



F-Categories and F-Functors in the Representation Theory of Lie Algebras

Author(s): Ben Cox

Source: *Transactions of the American Mathematical Society*, Vol. 343, No. 1 (May, 1994), pp. 433-453

Published by: American Mathematical Society

Stable URL: <http://www.jstor.org/stable/2154540>

Accessed: 03/05/2010 10:38

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=ams>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



American Mathematical Society is collaborating with JSTOR to digitize, preserve and extend access to *Transactions of the American Mathematical Society*.

<http://www.jstor.org>

\mathfrak{F} -CATEGORIES AND \mathfrak{F} -FUNCTORS IN THE REPRESENTATION THEORY OF LIE ALGEBRAS

BEN COX

ABSTRACT. The fields of algebra and representation theory contain abundant examples of functors on categories of modules over a ring. These include of course Hom , Ext , and Tor as well as the more specialized examples of completion and localization used in the setting of representation theory of a semisimple Lie algebra. In this article we let \mathfrak{a} be a Lie subalgebra of a Lie algebra \mathfrak{g} and Γ be a functor on some category of \mathfrak{a} -modules. We then consider the following general question: For a \mathfrak{g} -module E what hypotheses on Γ and E are sufficient to insure that $\Gamma(E)$ admits a canonical structure as a \mathfrak{g} -module? The article offers an answer through the introduction of the notion of \mathfrak{F} -categories and \mathfrak{F} -functors. The last section of the article treats various examples of this theory.

1. INTRODUCTION

Suppose \mathfrak{a} is a Lie subalgebra of a Lie algebra \mathfrak{g} and γ is a functor from the category of \mathfrak{a} -modules to itself. Now suppose A is a \mathfrak{g} -module. Then by restriction of the action from \mathfrak{g} to \mathfrak{a} we obtain an \mathfrak{a} -module which we denote by the same symbol A . Within representation theory of Lie algebras the following question has arisen naturally in a number of contexts (see [D, E, Jo1, Jo2, Jo3, Wa, Z]). What hypothesis on γ and A are sufficient to imply that γA admits a \mathfrak{g} -module structure? We offer an answer to this question in the form of the Lifting Theorem (see 4.6).

Let us review the contents of the paper. In §2 we set down the necessary notation and establish a few preliminary results. In §§3 and 5 we recall the definitions of some well-known functors in representation theory. Among some of the more interesting examples are Enright's completion functors and Zuckerman's derived functors of the \mathfrak{k} -finite functor. There is a second group of functors (also discussed in §3) which includes some specialized versions of Tor . An essential property of the functors in the first group is that they commute with those in the second. This property, which is encapsulated in the notion of what we call an \mathfrak{F} -category and an \mathfrak{F} -functor, is the basis for a proof of the Lifting Theorem. The statement and proof of this theorem is then given in §4. Here the reader will also find a related result on the derived functors of \mathfrak{F} -functors and the proof that under some rather mild conditions the left adjoint to a right exact \mathfrak{F} -functor is an \mathfrak{F} -functor. We end the paper with some remarks on how the Lifting Theorem applies to a variety of examples.

Received by the editors May 28, 1991 and, in revised form, November 12, 1992.
1991 *Mathematics Subject Classification*. Primary 17B67.

At this point the author would like to thank his thesis advisor Professor Thomas J. Enright for his patience, encouragement, and guidance. Professor Enright's enthusiasm and deep insight into the representation theory of Lie algebras has been an inspiration to the author.

