S \otimes E_{i-1}$ . Thus for  $\{e_k\}$  a basis of  $E$  with  $e_{ij} \in E_i$

$$\begin{aligned} x(s \otimes e_{ij}) &\equiv xs \otimes e_{ij} + s \otimes \lambda_i e_{ij} \pmod{S \otimes E_{i-1}} \\ &\equiv (x + \lambda_i)s \otimes e_{ij} \pmod{S \otimes E_{i-1}}. \end{aligned}$$

This proves the lemma.

$S$  is an  $R$ -bimodule. We shall need other bimodules and so introduce the notation  $E \otimes \mathbb{C}$  to denote the  $R$ -bimodule with left action given on  $E$  and trivial right action. Similarly, let  $E^\sigma$  be the right  $R$ -module defined by  $r(x) \cdot e = \sigma(x) \cdot e$ ,  $e \in E$ ,  $x \in R$ , and  $\sigma$  is the involutive antiautomorphism of  $U(\mathfrak{g})$  equal to  $-1$  on  $\mathfrak{g}$ . Then  $\mathbb{C} \otimes E^\sigma$  is an  $R$ -bimodule with trivial left action and right action given on  $E^\sigma$ .

**4.12. Lemma.** *As  $S$ -bimodules we have an isomorphism  $S \otimes (E \otimes \mathbb{C}) \cong S \otimes (\mathbb{C} \otimes E^\sigma)$ .*

*Proof.* Consider the map  $\phi: (R \otimes (E \otimes \mathbb{C})) \rightarrow R \otimes (\mathbb{C} \otimes E^\sigma)$  given by  $x \otimes e \otimes 1 \mapsto (1 \otimes 1 \otimes e) \cdot \sigma(x)$ . An easy induction argument using the filtration on  $U(\mathfrak{g})$  shows that  $\phi$  is surjective even at the filtered left; i.e.,  $\phi: R_i \otimes (E \otimes \mathbb{C}) \rightarrow R_i \otimes (\mathbb{C} \otimes E^\sigma)$  is surjective.

To prove injectivity suppose  $0 = \sum_i (1 \otimes 1 \otimes e_i) \cdot \sigma(x_i)$  where the  $e_i$  are as in the proof of Lemma 4.11 and let  $d$  be the maximal integer such that some  $x_j \in R_d$  but  $x_j \notin R_{d-1}$ . Then  $0 \equiv \sum_i \sigma(x_i) \otimes 1 \otimes e_i \pmod{R_{d-1} \otimes (\mathbb{C} \otimes E^\sigma)}$ . However, the  $e_i$  are linearly independent, so each  $\sigma(x_i) \in R_{d-1}$ . We clearly have a contradiction. This proves injectivity; thus  $\phi$  is an isomorphism.

Now we extend  $\phi$  to a map  $\phi: S \otimes (E \otimes \mathbb{C}) \rightarrow S \otimes (\mathbb{C} \otimes E^\sigma)$ . We must define  $\phi(x \setminus r \otimes e \otimes 1)$ . Recall that  $x$  acts on the right of  $S \otimes (\mathbb{C} \otimes E^\sigma)$  by an isomorphism  $i(x)$ . Set

$$\phi(x \setminus r \otimes e \otimes 1) = (1 \otimes 1 \otimes e)i(x)^{-1}a(r) = (1 \otimes 1 \otimes e)a(x \setminus r)$$

where  $a(-)$  denotes the right action of  $R$  on  $S \otimes (\mathbb{C} \otimes E^\sigma)$ . To see that  $\phi$  is well defined suppose  $x \setminus r = x' \setminus r'$ , i.e., there exists  $y \in X$  and  $s \in R$  such that  $yr = sr'$  and  $yx = sx'$ . Then we have  $a(yr) = a(sr')$  and  $a(yx) = a(sx')$ ; thus,  $\phi$  is well defined.

