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GENERALIZATIONS OF DEODHAR'S α-LOCALIZATION FUNCTOR

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ABSTRACT. In this paper we generalize the result of Deodhar (see Invent. Math. **57** (1980), 101–118) on α-localization functors. Namely, we show that localization with respect to a larger family of left denominator sets "intertwines" with tensoring by finite-dimensional representations. In the language of the author's previous work, localization with respect to such a left denominator set produces a new example of an $\mathfrak{F}$-functor and an $\mathfrak{F}$-category.

1. INTRODUCTION

Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. In [D], Deodhar discovered that Enright's completion functor (see [E1, E2]) is actually a subfunctor of an α-localization functor. This latter functor is obtained by noncommutative localization with respect to the left denominator set $S_\alpha = \{y_\alpha^n | n \in N\}$ where y_α is a nilpotent element in $\mathfrak{n}_-$. Localization with respect to this set "intertwines" with tensoring by finite-dimensional $\mathfrak{g}$-modules. Below we find new examples of noncommutative localization which also intertwine with tensoring by finite-dimensional representations.

2. PRELIMINARIES AND NOTATION

2.1. Recall that an *additive category* $\mathfrak{A}$ is a category satisfying the following three axioms:

(i) $\mathfrak{A}$ has a zero object;

(ii) any two objects in $\mathfrak{A}$ have a product; and

(iii) for all objects $A, B \in \mathrm{Ob}\,\mathfrak{A}$ the set of morphisms $\mathrm{Hom}_{\mathfrak{A}}(A, B)$ forms an abelian group such that the composition

$$\mathrm{Hom}_{\mathfrak{A}}(A, B) \times \mathrm{Hom}_{\mathfrak{A}}(B, C) \to \mathrm{Hom}_{\mathfrak{A}}(A, C)$$

is bilinear. One also has the following proposition.

Proposition [HS, Chapter 2]. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two additive categories and $F: \mathfrak{A} \to \mathfrak{B}$ a functor. Then the following are equivalent*:

(i) *F preserves sums* (*of two objects*).

(ii) *F preserves products* (*of two objects*).

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(iii) *For each* $A, A' \in \operatorname{Ob}\mathfrak{A}$ *one has that*

$$F\colon \operatorname{Hom}_{\mathfrak{A}}(A, A') \to \operatorname{Hom}_{\mathfrak{B}}(FA, FA')$$

is a group homomorphism.

A functor satisfying the above equivalent conditions is called an *additive* functor.

For a vector space V over a field k, let $T^n(V)$ denote the n-fold tensor product of V with itself, and let $T^0 = k$. Then $T(V) := \bigoplus_{n=0}^{\infty} T^n(V)$ is the *tensor algebra* of V. Elements in $T^n(V)$ are called *homogeneous of degree* n. If $\mathfrak{g}$ is a Lie algebra, let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$.

3. $\mathfrak{F}$-Categories and $\mathfrak{F}$-functors

3.1. For any Lie algebra $\mathfrak{g}$ (possibly infinite dimensional) defined over a field k of characteristic zero, let $M_{\mathfrak{g}}$ denote the category of all $\mathfrak{g}$-modules. Throughout we will assume that $\mathfrak{F}$ denotes an additive subcategory of $M_{\mathfrak{g}}$ satisfying the following two conditions:

(1) $\mathfrak{F}$ is closed under tensoring; i.e., if $E_j, F_j \in \mathfrak{F}$ and $f_j \in \operatorname{Hom}_{\mathfrak{F}}(E_j, F_j)$ for $j = 1, 2$, then $E_1 \otimes E_2$, $F_1 \otimes F_2 \in \operatorname{Ob}\mathfrak{F}$ and $f_1 \otimes f_2 \in \operatorname{Hom}_{\mathfrak{F}}(E_1 \otimes E_2, F_1 \otimes F_2)$.

(2) $\mathfrak{g} \in \operatorname{Ob}\mathfrak{F}$ as a $\mathfrak{g}$-module under the adjoint action.

For $F \in \operatorname{Ob}\mathfrak{F}$, $\mathfrak{g}$-modules A and B, and $h \in \operatorname{Hom}_{\mathfrak{g}}(A, B)$, let T_F denote the tensor product functor on $M_{\mathfrak{g}}$ given by $A \mapsto F \otimes A$ and $h \mapsto 1_F \otimes h$. If $\mathfrak{n} \subset \mathfrak{g}$ is a Lie subalgebra and $\mathfrak{C}$ is a subcategory of $M_{\mathfrak{n}}$, we shall use the symbol T_F to denote the tensor product functor on $\mathfrak{C}$ when no confusion is likely to arise. We call the category $\mathfrak{C}$ an *$\mathfrak{F}$-category* if it is additive and T_F carries $\mathfrak{C}$ into itself for all $F \in \operatorname{Ob}\mathfrak{F}$.

Now let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie subalgebras of $\mathfrak{g}$, and let $\mathfrak{A}$ (resp. $\mathfrak{B}$) be an additive subcategory of $M_{\mathfrak{a}}$ (resp. $M_{\mathfrak{b}}$). Suppose further that both $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathfrak{F}$-categories and τ is a functor from $\mathfrak{A}$ to $\mathfrak{B}$. We call τ an *intertwining functor* (or *$\mathfrak{F}$-intertwining functor* when more precision is necessary) if τ is additive and there exists a natural equivalence, for each $F \in \operatorname{Ob}\mathfrak{F}$, $i_F\colon T_F \circ \tau \to \tau \circ T_F$.