2. PRELIMINARIES AND NOTATION

2.1. If $G: \mathcal{C} \rightarrow \mathcal{D}$ is a functor then let $\text{im } G$ be the class of objects $G(C)$, $C \in \text{Ob } \mathcal{C}$, together with the collection of morphisms $G(h)$, $h \in \text{Hom}_{\mathcal{C}}(A, B)$ where $A, B \in \text{Ob } \mathcal{C}$. If \mathfrak{B} is a subcategory of \mathcal{D} let $G^{-1}(\mathfrak{B})$ denote the subcategory of \mathcal{C} where $A \in \text{Ob } G^{-1}(\mathfrak{B})$ if $G(A) \in \text{Ob } \mathfrak{B}$ and

$$\text{Hom}_{G^{-1}(\mathfrak{B})}(A, B) := G^{-1}(\text{Hom}_{\mathfrak{B}}(GA, GB)) \subseteq \text{Hom}_{\mathcal{C}}(A, B)$$

for $A, B \in \text{Ob } G^{-1}(\mathfrak{B})$. One can see that $G^{-1}(\mathfrak{B})$ is a subcategory as follows: Composition in $G^{-1}(\mathfrak{B})$ is just given by composition in \mathcal{C} , however one needs to see that the image lies in the appropriate space, i.e.,

$$\text{Hom}_{G^{-1}(\mathfrak{B})}(A, B) \times \text{Hom}_{G^{-1}(\mathfrak{B})}(B, C) \rightarrow \text{Hom}_{G^{-1}(\mathfrak{B})}(A, C)$$

where $(\varphi, \psi) \mapsto \psi \circ \varphi$. This is true as $f \in \text{Hom}_{G^{-1}(\mathfrak{B})}(A, B)$ and $g \in \text{Hom}_{G^{-1}(\mathfrak{B})}(B, C)$ implies that $G(g \circ f) = G(g) \circ G(f) \in \text{Hom}_{\mathfrak{B}}(GA, GC)$. It is clear that composition is associative and that for each $A \in \text{Ob } G^{-1}(\mathfrak{B})$ one has $1_A \in \text{Hom}_{G^{-1}(\mathfrak{B})}(A, A)$. Hence $G^{-1}(\mathfrak{B})$ is a subcategory of \mathcal{C} . If we write $\text{im } G \subseteq \mathfrak{B}$ we mean both $G(C) \in \text{Ob } \mathfrak{B}$ for all $C \in \text{Ob } \mathcal{C}$ and $G(\text{Hom}_{\mathcal{C}}(A, B)) \subseteq \text{Hom}_{\mathfrak{B}}(GA, GB)$ for all $A, B \in \text{Ob } \mathcal{C}$.

2.2. Recall that an *additive category* \mathfrak{A} is a category satisfying the following three axioms: (i) \mathfrak{A} has a zero object, (ii) any two objects in \mathfrak{A} have a product, and (iii) for all objects $A, B \in \text{Ob } \mathfrak{A}$ the set of morphisms $\text{Hom}_{\mathfrak{A}}(A, B)$ forms an abelian group such that the composition $\text{Hom}_{\mathfrak{A}}(A, B) \times \text{Hom}_{\mathfrak{A}}(B, C) \rightarrow \text{Hom}_{\mathfrak{A}}(A, C)$ is bilinear. One also has the following proposition.

2.3. **Proposition** [HS, p. 77]. *Let \mathfrak{A} and \mathfrak{B} be two additive categories and $F: \mathfrak{A} \rightarrow \mathfrak{B}$ a functor. Then the following are equivalent:*

- (i) F preserves sums (of two objects).
- (ii) F preserves products (of two objects).
- (iii) For each $A, A' \in \text{Ob } \mathfrak{A}$ one has that

$$F: \text{Hom}_{\mathfrak{A}}(A, A') \rightarrow \text{Hom}_{\mathfrak{B}}(FA, FA')$$

is a group homomorphism.

A functor satisfying the above equivalent conditions is called an *additive functor*.