Next we check that  $\phi$  is an  $R$ -bimodule isomorphism. Suppose  $x \in \mathfrak{g}$ , and let  $l(x)$  ( $r(x)$ ) denote the left (resp. right) action. Then for  $y \in X$ ,  $z \in R$ ,  $e \in E$  we have

$$\begin{aligned} l(x)\phi(y \setminus z \otimes e \otimes 1) &= l(x)(1 \otimes 1 \otimes e)r(y)^{-1}r(z) = (x \otimes 1 \otimes e)r(y)^{-1}r(z) \\ &= (x \otimes 1 \otimes e)r(y)^{-1}r(z) + (1 \otimes 1 \otimes \sigma(x) \cdot e)r(y)^{-1}r(z) \\ &\quad + (1 \otimes 1 \otimes x \cdot e)r(y)^{-1}r(z) \\ &= (1 \otimes 1 \otimes e)r(x)r(y)^{-1}r(z) + (1 \otimes 1 \otimes x \cdot e)r(y)^{-1}r(z) \\ &= \phi(x(y \setminus z) \otimes e \otimes 1) + \phi(y \setminus z \otimes x \cdot e \otimes 1). \end{aligned}$$

Now for the right action we have

$$\begin{aligned} r(x)\phi(y \setminus z \otimes e \otimes 1) &= (1 \otimes 1 \otimes e)r(y)^{-1}r(z)r(x) \\ &= \phi(y \setminus zx \otimes e \otimes 1) = \phi(y \setminus z \otimes e \otimes 1)r(x). \end{aligned}$$

Finally we check that  $\phi$  is an  $S$ -bimodule map. Suppose  $x \in X$ ; then from above  $l(x) \circ \phi = \phi \circ l(x)$ . By Lemma 4.11 we can invert these left actions. Multiplying out we have  $\phi \circ l(x)^{-1} = l(x)^{-1} \circ \phi$ ; so  $\phi$  intertwines with the left action of  $S$ . Similarly for the right action. This proves the lemma.

**4.13. Lemma.** *Let  $F$  be any left  $R$ -module. As left  $S$ -modules we have the isomorphism*

$$S \otimes_R (E \otimes F) \cong S \otimes (\mathbb{C} \otimes E^\sigma) \otimes_R F.$$

*Proof.* The left tensor product above is the quotient of  $S \otimes E \otimes F$  by the relations determined by a  $s \cdot x \otimes e \otimes f = s \otimes x \cdot (e \otimes f)$  where  $s \in S$ ,  $e \in E$ ,  $f \in F$ , and  $x \in \mathfrak{g}$ . However, this is precisely the same set of relations as  $(s \otimes e')x \otimes f = s \otimes e' \otimes x \cdot f$  for  $e' \in E^\sigma$ . This proves the lemma.

In the next lemma we introduce another multiplicative subset of  $U(\mathfrak{g})$ . We will see later (4.18) that this gives us another example of an  $\mathfrak{F}$ -functor. First we define an  $\mathfrak{h}$ -graded multiplicative subset of  $U(\mathfrak{g})$  to be a multiplicative subset  $Y$  of  $U(\mathfrak{g})$  such that  $Y = p \cup_{\eta \in \mathfrak{h}} Y_\eta$  and  $Y_\eta \cdot Y_\beta \subset Y_{\eta+\beta}$  where  $Y_\eta := \{y \in Y \mid [h, y] = \eta(h)y \text{ for all } h \in \mathfrak{h}^*\}$ .

**4.14. Lemma.** *Let  $Y$  be an  $\mathfrak{h}$ -graded multiplicative subset of  $U(\mathfrak{n}_-)$ ,  $W$  an  $\mathfrak{h}$ -semisimple  $Y$ -divisible  $\mathfrak{g}$ -module, and  $E$  a finite-dimensional  $\mathfrak{g}$ -module. Then  $E \otimes W$  is an  $\mathfrak{h}$ -semisimple  $Y$ -divisible  $\mathfrak{g}$ -module.*

*Proof.* The fact that  $E \otimes W$  is  $\mathfrak{h}$ -semisimple is obvious. Thus we need only show that  $x(E \otimes W) = E \otimes W$  for all  $x \in Y$ . We may assume  $x$  is not an element in  $Y_0 = \{y \in Y \mid h.y = 0 \text{ for all } h \in \mathfrak{h}\}$ . The result will follow if we can show that, for arbitrary  $e \in E_\gamma$  and  $w \in W_\lambda$ ,  $e \otimes w$  is in  $x(E \otimes W)$ . For  $\lambda \in \mathfrak{h}^*$

let  $E_\lambda$  denote the  $\lambda$ th weight space of  $E$ . Since  $E$  is finite dimensional, the set  $\Lambda(E) = \{\lambda \in \mathfrak{h}^* | E_\lambda \neq 0\}$  is finite; hence there exists a minimal  $\mu \in \Lambda(E)$ . If  $\gamma \in \Lambda(E)$  is minimal then  $xe = 0$  for  $e \in E_\gamma$ . Now as  $W$  is  $Y$ -divisible there exists for any  $w \in W$  an  $w' \in W$  such that  $xw' = w$  and  $x(e \otimes w') = e \otimes w$ . Thus we may assume as an induction hypothesis that  $\gamma$  is not minimal, and the previous statement is true for all  $e \in E_\mu$ ,  $w \in W$  with  $\mu < \gamma$ . Now if  $e' \in E_\gamma$  and  $w \in W$  then