Suppose τ is an intertwining functor, and let $\mathcal{I} = \{i_F \mid F \in \operatorname{Ob}\mathcal{F}\}$ denote the family of natural equivalences above. Recall that a natural equivalence $i_E\colon T_E \circ \tau \to \tau \circ T_E$ is a rule that assigns to each object A of $\mathfrak{A}$ an isomorphism $i_E(A)\colon T_E \circ \tau(A) \to \tau \circ T_E(A)$ such that for every homomorphism $f\colon A \to B$ in $\mathfrak{A}$ one has $i_E(B) \circ ((T_E \circ \tau)(f)) = (\tau \circ T_E)(f) \circ i_E(A)$. For convenience we set $i_{E,A} = i_E(A)$ for all $A \in \operatorname{Ob}\mathfrak{A}$, $E \in \operatorname{Ob}\mathfrak{F}$.

Suppose now that for every $E, F \in \operatorname{Ob}\mathfrak{F}$ and $h \in \operatorname{Hom}_{\mathfrak{F}}(E, F)$ one has $h \otimes 1_A \in \operatorname{Hom}_{\mathfrak{A}}(E \otimes A, F \otimes A)$ for $A \in \operatorname{Ob}\mathfrak{A}$. Assume the category $\mathfrak{B}$ has this same property. Then we say that $\mathcal{I}$ is *natural in the $\mathfrak{F}$-variable* (or *natural in* $\mathfrak{F}$) if the following diagram is commutative for all $E, F \in \operatorname{Ob}\mathfrak{F}$, $A \in \operatorname{Ob}\mathfrak{A}$, and $f \in \operatorname{Hom}_{\mathfrak{F}}(E, F)$:

$$\begin{tikzcd} E \otimes \tau A \arrow[r, "i_{E,A}"] \arrow[d, "f\otimes 1_{\tau A}"'] & \tau(E \otimes A) \arrow[d, "\tau(f\otimes 1_A)"] \\ F \otimes \tau A \arrow[r, "i_{F,A}"] & \tau(F \otimes A) \end{tikzcd} \tag{1}$$

We call $\mathscr{I}$ *distributive* if the following diagram is cummutative for $E, F \in \operatorname{Ob}\mathfrak{F}$ and $A \in \operatorname{Ob}\mathfrak{A}$:

$$(2)\qquad \begin{array}{ccc} (E\oplus F)\otimes \tau A & \xrightarrow{i_{E\oplus F,A}} & \tau((E\oplus F)\otimes A) \\ \downarrow & & \downarrow \\ (E\otimes \tau A)\oplus(F\otimes\tau A) & \xrightarrow{i_{E,A}\oplus i_{F,A}} & \tau(E\otimes A)\oplus\tau(F\otimes A) \end{array}$$

The left map (2) expresses the bilinearity of $\otimes$, and the right map expresses this bilinearity combined with additivity of τ.

We say that $\mathscr{I}$ is *associative* if the following diagram is commutative for all $E, F \in \operatorname{Ob}\mathfrak{F}$ and $A \in \operatorname{Ob}\mathfrak{A}$:

$$(3)\qquad \begin{tikzcd} E\otimes F\otimes \tau A \arrow[r, "1\otimes i_{F,A}"] \arrow[dr, "i_{E\otimes F,A}"'] & E\otimes\tau(F\otimes A) \arrow[d, "i_{E,F\otimes A}"] \\ & \tau(E\otimes F\otimes A) \end{tikzcd}$$

3.2 **Lemma** [C]. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathfrak{F}$-categories and $\tau\colon \mathfrak{A}\to\mathfrak{B}$ is an intertwining functor. If the family $\mathscr{I} = \{i_{F,A} | F \in \operatorname{Ob}\mathfrak{F},\ A \in \operatorname{Ob}\mathfrak{A}\}$ is natural in the $\mathfrak{F}$-variable then $\mathscr{I}$ is distributive.*

Suppose τ is an intertwining functor with the family of natural equivalences $\mathscr{I} = \{i_F | F \in \operatorname{Ob}\mathfrak{F})$. We call the pair $(\tau, \mathscr{I})$ an $\mathfrak{F}$-*functor* whenever $\mathscr{I}$ is both distributive and associative. When $\mathscr{I}$ is understood to be fixed we call τ an $\mathfrak{F}$-*functor.* (See [C] and the references listed there for several examples of $\mathfrak{F}$-functors in the representation theory of Lie algebras.)

4. Deodhar's α-localization functor and new examples of $\mathfrak{F}$-functors

4.1. Let $\mathfrak{g}$ be a Kac-Moody algebra, and let $\mathfrak{F}$ be the category of integrable $\mathfrak{g}$-modules. In this section we review Deodhar's α-localization functor D_α for α a real root of $\mathfrak{g}$. Our main result, Theorem 4.18, is a substantial generalization of Deodhar's results on localization and will provide us with new examples of $\mathfrak{F}$-functors. Let us begin by reviewing some noncommutative ring theory (our general reference will by [GW]).