2.4. For a vector space V over a field k , let $T^n(V)$ denote the n -fold tensor product of V with itself and let $T^0 = k$. Then $T(V) := \bigoplus_{n=0}^{\infty} T^n(V)$ is the *tensor algebra* of V . Elements in $T^n(V)$ are called *homogeneous of degree n* . If \mathfrak{g} is a Lie algebra let $U(\mathfrak{g})$ denote the universal enveloping algebra of \mathfrak{g} . Furthermore set $T_n(V) := \bigoplus_{0 \leq m \leq n} T^m(V)$ and let $U_n(\mathfrak{g})$ denote the image of $T_n(\mathfrak{g})$ under the canonical projection $T(\mathfrak{g}) \rightarrow U(\mathfrak{g})$.

Let \mathfrak{g} be a Kac-Moody algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (see [Ka, Chapter 1] for definitions and notation) and let $\lambda \in \mathfrak{h}^*$. Let $\mathbb{C}_\lambda = \mathbb{C}v$ denote the one-dimensional $\mathfrak{h} \oplus \mathfrak{n}_+$ -module defined by

$$h, v = \lambda(h)v \quad \text{and} \quad n_+.v = 0.$$

Then the Verma module $M(\lambda)$ of highest weight $\lambda - \rho$ is the \mathfrak{g} -module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda - \rho}$ where $\rho(\alpha) = 1$ for all positive simple roots α .

2.5. Let \mathfrak{a} be an arbitrary Lie algebra over a field k . For V a finite-dimensional trivial \mathfrak{a} -module and P and E arbitrary \mathfrak{a} -modules, let

$$\xi = \xi_{V, P, E}: V^* \otimes \text{Hom}_{\mathfrak{a}}(P, E) \rightarrow \text{Hom}_{\mathfrak{a}}(V \otimes P, E)$$

be given by the canonical homomorphism $\xi(\nu^* \otimes \varphi)(v \otimes p) = \nu^*(v)\varphi(p)$ for all $\nu^* \in V^*$, $v \in V$, $p \in P$, and $\varphi \in \text{Hom}_{\mathfrak{a}}(P, E)$.

Lemma. ξ is an \mathfrak{a} -module isomorphism and is natural in all variables

Proof. First we show that ξ is an \mathfrak{a} -module homomorphism. Let $x \in \mathfrak{a}$. Then

$$\xi(x.(\nu^* \otimes \varphi))(v \otimes p) = (x.\nu^*)(v)\varphi(p) + \nu^*(v)(x.\varphi)(p) = 0$$

as $\varphi \in \text{Hom}_{\mathfrak{a}}(P, E)$ and V^* has trivial \mathfrak{a} -module structure. On the other hand,

$$\begin{aligned} (x.\xi(\nu^* \otimes \varphi))(v \otimes p) &= x.(\xi(\nu^* \otimes \varphi)(v \otimes p)) - \xi(\nu^* \otimes \varphi)(x.(v \otimes p)) \\ &= \nu^*(v)(x.\varphi)(p) = 0 \end{aligned}$$

as $\varphi \in \text{Hom}_{\mathfrak{a}}(P, E)$.

We now show that ξ is natural in the first variable V , and leave the verification for the other variables to the interested reader. Let V and W be two finite-dimensional trivial \mathfrak{a} -modules and $f \in \text{Hom}_{\mathfrak{a}}(V, W)$. For $w^* \in W^*$, $\varphi \in \text{Hom}_{\mathfrak{a}}(P, E)$ one has

$$\begin{aligned} (\xi \circ f^* \otimes 1)(w^* \otimes \varphi)(v \otimes p) &= (f^* \circ w^*)(v)\varphi(p) = w^*(f(v))\varphi(p) \\ &= \xi(w^* \otimes \varphi)(f(v) \otimes p) = ((f \otimes 1)^* \circ \xi)(w^* \otimes \varphi)(v \otimes p). \end{aligned}$$

This proves the claim that ξ is natural in V .

We still have to prove that ξ is an isomorphism. To that end let $\{v_i\}$ be a basis of V and $\{v_i^*\}$ be its dual basis. If $\xi(\sum_i a_i v_i^* \otimes \varphi_i) = 0$ with $a_i \in k$ then $\xi(\sum_i a_i v_i^* \otimes \varphi_i)(v_j \otimes p) = a_j \varphi_j(p) = 0$ for all $p \in P$ and all j . Thus ξ is injective.