$$x(e' \otimes w') = e' \otimes w + \sum_i e'_i \otimes w'_i$$

where  $e'_i \in E_{\mu_i}$  with  $\mu_i < \gamma$  and  $xw' = w$ . By induction  $e'_i \otimes w'_i \in x(E \otimes W)$ ; thus  $e' \otimes w \in x(E \otimes W)$ . This completes the proof of the lemma.

**4.15. Lemma.** *Let  $Y$  be an  $\mathfrak{h}$ -graded multiplicative subset of  $U(\mathfrak{n}_-)$ ,  $W$  an  $\mathfrak{h}$ -semisimple  $Y$ -torsionfree  $\mathfrak{g}$ -module, and  $E$  a finite-dimensional  $\mathfrak{g}$ -module. Then  $E \otimes W$  is an  $\mathfrak{h}$ -semisimple  $Y$ -torsionfree  $\mathfrak{g}$ -module.*

*Proof.* Let  $\sum_i e_{\mu_i} \otimes w_{\lambda_i} \neq 0$  with  $e_{\mu_i} \in E_{\mu_i}$ ,  $w_{\lambda_i} \in W_{\lambda_i}$  linearly independent and  $x(\sum_i e_{\mu_i} \otimes w_{\lambda_i}) = 0$  for some  $x \in Y_\eta$  where  $\eta \neq 0$ . Lexicographically order the set  $\Lambda(E) \times \Lambda(W)$  (notation as in the previous lemma), and choose  $(\mu_j, \lambda_j)$  maximal with respect to this ordering and such that  $e_{\mu_j} \otimes w_{\lambda_j} \neq 0$ . Then  $0 = \sum_i e_{\mu_i} \otimes xw_{\lambda_i} + \sum_i e_{\mu'_i} \otimes w_{\lambda'_i}$  where  $\mu'_k < \mu_j$  for all  $k$ . But then  $e_{\mu_j} \otimes xw_{\lambda_j} = 0$ ; thus  $xw_{\lambda_j} = 0$ . Since  $W$  is  $Y$ -torsionfree,  $x = 0$ . Hence  $E \otimes W$  is  $Y$ -torsionfree.

**4.16. Lemma.** *If  $E$  and  $W$  are  $\mathfrak{g}$ -modules with  $E$  finite dimensional then*

$$\text{Hom}(E, W) \cong E^* \otimes W.$$

**4.17. Theorem** (Mackey Isomorphism Theorem). *Suppose  $V$  and  $E$  are  $\mathfrak{g}$ -modules with  $E$  finite dimensional and  $V$   $\mathfrak{h}$ -semisimple. Let  $Y \subseteq U(\mathfrak{n}_-)$  be an  $\mathfrak{h}$ -graded left denominator set. Then we have a natural isomorphism*

$$Y^{-1}(V \otimes E) \cong (Y^{-1}V) \otimes E$$

*of  $\mathfrak{g}$ -modules.*

*Proof.* For the proof of this theorem set  $R = U(\mathfrak{n}_-)$ , and let  $W$  be an  $\mathfrak{h}$ -semisimple  $Y^{-1}R$ -module. Now consider the following sequence of isomorphisms:

$$\begin{aligned} & \text{Hom}_{Y^{-1}R}(Y^{-1}(E \otimes V), W) \\ & \cong \text{Hom}_R(E \otimes V, W) \cong \text{Hom}_R(V, \text{Hom}(E, W)) \\ & \cong \text{Hom}_{Y^{-1}R}(Y^{-1}V, \text{Hom}(E, W)) \cong \text{Hom}_{Y^{-1}R}(E \otimes Y^{-1}V, W). \end{aligned}$$