4.2. All rings in this article are assumed to have an identity. If X is a multiplicative subset then a *left ring of fractions for R with respect to X* is a ring homomorphism $\phi\colon R\to S$ such that

(a) $\phi(x)$ is a unit in S for all $x \in X$,
(b) each element of S can be written in the form $\phi(x)^{-1}\phi(r)$ for some $x\in X$ and $r \in R$, and
(c) $\ker(\phi) = \{r\in R | xr = 0$ for some $x\in X\}$.

A multiplicative set X in a ring R that satisfies the following two conditions is called a *left denominator set*:

(Da) $Xr\cap Rx \neq \varnothing$ for all $r\in R$ and $x\in X$;
(Db) if $r\in R$ and $x\in X$ are such that $xr = 0$ then there exists $x'\in X$ such that $rx' = 0$.

A well-known result due to Goldie is

4.3. **Theorem** [GW, Proposition 9.7]. *Let X be a multiplicative set in a ring R. Then there exists a left ring of fractions for R with respect to X if and only if X is a left denominator set.*

Moreover, if a left ring of fractions exists then it is unique up to isomorphism (see [GW, Corollary 9.5]). If X is a left denominator set then we let $X^{-1}R$ denote its unique ring of fractions. This (as is well known) can be constructed as follows: Define on $X \times R$ an equivalence relation $\sim$ where $(x, r) \sim (x', r')$ if there exists $s \in R$ and $y \in X$ such that $yr = sr'$ and $yx = sx'$. The set of equivalence classes of $X \times R$ has an obvious ring structure which gives us $X^{-1}R$. We let $x \setminus r$ denote the equivalence class of (x, r).

4.4. Let X be a left denominator set, and let A be a left R-module. A *module of fractions for A with respect to X* is an R-module homomorphism $f\colon A \to B$ where B is a left $X^{-1}R$-module such that

(a) every element of B can be written in the form $x^{-1}f(a)$ for some $x \in X$ and $a \in A$, and
(b) $\ker f = \{a \in A | ax = 0 \text{ for some } x \in X\}$.

Another basic fact is

Theorem [GW, Corollary 9.11 and Theorem 9.13]. *If X is a left denominator set in a ring R then there exists a unique* (*up to isomorphism*) *module of fractions for any left R-module A with respect to X.*

4.5. When X is a left denominator set and A is a left R-module then we let $X^{-1}A$ denote its unique module of fractions with respect to X. In addition there is the following useful construction of $X^{-1}A$.

4.6. **Theorem** [GW, Proposition 9.14]. *For X a left denominator set in R and A a left R-module we have*

$$X^{-1}R \otimes_R A \cong X^{-1}A$$

where the map is given by $s \otimes a \mapsto sa$ for $a \in A$ and $s \in X^{-1}R$.

One has the following universal mapping property of localizations.

Proposition [GW, Proposition 9.10]. *Let X be a left denominator set in a ring R, let A be a left R-module, and suppose C is a left $X^{-1}R$-module with $g\colon A \to C$ an R-module homomorphism. If $f\colon A \to X^{-1}A$ is the module of fractions for A with respect to X, then there exists a unique $X^{-1}R$-module homomorphism $h\colon X^{-1}A \to C$ such that $g = h \circ f$.*

4.7. Next we define Deodhar's α-localization functor and record the well-known fact that it is just "localization" with respect to the multiplicative set $S_\alpha = \{y_\alpha^n | n \in \mathbb{N}\}$ where $y_\alpha \in \mathfrak{g}_{-\alpha}$ is nonzero and α is a positive root.

Let $\mathscr{A}$ denote the category of $\mathfrak{h}$-semisimple $U(\mathfrak{n}_-)$-torsionfree $\mathfrak{g}$-modules. If $A \in \operatorname{Ob}\mathscr{A}$, let $A' = \{y_\alpha^{-n} | n \in \mathbb{N}\} \times A$, and define an equivalence relation on A' by $(y_\alpha^{-n}, a) \sim (y_\alpha^{-m}, a')$ if and only if $y_\alpha^n a' = y_\alpha^m a$. Set $D_\alpha(A) = A'/\sim$. $D_\alpha(A)$ is given a $\mathfrak{g}$-module structure as follows.

4.8. **Lemma** [D, Lemma 2.1]. *If $z \in \mathfrak{g}$ and $0 \leq r \in \mathbb{Z}$ then there exists $0 \leq s \in \mathbb{Z}$ such that $y_\alpha^s z = u y_\alpha^r$ for some $u \in U(\mathfrak{g})$.*

Remark. Deodhar's proof is based on a straightforward $\mathrm{sl}(2, \mathbb{C})$ computation, and from the proof it is easy to see that this lemma is also true for $z \in U(\mathfrak{g})$

so that S_α is a left Ore set, i.e., it satisfies (Da). It then follows from a result of Goldie (see [GW, Proposition 9.9]) that S_α is a left denominator set.

Now if $(y_\alpha^{-r}, a) \in D_\alpha(A)$ and $z \in \mathfrak{g}$ then $z(y_\alpha^{-r}, a)$ is defined to be (y_α^{-s}, ua) where u is in the lemma above.

4.9. **Proposition** [D, Proposition 2.2]. *Under the action of* $\mathfrak{g}$ *on* D_α *given above,* $D_\alpha(A)$ *becomes a well-defined* $\mathfrak{g}$*-module.*

Remark. By definition D_α is Deodhar's α-localization functor.