Suppose now $\varphi \in \text{Hom}_{\mathfrak{a}}(V \otimes P, E)$ then if $\{p_k\}$ is a basis of P we can set $\bar{\varphi}_i(p_k) := \varphi(v_i \otimes p_k)$. It is easy to see that $\bar{\varphi}_i \in \text{Hom}_{\mathfrak{a}}(P, E)$ using the hypothesis that V is a trivial \mathfrak{a} -module. Now $\xi(\sum_i v_i^* \otimes \bar{\varphi}_i)(v_j \otimes p_k) = \bar{\varphi}_j(p_k) = \varphi(v_j \otimes p_k)$ so that ξ is surjective as well.

2.6. Let A, B and F be \mathfrak{a} -modules where $\mathfrak{a} = \mathfrak{a}(A)$ is a Kac-Moody algebra with triangular decomposition $\mathfrak{a}(A) = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$. For any \mathfrak{a} -module M we let $M[h]$ denote the sum of the \mathfrak{h} weight spaces of M . Let $\sigma: \mathfrak{a} \rightarrow \mathfrak{a}$ be the involutive antiautomorphism defined by $\sigma(x) = -x$ and let $\sigma_c: \mathfrak{a} \rightarrow \mathfrak{a}$ be the compact Chevalley involutive antiautomorphism defined $\sigma_c(x_\alpha) = x_{-\alpha}$ for α a root. Let now M^* (resp. M_c^*) be the \mathfrak{a} -module $\text{Hom}(M, \mathbb{C})$ with underlying

action $(x.\varphi)(m) = \varphi(\sigma(m))$ (resp. $(x.\varphi)(m) = \varphi(\sigma_c(m))$). Set $M^\sharp = M^*[\mathfrak{h}]$ and $M_c^\sharp = M_c^*[\mathfrak{h}]$. Let

$$\eta = \eta_{A, M, B}: \text{Hom}_\alpha(A, M \otimes B) \rightarrow \text{Hom}_\alpha(A \otimes M^\sharp, B)$$

be given by $\eta(\varphi)(a \otimes m^*) = \sum_i m^*(m_i)b_i$ where $\varphi(a) = \sum_i m_i \otimes b_i$ and $m_i \in M, b_i \in B$. For $f \in \text{Hom}_\alpha(M, N)$ let $f^\sharp: N^\sharp \rightarrow M^\sharp$ denote the obvious induced map.

Lemma. *Let H be a trivial finite-dimensional α -module with A, B, M arbitrary α -modules. Then the following diagram is commutative and η is a natural transformation:*

$$\begin{CD} H^* \otimes \text{Hom}_\alpha(A, M \otimes B) @>\xi>> \text{Hom}_\alpha(H \otimes A, M \otimes B) \\ @V\eta VV @VV\eta V \\ H^* \otimes \text{Hom}_\alpha(A \otimes M^\sharp, B) @>\xi>> \text{Hom}_\alpha(H \otimes A \otimes M^\sharp, B) \end{CD}$$

Proof. We first check commutativity of the diagram. Let $\chi \in H^*, h \in H, a \in A, m^* \in M^\sharp$ and $\varphi \in \text{Hom}_\alpha(A, M \otimes B)$. Then

$$\begin{aligned} (\xi \circ 1 \otimes \eta)(\chi \otimes \varphi)(h \otimes a \otimes m^*) &= \xi(\chi \otimes \eta(\varphi))(h \otimes a \otimes m^*) \\ &= \chi(h)\eta(\varphi)(a \otimes m^*) = \chi(h)(m^* \otimes 1)(\varphi(a)) \\ &= (m^* \otimes 1)(\chi(h)\varphi(a)) = (m^* \otimes 1)(\xi(\chi \otimes \varphi)(h \otimes a)) \\ &= \eta(\xi(\chi \otimes \varphi))(h \otimes a \otimes m^*) = (\eta \circ \xi)(\chi \otimes \varphi)(h \otimes a \otimes m^*). \end{aligned}$$

This proves the commutativity.