The first isomorphism is just given by Proposition 4.6. The second and fourth isomorphisms are derived from the adjoint associativity property of  $\text{Hom}$ . Now an  $R$ -module is a  $Y^{-1}R$ -module if and only if it is  $Y$ -divisible and  $Y$ -torsionfree (see [GW, Proposition 9.12]). Thus Lemmas 4.14 through 4.16 imply that  $\text{Hom}(E, W)$  is a  $Y^{-1}$ -module so that we can apply Proposition 4.6 to obtain the third isomorphism. We now proceed as in [Kn, Proposition 5.14] and let  $W = Y^{-1}(E \otimes V)$ . The identity in  $\text{Hom}_{Y^{-1}R}(Y^{-1}(E \otimes V), Y^{-1}(E \otimes V))$  induces a map in  $\text{Hom}_{Y^{-1}R}(E \otimes Y^{-1}V, Y^{-1}(E \otimes Y^{-1}V))$  which we will denote by  $\varphi$ . Similarly the identity morphism in  $\text{Hom}_{Y^{-1}R}(E \otimes Y^{-1}V, E \otimes Y^{-1}V)$  induces a map  $\psi \in \text{Hom}_{Y^{-1}R}(Y^{-1}(E \otimes V), E \otimes Y^{-1}V)$ . One can check as in



[Kn, Proposition 5.14] that  $\psi \circ \phi = 1$  and  $\phi \circ \psi = 1$ . This completes the proof of the theorem.

We now consider localization as a functor on the category  $M_{\mathfrak{g}}$  of all  $\mathfrak{g}$ -modules. Let  $\mathfrak{F}$  denote the subcategory of all finite-dimensional  $\mathfrak{g}$ -modules. Set  $\tau_X M = X^{-1}M$  and  $\tau_Y M = Y^{-1}M$  where  $X$  satisfies 4.10.1, and  $Y \subseteq U(\mathfrak{n}_-)$  is an  $\mathfrak{h}$ -graded left denominator set. The main result of the article is

**4.18. Theorem.**  $\tau_X$  and  $\tau_Y$  are intertwining  $\mathfrak{F}$ -functors that are natural in  $\mathfrak{F}$ .

*Proof.* Theorem 4.17 proves the theorem for  $Y$ . For  $X$  we let  $E \in \text{Ob } \mathfrak{F}$ ,  $M \in M_{\mathfrak{g}}$ . Then

$$\begin{aligned} \tau_X(E \otimes M) &= S \otimes_R (E \otimes M) \cong S \otimes (\mathbb{C} \otimes E^\sigma) \otimes_R M \quad (\text{by Lemma 4.13}) \\ &\cong S \otimes (E \otimes \mathbb{C}) \otimes_R M \cong E \otimes \tau_X M \quad (\text{by } \phi^{-1} \text{ and Lemma 4.12}). \end{aligned}$$

The first isomorphism is induced by the identity map, so we shall identify these spaces. For the second isomorphism we have  $\phi^{-1}$  where  $\phi$  is given by  $\phi = \phi_E$  where  $\phi_E(x \setminus r \otimes e \otimes 1) = (1 \otimes 1 \otimes e)a(x)^{-1}a(r)$ .

Now suppose  $\gamma \in \text{Hom}_{\mathfrak{g}}(E, F)$ ;  $E, F \in \text{Ob } \mathfrak{F}$ . Since  $1 \otimes \gamma$  intertwines the (right) action of  $\mathfrak{g}$  from  $S \otimes E^\sigma$  to  $S \otimes F^\sigma$ ,  $1 \otimes \gamma$  intertwines that of  $a(x)^{-1}$  as well. This implies that the family of equivalences  $\{\phi_E | E \in \text{Ob } \mathfrak{F}\}$  is natural in  $\mathfrak{F}$ . This proves the theorem.

*Remark.* For the case  $X = S_\alpha$  the result above is due to Deodhar (see [D, Theorem 3.1]) where again the proof is based on an  $\mathfrak{sl}(2, \mathbb{C})$  calculation.

**4.19. Corollary.**  $\tau_X$  and  $\tau_Y$  are  $\mathfrak{F}$ -functors.

*Proof.* By Theorem 4.18 and Lemma 3.2 we need only show that  $\tau = \tau_X$  and  $\tau = \tau_Y$  are associative. First consider  $\tau = \tau_X$ . Suppose  $E, F \in \text{Ob } \mathfrak{F}$ ,  $M \in \text{Ob } M_{\mathfrak{g}}$ . Since the maps involved are all left  $S$ -module maps, we need only verify the identity on a set of  $S$ -generators for the space. Clearly  $\mathbb{C} \otimes \mathbb{C} \otimes E^\sigma \otimes F^\sigma \otimes_R M$  is a set of generators; so  $\mathbb{C} \otimes E \otimes F \otimes \mathbb{C} \otimes_R M$  is a set of  $S$ -generators for  $S \otimes E \otimes F \otimes \mathbb{C} \otimes_R M$ . But  $\phi_{E \otimes F}$  and also  $\phi_E \circ (1 \otimes \phi_F)$  essentially equal the identity map on these generators;

$$\begin{aligned} \phi_{E \otimes F}(1 \otimes e \otimes f \otimes 1 \otimes m) &= 1 \otimes 1 \otimes e \otimes f \otimes m \\ &= \phi_E \circ (1 \otimes \phi_F)(1 \otimes e \otimes f \otimes 1 \otimes m). \end{aligned}$$