Next for $A \in \mathscr{A}$, $A \hookrightarrow D_\alpha(A)$ as a $\mathfrak{g}$-module so that by the universal mapping property of the module of fractions there exists a unique $S_\alpha^{-1}U(\mathfrak{g})$-module map $S_\alpha^{-1}A \to D_\alpha A$. In fact this map is given by $y_\alpha^{-n}a \mapsto (y_\alpha^{-n}, a)$. This map is clearly a bijection. Hence $S_\alpha^{-1}|_{\mathscr{A}} \cong D_\alpha$.

4.10. We now turn to our generalization of these results of Deodhar. Let $X_0 \subset \mathfrak{g}\oplus\mathbb{C} = U(\mathfrak{g})_1$, and let $X \subset U(\mathfrak{g})$ be a left denominator set generated by 1 and X_0 with the following property: for $x \in X_0$ and E any finite-dimensional $\mathfrak{g}$-module

(1) if λ is an eigenvalue of the action of x on E then $x + \lambda \cdot 1 \in X$.

We shall suppose throughout this section that X satisfies (1).

For convenience we set $R = U(\mathfrak{g})$ and $S = X^{-1}R$.

4.11. **Lemma.** *Let* $x \in X$. *Then* x *acts by an isomorphism on* $S \otimes E$.

Proof. It is sufficient to check the lemma for $x \in X_0$. Introduce an x-stable filtration $0 = E_0 \subset \cdots \subset E_d$ on E with x acting by $\lambda_i.1$ on E_i/E_{i-1}. Then suppose x acts by isomorphism on $S\otimes E_{i-1}$. Thus for $\{e_k\}$ a basis of E with $e_{i_j} \in E_i$

$$\begin{aligned} x(s \otimes e_{i_j}) &\equiv xs \otimes e_{i_j} + s \otimes \lambda_i e_{i_j} \bmod S \otimes E_{i-1} \\ &\equiv (x + \lambda_i)s \otimes e_{i_j} \bmod S \otimes E_{i-1}. \end{aligned}$$

This proves the lemma.

S is an R-bimodule. We shall need other bimodules and so introduce the notation $E \otimes \mathbb{C}$ to denote the R-bimodule with left action given on E and trivial right action. Similarly, let E^σ be the right R-module defined by $r(x)\cdot e = \sigma(x) \cdot e$, $e \in E$, $x \in R$, and σ is the involutive antiautomorphism of $U(\mathfrak{g})$ equal to -1 on $\mathfrak{g}$. Then $\mathbb{C}\otimes E^\sigma$ is an R-bimodule with trivial left action and right action given on E^σ

4.12. **Lemma.** *As* S*-bimodules we have an isomorphism* $S \otimes (E \otimes \mathbb{C}) \cong S \otimes (\mathbb{C} \otimes E^\sigma)$.

Proof. Consider the map $\underline{\phi}$: $(R\otimes(E\otimes\mathbb{C}) \to R\otimes(\mathbb{C}\otimes E^\sigma)$ given by $x\otimes e\otimes 1 \mapsto (1\otimes 1\otimes e)\cdot\sigma(x)$. An easy induction argument using the filtration on $U(\mathfrak{g})$ shows that $\underline{\phi}$ is surjective even at the filtered left; i.e., $\underline{\phi}$: $R_i\otimes(E\otimes\mathbb{C}) \to R_i\otimes(\mathbb{C}\otimes E^\sigma)$ is surjective.

To prove injectivity suppose $0 = \sum_i(1 \otimes 1 \otimes e_i) \cdot \sigma(x_i)$ where the e_i are as in the proof of Lemma 4.11 and let d be the maximal integer such that some $x_j \in R_d$ but $x_j \notin R_{d-1}$. Then $0 \equiv \sum_i \sigma(x_i) \otimes 1 \otimes e_i \mod R_{d-1} \otimes (\mathbb{C} \otimes E^\sigma)$. However, the e_i are linearly independent, so each $\sigma(x_i) \in R_{d-1}$. We clearly have a contradiction. This proves injectivity; thus $\underline{\phi}$ is an isomorphism.

Now we extend ϕ to a map $\phi\colon S\otimes(E\otimes\mathbb{C})\to S\otimes(\mathbb{C}\otimes E^\sigma)$. We must define $\phi(x\setminus r\otimes e\otimes 1)$. Recall that x acts on the right of $S\otimes(\mathbb{C}\otimes E^\sigma)$ by an isomorphism $i(x)$. Set

$$\phi(x\setminus r\otimes e\otimes 1)=(1\otimes 1\otimes e)i(x)^{-1}a(r)=(1\otimes 1\otimes e)a(x\setminus r)$$

where $a(-)$ denotes the right action of R on $S\otimes(\mathbb{C}\otimes E^\sigma)$. To see that ϕ is well defined suppose $x\setminus r=x'\setminus r'$, i.e., there exists $y\in X$ and $s\in R$ such that $yr=sr'$ and $yx=sx'$. Then we have $a(yr)=a(sr')$ and $a(yx)=a(sx')$; thus, ϕ is well defined.