We now check that η is natural in the variable A and leave the verification of the other variables to the reader. Let $f \in \text{Hom}_\alpha(A, C), \varphi \in \text{Hom}_\alpha(C, M \otimes B)$. Then $\varphi(f(a)) = \sum_i m_i \otimes b_i$ with $m_i \in M$ and $b_i \in B$. Then for $m^* \in M^\sharp$ and $a \in A$

$$\begin{aligned} \eta_{A, M, B}(f^*(\varphi))(a \otimes m^*) &= \sum_i m^*(m_i)b_i = \eta_{C, M, B}(\varphi)(f(a) \otimes m^*) \\ &= ((f \otimes 1)^* \circ \eta_{C, M, B})(\varphi)(a \otimes m^*). \end{aligned}$$

Hence $\eta_{A, M, B} \circ f^* = (f \otimes 1)^* \circ \eta_{C, M, B}$ and η is natural in the first variable.

Define further $\beta: A^\sharp \otimes B^\sharp \rightarrow (A \otimes B)^\sharp$ by the formula $\beta(a^* \otimes b^*)(a \otimes b) = a^*(a)b^*(b)$. If A and B are finite dimensional, it is easy to check that β is an isomorphism.

2.7. We keep the same notation as in the previous sections.

Lemma. *For $Q, A, F,$ and E α -modules the following diagram is commutative:*

$$\begin{CD} \text{Hom}_\alpha(Q, F \otimes E \otimes A) @>\eta>> \text{Hom}_\alpha(Q \otimes (F \otimes E)^\sharp, A) \\ @V\eta VV @VV\text{Hom}_\alpha(\beta) V \\ \text{Hom}_\alpha(Q \otimes F^\sharp, E \otimes A) @>\eta>> \text{Hom}_\alpha(Q \otimes F^\sharp \otimes E^\sharp, A) \end{CD}$$

Proof. The proof is straightforward and is left to the reader.

2.8. The next result will be used in §4. First a definition. Let R be a ring with identity and let M_R denote the category of all R -modules. An additive

full subcategory \mathcal{C} of M_R is said to be a *good* category if its class of objects is closed under passage to submodules and quotients.

Proposition [Kn, Proposition 5.14]. *Let \mathcal{C} and \mathcal{D} be two good categories and $G: \mathcal{C} \rightarrow \mathcal{D}$ and $G': \mathcal{C} \rightarrow \mathcal{D}$ two functors such that there are isomorphisms*

$$\text{Hom}_R(G(A), B) \cong \text{Hom}_R(G'(A), B)$$

natural for all $A \in \text{Ob } \mathcal{C}$ and $B \in \text{Ob } \mathcal{D}$. Then G is naturally equivalent to G' .

Remarks. A similar result holds if one reverses the roles of the two variables in Hom . Let $T_{B,A}: \text{Hom}_R(G(A), B) \cong \text{Hom}_R(G'(A), B)$ denote the above given isomorphism. Then the natural equivalence from G to G' is implemented by the map $T_{G'(A),A}^{-1}(1_{G'(A)}): G(A) \rightarrow G'(A)$ where $A \in \text{Ob } \mathcal{C}$. For more details see [Kn, Chapter 5].