If  $\tau = \tau_Y$ , one uses the identification  $Y^{-1}R \otimes_R A \cong Y^{-1}A$  of Theorem 4.6 and observes that the isomorphism of 4.17 is given by  $1 \otimes v \otimes e \mapsto 1 \otimes v \otimes e$  on a set of  $S$  generators of  $Y^{-1}(V \otimes E)$ . Now one checks as before that the appropriate identity is satisfied on the set  $\mathbb{C} \otimes E \otimes F \otimes A$  of  $S$ -generators of  $\tau_Y(E \otimes F \otimes A)$ .

**4.20. Examples.** Here we describe some multiplicative sets  $X$  other than Deodhar's  $S_\alpha$ , for which the hypothesis of 4.18 is satisfied. Set  $\mathbb{Z}_{\geq 0} = \{k \in \mathbb{Z} | k \geq 0\}$ .

(a) If  $A$  is a  $\mathbb{C}$ -algebra and  $x \in A$  then let  $d_x: A \rightarrow A$  denote the  $\mathbb{C}$ -algebra map given by  $d_x(y) = xy - yx$  for all  $y \in A$ . For this first example we need

**Theorem [BR, Satz 2.2].** *Let  $A$  be a prime Noetherian  $\mathbb{C}$ -algebra. Suppose  $x \in A$  is an element such that  $d_x$  is locally nilpotent on  $A$ . If  $x$  is not*

nilpotent, then  $x$  is a nonzero divisor in  $A$  and the localization  $S^{-1}A$  exists where  $S = \{x^n | n \in \mathbb{Z}_{\geq 0}\}$ .

First let  $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ ,  $\mathfrak{h}$  a Cartan subalgebra of  $\mathfrak{g}$ ,  $R \subset \mathfrak{h}^*$  the set of nonzero roots,  $R^+ \subset R$  a set of positive roots, and  $B = \{\alpha, \beta\} \subset R^+$  a basis for  $R$ . Fix a Chevalley basis  $\{x_\gamma, h_\alpha, h_\beta | \gamma \in R\}$  of  $\mathfrak{g}$  where  $x_\gamma$  denotes the element in this basis with weight  $\gamma \in R$ . Let  $\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathbb{C}x_\alpha$ . For  $\gamma \in R_+$  set  $y_\gamma = x_{-\gamma}$ .

Fix  $\varepsilon \in \mathbb{C} \setminus \{0, 1\}$ , and define  $S_\varepsilon = \{(y_\alpha y_\beta - \varepsilon[y_\alpha, y_\beta])^n | n \in \mathbb{Z}_{\geq 0}\}$ . By the Theorem above  $S_\varepsilon$  is a multiplicative left denominator subset of  $U(\mathfrak{g})$ . It is also clear that  $S_\varepsilon$  is  $\mathfrak{h}$ -graded. Consequently  $S_\varepsilon$  satisfies the hypotheses of 4.18. Our goal is to show that localization with respect to  $S_\varepsilon$  is not the same as localization with respect to the denominator set  $S_\gamma^\mu := \{(\mu y_\gamma)^n | n \in \mathbb{Z}_{\geq 0}\}$  for any  $\gamma \in R$  and  $\mu \in \mathbb{C} \setminus \{0\}$ .

We first consider the case that  $\gamma = \alpha$ . Suppose now that localization with respect to  $S_\alpha^\mu$  is the same as localization with respect to  $S_\varepsilon$ . Then  $y_\alpha$  is invertible in  $S_\varepsilon^{-1}U(\mathfrak{g})$ ; thus,  $(s \setminus m)y_\alpha = 1$  for some  $s \in S_\varepsilon$  and  $m \in U(\mathfrak{g})$ . Consequently there exists  $a \in U(\mathfrak{g})$  and  $b \in S_\varepsilon$  with  $as = b \in S_\varepsilon$  and  $amy_\alpha = b$ . This implies that  $my_\alpha = s = (y_\alpha y_\beta - \varepsilon[y_\alpha, y_\beta])^n$  for some  $n \geq 0$  ( $a \neq 0$  since otherwise  $0 = as = b \in S_\varepsilon$ ).