Next we check that ϕ is an R-bimodule isomorphism. Suppose $x\in\mathfrak{g}$, and let $l(x)$ $(r(x))$ denote the left (resp. right) action. Then for $y\in X$, $z\in R$, $e\in E$ we have

$$\begin{aligned}l(x)\phi(y\setminus z\otimes e\otimes 1)&=l(x)(1\otimes 1\otimes e)r(y)^{-1}r(z)=(x\otimes 1\otimes e)r(y)^{-1}r(z)\\&=(x\otimes 1\otimes e)r(y)^{-1}r(z)+(1\otimes 1\otimes\sigma(x)\cdot e)r(y)^{-1}r(z)\\&\quad+(1\otimes 1\otimes x\cdot e)r(y)^{-1}r(z)\\&=(1\otimes 1\otimes e)r(x)r(y)^{-1}r(z)+(1\otimes 1\otimes x\cdot e)r(y)^{-1}r(z)\\&=\phi(x(y\setminus z)\otimes e\otimes 1)+\phi(y\setminus z\otimes x\cdot e\otimes 1).\end{aligned}$$

Now for the right action we have

$$\begin{aligned}r(x)\phi(y\setminus z\otimes e\otimes 1)&=(1\otimes 1\otimes e)r(y)^{-1}r(z)r(x)\\&=\phi(y\setminus zx\otimes e\otimes 1)=\phi(y\setminus z\otimes e\otimes 1)r(x).\end{aligned}$$

Finally we check that ϕ is an S-bimodule map. Suppose $x\in X$; then from above $l(x)\circ\phi=\phi\circ l(x)$. By Lemma 4.11 we can invert these left actions. Multiplying out we have $\phi\circ l(x)^{-1}=l(x)^{-1}\circ\phi$; so ϕ intertwines with the left action of S. Similarly for the right action. This proves the lemma.

4.13. **Lemma.** *Let F be any left R-module. As left S-modules we have the isomorphism*

$$S\otimes_R(E\otimes F)\cong S\otimes(\mathbb{C}\otimes E^\sigma)\otimes_R F.$$

Proof. The left tensor product above is the quotient of $S\otimes E\otimes F$ by the relations determined by a $s\cdot x\otimes e\otimes f=s\otimes x\cdot(e\otimes f)$ where $s\in S$, $e\in E$, $f\in F$, and $x\in\mathfrak{g}$. However, this is precisely the same set of relations as $(s\otimes e')x\otimes f=s\otimes e'\otimes x\cdot f$ for $e'\in E^\sigma$. This proves the lemma.

In the next lemma we introduce another multiplicative subset of $U(\mathfrak{g})$. We will see later (4.18) that this gives us another example of an $\mathfrak{F}$-functor. First we define an $\mathfrak{h}$*-graded multiplicative subset* of $U(\mathfrak{g})$ to be a multiplicative subset Y of $U(\mathfrak{g})$ such that $Y=p\bigcup_{\eta\in\mathfrak{h}^*}Y_\eta$ and $Y_\eta\cdot Y_\beta\subset Y_{\eta+\beta}$ where $Y_\eta:=\{y\in Y|[h,y]=\eta(h)y$ for all $h\in\mathfrak{h}^*\}$.

4.14. **Lemma.** *Let Y be an $\mathfrak{h}$-graded multiplicative subset of $U(\mathfrak{n}_-)$, W an $\mathfrak{h}$-semisimple Y-divisible $\mathfrak{g}$-module, and E a finite-dimensional $\mathfrak{g}$-module. Then $E\otimes W$ is an $\mathfrak{h}$-semisimple Y-divisible $\mathfrak{g}$-module.*

Proof. The fact that $E\otimes W$ is $\mathfrak{h}$-semisimple is obvious. Thus we need only show that $x(E\otimes W)=E\otimes W$ for all $x\in Y$. We may assume x is not an element in $Y_0=\{y\in Y|h.y=0$ for all $h\in\mathfrak{h}\}$. The result will follow if we can show that, for arbitrary $e\in E_\gamma$ and $w\in W_\lambda$, $e\otimes w$ is in $x(E\otimes W)$. For $\lambda\in\mathfrak{h}^*$

let E_λ denote the λth weight space of E. Since E is finite dimensional, the set $\Lambda(E) = \{\lambda \in \mathfrak{h}^* | E_\lambda \neq 0\}$ is finite; hence there exists a minimal $\mu \in \Lambda(E)$. If $\gamma \in \Lambda(E)$ is minimal then $xe = 0$ for $e \in E_\gamma$. Now as W is Y-divisible there exists for any $w \in W$ an $w' \in W$ such that $xw' = w$ and $x(e \otimes w') = e \otimes w$. Thus we may assume as an induction hypothesis that γ is not minimal, and the previous statement is true for all $e \in E_\mu$, $w \in W$ with $\mu < \gamma$. Now if $e' \in E_\gamma$ and $w \in W$ then

$$x(e' \otimes w') = e' \otimes w + \sum_i e'_i \otimes w'_i$$

where $e'_i \in E_{\mu_i}$ with $\mu_i < \gamma$ and $xw' = w$. By induction $e'_i \otimes w'_i \in x(E \otimes W)$; thus $e' \otimes w \in x(E \otimes W)$. This completes the proof of the lemma.