3. \mathfrak{F} -CATEGORIES AND \mathfrak{F} -FUNCTORS

3.0. Let \mathfrak{g} be a Lie algebra defined over a field \mathbb{K} of characteristic zero. If \mathfrak{F} and \mathfrak{A} are two additive categories of modules for \mathfrak{g} such that \mathfrak{F} is closed under tensoring and \mathfrak{A} is stable under tensoring by objects in \mathfrak{F} , then we say \mathfrak{A} is an *\mathfrak{F} -category* (see 3.2). In particular if \mathfrak{A} and \mathfrak{F} are also abelian categories then the Grothendieck group (see, for example, [HS, p. 75] for the definition of the Grothendieck group of an abelian category) of \mathfrak{F} , $G(\mathfrak{F})$, has the structure of an associative algebra over \mathbb{K} and the Grothendieck group of \mathfrak{A} becomes a $G(\mathfrak{F})$ -module. Hence an \mathfrak{F} -category can be viewed as a generalization of the notion of a module over a ring.

Suppose now that $\tau: \mathfrak{A} \rightarrow \mathfrak{B}$ is an additive functor between two \mathfrak{F} -categories of \mathfrak{g} -modules. If the operation of tensoring by objects in \mathfrak{F} “intertwines” with τ (i.e., $T_F \circ \tau$ is naturally equivalent to $\tau \circ T_F$ for every object F in \mathfrak{F} where $T_F = F \otimes \text{---}$), then we say τ is an *intertwining functor* (see 3.3). Note in particular that if \mathfrak{A} and \mathfrak{B} are also abelian categories then τ induces a $G(\mathfrak{F})$ -module homomorphism from $G(\mathfrak{A})$ to $G(\mathfrak{B})$. In this way we view an \mathfrak{F} -intertwining functor as a generalization of a module homomorphism.

In §5 we will give several examples of \mathfrak{F} -functors and \mathfrak{F} -categories, but for now we just make the above ideas more precise.

3.1. For any Lie algebra \mathfrak{g} (possibly infinite dimensional) defined over a field k of characteristic zero, let $M_{\mathfrak{g}}$ denote the category of all \mathfrak{g} -modules. Throughout we will assume that \mathfrak{F} denotes an additive subcategory of $M_{\mathfrak{g}}$ satisfying the following two conditions:

- (1) \mathfrak{F} is closed under tensoring; i.e., if $E_j, F_j \in \mathfrak{F}$, $f_j \in \text{Hom}_{\mathfrak{F}}(E_j, F_j)$ for $j = 1, 2$ then $E_1 \otimes E_2, F_1 \otimes F_2 \in \text{Ob } \mathfrak{F}$, and

$$f_1 \otimes f_2 \in \text{Hom}_{\mathfrak{F}}(E_1 \otimes E_2, F_1 \otimes F_2).$$

- (2) $\mathfrak{g} \in \text{Ob } \mathfrak{F}$ as a \mathfrak{g} -module under the adjoint action.

3.2. For $F \in \text{Ob } \mathfrak{F}$ let T_F denote the tensor product functor on $M_{\mathfrak{g}}$ given by $A \mapsto F \otimes A$ and $h \mapsto 1_F \otimes h$ for \mathfrak{g} -modules A and B and $h \in \text{Hom}_{\mathfrak{g}}(A, B)$. If $\mathfrak{n} \subset \mathfrak{g}$ is a Lie subalgebra and \mathcal{C} is a subcategory of $M_{\mathfrak{n}}$ we shall use the symbol T_F to denote the tensor product functor on \mathcal{C} when no confusion is likely to arise. The category \mathcal{C} is called an *\mathfrak{F} -category* if it is additive and T_F carries \mathcal{C} into itself for all $F \in \text{Ob } \mathfrak{F}$.

3.3. **Example.** Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac-Moody algebra associated to a generalized Cartan matrix A (see [Ka, Chapter 1]). Let $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Theta} \mathfrak{g}_\alpha$ be a root space decomposition of \mathfrak{g} with \mathfrak{h} a Cartan subalgebra, $\Theta \subseteq \mathfrak{h}^*$ a set of roots, Θ^+ a set of positive roots, and Π the set of simple roots. Fix $\alpha \in \Theta^+$ and let $\mathfrak{a} = \mathbb{C}x_\alpha \oplus \mathbb{C}h_\alpha \oplus \mathbb{C}y_\alpha$, $x_\alpha \in \mathfrak{g}_\alpha$, $h_\alpha = [x_\alpha, y_\alpha]$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, be a copy of $sl(2, \mathbb{C})$ inside of \mathfrak{g} . The full subcategory \mathcal{F}_α of M_α is defined to have as objects, \mathfrak{a} -modules M such that

- (i) h_α acts semisimply on M ,
- (ii) the $\mathbb{C}[y_\alpha]$ -action on M is torsion free, and
- (iii) M is $\mathbb{C}[x_\alpha]$ -finite.