Define an ordering of the Chevalley basis by

$$x_\alpha < x_\beta < x_{\alpha+\beta} < h_\alpha < h_\beta < y_{\alpha+\beta} < y_\beta < y_\alpha.$$

Then the Poincaré-Birkhoff-Witt Theorem tells us that  $U(\mathfrak{g})$  has a basis of monomials  $\bar{x}\bar{h}\bar{y}$  where

$$\bar{x} = x_\alpha^{n_1} x_\beta^{n_2} x_{\alpha+\beta}^{n_3}, \quad \bar{h} = h_\alpha^{n_4} h_\beta^{n_5}, \quad \bar{y} = y_{\alpha+\beta}^{n_6} y_\beta^{n_7} y_\alpha^{n_8},$$

and  $n_i \in \mathbb{Z}_{\geq 0}$ . Consequently  $m = \sum u_{\bar{n}} \bar{x}\bar{h}\bar{y}$  where  $u_{\bar{n}} \in \mathbb{C}$  and  $\bar{n} = (n_1, \dots, n_8) \in \mathbb{Z}_{\geq 0}^8$ .

Thus

$$my_\alpha = \sum u_{\bar{n}} \bar{x}\bar{h}\bar{y}y_\alpha = (y_\alpha y_\beta - \varepsilon[y_\alpha, y_\beta])^n \in U(\mathfrak{n}_-) \mathfrak{n}_-.$$

By the Poincaré-Birkhoff-Witt Theorem we have that  $m \in U(\mathfrak{n}_-) \mathfrak{n}_-$ . A straightforward calculation shows that

$$my_\alpha = \sum u_{\bar{n}} \bar{x}\bar{h}\bar{y}y_\alpha = (1 - \varepsilon)^n y_{\alpha+\beta}^n + p(y_\alpha, y_\beta, y_{\alpha+\beta})y_\alpha$$

for some polynomial  $p$ . Thus

$$(m - p(y_\alpha, y_\beta, y_{\alpha+\beta}))y_\alpha = (1 - \varepsilon)^n y_{\alpha+\beta}^n.$$

If  $\varepsilon \neq 1$ , this is impossible by the Poincaré-Birkhoff-Witt Theorem. This proves that localization with respect to  $S_\alpha^\mu$  is not the same as localization with respect to  $S_\varepsilon$  if  $\varepsilon \neq 1$ . A very similar argument (but with a different ordering on the basis) shows that localization with respect to  $S_\gamma^\mu$  where  $\gamma \in \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$  is not the same as localization with respect to  $S_\varepsilon$  if  $\varepsilon \neq 0, 1$ .

(b) Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  with  $\{x, h, y\}$  a Chevalley basis of  $\mathfrak{g}$  such that  $[h, x] = 2x$ ,  $[x, y] = h$ , and  $[h, y] = -2y$ . Let  $X_0 = \{h - n1 | n \in \mathbb{Z}\}$  and  $X = \{(h - n1)^k | n, k \in \mathbb{Z}, k \geq 0\}$  be subsets of  $U(\mathfrak{g})$ . This is Example 1.8 in [BR] of an Ore subset of  $U(\mathfrak{g})$ . Now it is straightforward to check that  $X$  satisfies condition 4.10(1). Thus we only need to see that localization with respect to  $X$  is not the same as localization with respect to  $S_\alpha = \{x^n | n \in \mathbb{Z}_{\geq 0}\}$  or

$S_{-\alpha} = \{y^n | n \in \mathbb{Z}_{\geq 0}\}$ . We will prove the case  $S_{\alpha} = \{x^n | n \in \mathbb{Z}_{\geq 0}\}$  and leave the other case to the reader. As in the previous example we assume the contrapositive so that there exists  $m \in U(\mathfrak{sl}(2, \mathbb{C}))$ ,  $n_1, \dots, n_t \in \mathbb{Z}$ , and  $k_1, \dots, k_t \in \mathbb{Z}_{\geq 0}$  such that

$$my = (h - n_1)^{k_1} \cdots (h - n_t)^{k_t}.$$

Using the Poincaré-Birkhoff-Witt Theorem we see that this is impossible. Thus, localization with respect to  $X$  is not the same as localization with  $x$  or  $y$ .

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