4.15. **Lemma.** *Let Y be an $\mathfrak{h}$-graded multiplicative subset of $U(\mathfrak{n}_-)$, W an $\mathfrak{h}$-semisimple Y-torsionfree $\mathfrak{g}$-module, and E a finite-dimensional $\mathfrak{g}$-module. Then $E \otimes W$ is an $\mathfrak{h}$-semisimple Y-torsionfree $\mathfrak{g}$-module.*

Proof. Let $\sum_i e_{\mu_i} \otimes w_{\lambda_i} \neq 0$ with $e_{\mu_i} \in E_{\mu_i}$, $w_{\lambda_i} \in W_{\lambda_i}$ linearly independent and $x(\sum_i e_{\mu_i} \otimes w_{\lambda_i}) = 0$ for some $x \in Y_\eta$ where $\eta \neq 0$. Lexicographically order the set $\Lambda(E) \times \Lambda(W)$ (notation as in the previous lemma), and choose (μ_j, λ_j) maximal with respect to this ordering and such that $e_{\mu_i} \otimes w_{\lambda_i} \neq 0$. Then $0 = \sum_i e_{\mu_i} \otimes xw_{\lambda_i} + \sum_i e_{\mu'_i} \otimes w_{\lambda'_i}$ where $\mu'_k < \mu_j$ for all k. But then $e_{\mu_j} \otimes xw_{\lambda_j} = 0$; thus $xw_{\lambda_j} = 0$. Since W is Y-torsionfree, $x = 0$. Hence $E \otimes W$ is Y-torsionfree.

4.16. **Lemma.** *If E and W are $\mathfrak{g}$-modules with E finite dimensional then*

$$\operatorname{Hom}(E, W) \cong E^* \otimes W.$$

4.17. **Theorem** (Mackey Isomorphism Theorem). *Suppose V and E are $\mathfrak{g}$-modules with E finite dimensional and V $\mathfrak{h}$-semisimple. Let $Y \subseteq U(\mathfrak{n}_-)$ be an $\mathfrak{h}$-graded left denominator set. Then we have a natural isomorphism*

$$Y^{-1}(V \otimes E) \cong (Y^{-1}V) \otimes E$$

of $\mathfrak{g}$-modules.

Proof. For the proof of this theorem set $R = U(\mathfrak{n}_-)$, and let W be an $\mathfrak{h}$-semisimple $Y^{-1}R$-module. Now consider the following sequence of isomorphisms:

$$\begin{aligned}
&\operatorname{Hom}_{Y^{-1}R}(Y^{-1}(E \otimes V), W) \\
&\quad \cong \operatorname{Hom}_R(E \otimes V, W) \cong \operatorname{Hom}_R(V, \operatorname{Hom}(E, W)) \\
&\quad \cong \operatorname{Hom}_{Y^{-1}R}(Y^{-1}V, \operatorname{Hom}(E, W)) \cong \operatorname{Hom}_{Y^{-1}R}(E \otimes Y^{-1}V, W).
\end{aligned}$$

The first isomorphism is just given by Proposition 4.6. The second and fourth isomorphisms are derived from the adjoint associativity property of Hom. Now an R-module is a $Y^{-1}R$-module if and only if it is Y-divisible and Y-torsionfree (see [GW, Proposition 9.12]). Thus Lemmas 4.14 through 4.16 imply that $\operatorname{Hom}(E, W)$ is a Y^{-1}-module so that we can apply Proposition 4.6 to obtain the third isomorphism. We now proceed as in [Kn, Proposition 5.14] and let $W = Y^{-1}(E \otimes V)$. The identity in $\operatorname{Hom}_{Y^{-1}R}(Y^{-1}(E \otimes V), Y^{-1}(E \otimes V))$ induces a map in $\operatorname{Hom}_{Y^{-1}R}(E \otimes Y^{-1}V, Y^{-1}(E \otimes Y^{-1}V))$ which we will denote by φ. Similarly the identity morphism in $\operatorname{Hom}_{Y^{-1}R}(E \otimes Y^{-1}V, E \otimes Y^{-1}V)$ induces a map $\psi \in \operatorname{Hom}_{Y^{-1}R}(Y^{-1}(E \otimes V), E \otimes Y^{-1}V)$. One can check as in

[Kn, Proposition 5.14] that $\psi \circ \varphi = 1$ and $\varphi \circ \psi = 1$. This completes the proof of the theorem.

We now consider localization as a functor on the category $M_{\mathfrak{g}}$ of all $\mathfrak{g}$-modules. Let $\mathfrak{F}$ denote the subcategory of all finite-dimensional $\mathfrak{g}$-modules. Set $\tau_X M = X^{-1}M$ and $\tau_Y M = Y^{-1}M$ where X satisfies 4.10.1, and $Y \subseteq U(\mathfrak{n}_-)$ is an $\mathfrak{h}$-graded left denominator set. The main result of the article is

4.18. **Theorem.** *τ_X and τ_Y are intertwining $\mathfrak{F}$-functors that are natural in $\mathfrak{F}$.*

Proof. Theorem 4.17 proves the theorem for Y. For X we let $E \in \operatorname{Ob}\mathfrak{F}$, $M \in M_{\mathfrak{g}}$. Then

$$\begin{aligned}\tau_X(E \otimes M) &= S \otimes_R (E \otimes M) \cong S \otimes (\mathbb{C} \otimes E^\sigma) \otimes_R M \quad \text{(by Lemma 4.13)}\\ &\cong S \otimes (E \otimes \mathbb{C}) \otimes_R M \cong E \otimes \tau_X M \quad (\text{by } \phi^{-1} \text{ and Lemma 4.12}).\end{aligned}$$

The first isomorphism is induced by the identity map, so we shall identify these spaces. For the second isomorphism we have ϕ^{-1} where ϕ is given by $\phi = \phi_E$ where $\phi_E(x \setminus r \otimes e \otimes 1) = (1 \otimes 1 \otimes e)a(x)^{-1}a(r)$.