This category was first introduced in [E, §3].

For convenience we collect here the definitions of several related categories which will appear later. Set $\mathfrak{n}_+ = \sum_{\alpha \in \Theta^+} \mathfrak{g}_\alpha$ and put $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$. For finite-dimensional \mathfrak{g} , \mathfrak{b} is a Borel subalgebra. Let \mathcal{O} denote the category of $U(\mathfrak{g})$ -modules for which \mathfrak{h} acts semisimply and \mathfrak{b} acts locally finitely. Let \mathfrak{F} be the category of integrable \mathfrak{g} -modules; i.e., $N \in \text{Ob } \mathfrak{F}$ if and only if N is \mathfrak{h} -diagonalizable and y_α and x_α act locally nilpotently for all $\alpha \in \Theta$. Then \mathfrak{F} is closed under tensoring and $\mathfrak{g} \in \text{Ob } \mathfrak{F}$ by [Ka, Lemma 3.5], so \mathfrak{F} satisfies 3.1. Moreover it is not too hard to check that \mathcal{F}_α is stable under tensoring by objects and morphisms in \mathfrak{F} . Hence \mathcal{F}_α is an \mathfrak{F} -category (see [D, Remark 2]).

3.4. Now let \mathfrak{a} and \mathfrak{b} be two Lie subalgebras of \mathfrak{g} and let \mathfrak{A} (resp. \mathfrak{B}) be an additive subcategory of M_α (resp. M_β). Suppose further that both \mathfrak{A} and \mathfrak{B} are \mathfrak{F} -categories and τ is a functor from \mathfrak{A} to \mathfrak{B} . We call τ an *intertwining functor* (or *\mathfrak{F} -intertwining functor* when more precision is necessary) if τ is additive and there exists a natural equivalence for each $F \in \text{Ob } \mathfrak{F}$, $i_F: T_F \circ \tau \rightarrow \tau \circ T_F$.

3.5. Suppose τ is an intertwining functor and let $\mathcal{S} = \{i_F | F \in \text{Ob } \mathfrak{F}\}$ denote the family of natural equivalences above. Recall that a natural equivalence $i_E: T_E \circ \tau \rightarrow \tau \circ T_E$ is a rule that assigns to each object A of \mathfrak{A} an isomorphism $i_E(A): T_E \circ \tau(A) \rightarrow \tau \circ T_E(A)$ such that for every homomorphism $f: A \rightarrow B$ in \mathfrak{A} one has $i_E(B) \circ ((T_E \circ \tau)(f)) = (\tau \circ T_E)(f) \circ i_E(A)$. For convenience we set $i_{E,A} = i_E(A)$ for all $A \in \text{Ob } \mathfrak{A}$, $E \in \text{Ob } \mathfrak{F}$.

Suppose now that for every $E, F \in \text{Ob } \mathfrak{F}$ and $h \in \text{Hom}_{\mathfrak{F}}(E, F)$ one has $h \otimes 1_A \in \text{Hom}_{\mathfrak{A}}(E \otimes A, F \otimes A)$ for $A \in \text{Ob } \mathfrak{A}$. Assume \mathfrak{B} has this same property. Then we say that \mathcal{S} is *natural in the \mathfrak{F} -variable* (or *natural in \mathfrak{F}*) if the following diagram is commutative for all $E, F \in \text{Ob } \mathfrak{F}$, $A \in \text{Ob } \mathfrak{A}$, and $f \in \text{Hom}_{\mathfrak{F}}(E, F)$:

$$(1) \quad \begin{array}{ccc} E \otimes \tau A & \xrightarrow{i_{E,A}} & \tau(E \otimes A) \\ f \otimes 1_{\tau A} \downarrow & & \downarrow \tau(f \otimes 1_A) \\ F \otimes \tau A & \xrightarrow{i_{F,A}} & \tau(F \otimes A) \end{array}$$