Now suppose $\gamma \in \operatorname{Hom}_{\mathfrak{g}}(E, F)$; $E, F \in \operatorname{Ob}\mathfrak{F}$. Since $1 \otimes \gamma$ intertwines the (right) action of $\mathfrak{g}$ from $S \otimes E^\sigma$ to $S \otimes F^\sigma$, $1 \otimes \gamma$ intertwines that of $a(x)^{-1}$ as well. This implies that the family of equivalences $\{\phi_E | E \in \operatorname{Ob}\mathfrak{F}\}$ is natural in $\mathfrak{F}$. This proves the theorem.

Remark. For the case $X = S_\alpha$ the result above is due to Deodhar (see [D, Theorem 3.1]) where again the proof is based on an $\mathrm{sl}(2, \mathbb{C})$ calculation.

4.19. **Corollary.** *τ_X and τ_Y are $\mathfrak{F}$-functors.*

Proof. By Theorem 4.18 and Lemma 3.2 we need only show that $\tau = \tau_X$ and $\tau = \tau_Y$ are associative. First consider $\tau = \tau_X$. Suppose $E, F \in \operatorname{Ob}\mathfrak{F}$, $M \in \operatorname{Ob} M_{\mathfrak{g}}$. Since the maps involved are all left S-module maps, we need only verify the identity on a set of S-generators for the space. Clearly $\mathbb{C}\otimes\mathbb{C}\otimes E^\sigma \otimes F^\sigma \otimes_R M$ is a set of generators; so $\mathbb{C} \otimes E \otimes F \otimes \mathbb{C} \otimes_R M$ is a set of S-generators for $S \otimes E \otimes F \otimes \mathbb{C} \otimes_R M$. But $\phi_{E\otimes F}$ and also $\phi_E \circ (1 \otimes \phi_F)$ essentially equal the identity map on these generators;

$$\begin{aligned}\phi_{E\otimes F}(1 \otimes e \otimes f \otimes 1 \otimes m) &= 1 \otimes 1 \otimes e \otimes f \otimes m\\ &= \phi_E \circ (1 \otimes \phi_F)(1 \otimes e \otimes f \otimes 1 \otimes m).\end{aligned}$$

If $\tau = \tau_Y$, one uses the identification $Y^{-1}R \otimes_R A \cong Y^{-1}A$ of Theorem 4.6 and observes that the isomorphism of 4.17 is given by $1 \otimes v \otimes e \mapsto 1 \otimes v \otimes e$ on a set of S generators of $Y^{-1}(V \otimes E)$. Now one checks as before that the appropriate identity is satisfied on the set $\mathbb{C} \otimes E \otimes F \otimes A$ of S-generators of $\tau_Y(E \otimes F \otimes A)$.

4.20. **Examples.** Here we describe some multiplicative sets X other than Deodhar's S_α, for which the hypothesis of 4.18 is satisfied. Set $\mathbb{Z}_{\geq 0} = \{k \in \mathbb{Z} | k \geq 0\}$.

(a) If A is a $\mathbb{C}$-algebra and $x \in A$ then let $d_x \colon A \to A$ denote the $\mathbb{C}$-algebra map given by $d_x(y) = xy - yx$ for all $y \in A$. For this first example we need

Theorem [BR, Satz 2.2]. *Let A be a prime Noetherian $\mathbb{C}$-algebra. Suppose $x \in A$ is an element such that d_x is locally nilpotent on A. If x is not* *nilpotent, then* x *is a nonzero divisor in* A *and the localization* $S^{-1}A$ *exists where* $S=\{x^n|n\in\mathbb{Z}_{\geq 0}\}$.

First let $\mathfrak{g}=\mathrm{sl}(3,\mathbb{C})$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $R\subset\mathfrak{h}^*$ the set of nonzero roots, $R^+\subset R$ a set of positive roots, and $B=\{\alpha,\beta\}\subset R^+$ a basis for R. Fix a Chevalley basis $\{x_\gamma, h_\alpha, h_\beta|\gamma\in R\}$ of $\mathfrak{g}$ where x_γ denotes the element in this basis with weight $\gamma\in R$. Let $\mathfrak{n}_\pm=\bigoplus_{\alpha\in R_\pm}\mathbb{C}x_\alpha$. For $\gamma\in R_+$ set $y_\gamma=x_{-\gamma}$.

Fix $\varepsilon\in\mathbb{C}\setminus\{0,1\}$, and define $S_\varepsilon=\{(y_\alpha y_\beta-\varepsilon[y_\alpha, y_\beta])^n|n\in\mathbb{Z}_{\geq 0}\}$. By the Theorem above S_ε is a multiplicative left denominator subset of $U(\mathfrak{g})$. It is also clear that S_ε is $\mathfrak{h}$-graded. Consequently S_ε satisfies the hypotheses of 4.18. Our goal is to show that localization with respect to S_ε is not the same as localization with respect to the denominator set $S_\gamma^\mu:=\{(\mu y_\gamma)^n|n\in\mathbb{Z}_{\geq 0}\}$ for any $\gamma\in R$ and $\mu\in\mathbb{C}\setminus\{0\}$.

We first consider the case that $\gamma=\alpha$. Suppose now that localization with respect to S_α^μ is the same as localization with respect to S_ε. Then y_α is invertible in $S_\varepsilon^{-1}U(\mathfrak{g})$; thus, $(s\setminus m)y_\alpha=1$ for some $s\in S_\varepsilon$ and $m\in U(\mathfrak{g})$. Consequently there exists $a\in U(\mathfrak{g})$ and $b\in S_\varepsilon$ with $as=b\in S_\varepsilon$ and $amy_\alpha=b$. This implies that $my_\alpha=s=(y_\alpha y_\beta-\varepsilon[y_\alpha y_\beta])^n$ for some $n\geq 0$ ($a\neq 0$ since otherwise $0=as=b\in S_\varepsilon$).

Define an ordering of the Chevalley basis by

$$x_\alpha<x_\beta<x_{\alpha+\beta}<h_\alpha<h_\beta<y_{\alpha+\beta}<y_\beta<y_\alpha.$$

Then the Poincaré-Birkhoff-Witt Theorem tells us that $U(\mathfrak{g})$ has a basis of mononomials $\overline{x}\overline{h}\overline{y}$ where

$$\overline{x}=x_\alpha^{n_1}x_\beta^{n_2}x_{\alpha+\beta}^{n_3},\qquad \overline{h}=h_\alpha^{n_4}h_\beta^{n_5},\qquad \overline{y}=y_{\alpha+\beta}^{n_6}y_\beta^{n_7}y_\alpha^{n_8},$$

and $n_i\in\mathbb{Z}_{\geq 0}$. Consequently $m=\sum u_{\overline{n}}\overline{x}\overline{h}\overline{y}$ where $u_{\overline{n}}\in\mathbb{C}$ and $\overline{n}=(n_1,\dots,n_8)\in\mathbb{Z}_{\geq 0}^8$.

Thus

$$my_\alpha=\sum u_{\overline{n}}\overline{x}\overline{h}\overline{y}y_\alpha=(y_\alpha y_\beta-\varepsilon[y_\alpha, y_\beta])^n\in U(\mathfrak{n}_-)\mathfrak{n}_-.$$

By the Poincaré-Birkhoff-Witt Theorem we have that $m\in U(\mathfrak{n}_-)\mathfrak{n}_-$. A straightforward calculation shows that

$$my_\alpha=\sum u_{\overline{n}}\overline{x}\overline{h}\overline{y}y_\alpha=(1-\varepsilon)^n y_{\alpha+\beta}^n+p(y_\alpha, y_\beta, y_{\alpha+\beta})y_\alpha$$

for some polynomial p. Thus

$$(m-p(y_\alpha, y_\beta, y_{\alpha+\beta}))y_\alpha=(1-\varepsilon)^n y_{\alpha+\beta}^n.$$

If $\varepsilon\neq 1$, this is impossible by the Poincaré-Birkhoff-Witt Theorem. This proves that localization with respect to S_α^μ is not the same as localization with respect to S_ε if $\varepsilon\neq 1$. A very similar argument (but with a different ordering on the basis) shows that localization with respect to S_γ^μ where $\gamma\in\{\pm\alpha,\pm\beta,\pm(\alpha+\beta)\}$ is not the same as localization with respect to S_ε if $\varepsilon\neq 0, 1$.

(b) Let $\mathfrak{g}=\mathrm{sl}(2,\mathbb{C})$ with $\{x,h,y\}$ a Chevalley basis of $\mathfrak{g}$ such that $[h,x]=2x$, $[x,y]=h$, and $[h,y]=-2y$. Let $X_0=\{h-n1|n\in\mathbb{Z}\}$ and $X=\{(h-n1)^k|n,k\in\mathbb{Z},\ k\geq 0\}$ be subsets of $U(\mathfrak{g})$. This is Example 1.8 in [BR] of an Ore subset of $U(\mathfrak{g})$. Now it is straightforward to check that X satisfies condition 4.10(1). Thus we only need to see that localization with respect to X is not the same as localization with respect to $S_\alpha=\{x^n|n\in\mathbb{Z}_{\geq 0}\}$ or

$S_{-\alpha} = \{y^n | n \in \mathbb{Z}_{\geq 0}\}$. We will prove the case $S_\alpha = \{x^n | n \in \mathbb{Z}_{\geq 0}\}$ and leave the other case to the reader. As in the previous example we assume the contrapositive so that there exists $m \in U(\mathrm{sl}(2, \mathbb{C}))$, $n_1, \dots, n_t \in \mathbb{Z}$, and $k_1, \dots, k_t \in \mathbb{Z}_{\geq 0}$ such that

$$my = (h - n_1)^{k_1} \cdots (h - n_t)^{k_t}.$$

Using the Poincaré-Birkhoff-Witt Theorem we see that this is impossible. Thus, localization with respect to X is not the same as localization with x or y.

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