We call \mathcal{S} *distributive* if the following diagram is commutative for $E, F \in$

Ob \mathfrak{F} and $A \in \text{Ob } \mathfrak{A}$:

$$(2) \quad \begin{array}{ccc} (E \oplus F) \otimes \tau A & \xrightarrow{i_{E \oplus F, A}} & \tau((E \oplus F) \otimes A) \\ \downarrow & & \downarrow \\ (E \otimes \tau A) \oplus (F \otimes \tau A) & \xrightarrow{i_{E, A} \oplus i_{F, A}} & \tau(E \otimes A) \oplus \tau(F \otimes A) \end{array}$$

The left map in (2) expresses the bilinearity of \otimes and the right map expresses this bilinearity combined with additivity of τ .

We say that \mathcal{S} is *associative* if the following diagram is commutative for all $E, F \in \text{Ob } \mathfrak{F}$ and $A \in \text{Ob } \mathfrak{A}$:

$$(3) \quad \begin{array}{ccc} E \otimes F \otimes \tau A & \xrightarrow{1 \otimes i_{F, A}} & E \otimes \tau(F \otimes A) \\ & \searrow i_{E \otimes F, A} & \downarrow i_{E, F \otimes A} \\ & & \tau(E \otimes F \otimes A) \end{array}$$

3.6. To help set these ideas we include a proof of the following somewhat straightforward result.

Lemma. *Suppose \mathfrak{A} and \mathfrak{B} are \mathfrak{F} -categories and $\tau: \mathfrak{A} \rightarrow \mathfrak{B}$ is an intertwining functor. If the family $\mathcal{S} = \{i_{F, A} | F \in \text{Ob } \mathfrak{F}, A \in \text{Ob } \mathfrak{A}\}$ is natural in the \mathfrak{F} -variable then \mathcal{S} is distributive.*

Proof. The sum $X = U \oplus V$ in \mathfrak{A} can be expressed as a split exact sequence $0 \rightarrow U \rightarrow X \rightarrow V \rightarrow 0$. Thus the left map in 3.5.2 can be expressed as a split short exact sequence. Similarly since τ is additive, the right map in 3.5.2 can also be so expressed. We then obtain the following diagram:

$$(1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \tau(E \otimes A) & \xrightarrow{\tau(a \otimes 1_A)} & \tau((E \oplus F) \otimes A) & \xrightarrow{\tau(b \otimes 1_A)} & \tau(F \otimes A) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & (i) & & (ii) & & \\ 0 & \longrightarrow & E \otimes \tau A & \xrightarrow{a \otimes 1_{\tau A}} & (E \oplus F) \otimes \tau A & \xrightarrow{b \otimes 1_{\tau A}} & F \otimes \tau A \longrightarrow 0 \end{array}$$

Here the vertical maps come from \mathcal{S} while a (resp. b) is the obvious inclusion (resp. surjection). By naturality of \mathcal{S} in \mathfrak{F} both subdiagrams (i) and (ii) above are commutative. In turn this shows that 3.5.2 is commutative and thus the lemma is proved.

3.7. Suppose τ is an intertwining functor with the family of natural equivalences $\mathcal{S} = \{i_F | F \in \text{Ob } \mathfrak{F}\}$. We call the pair (τ, \mathcal{S}) an *\mathfrak{F} -functor* whenever \mathcal{S} is both distributive and associative. When \mathcal{S} is understood to be fixed we call τ an *\mathfrak{F} -functor*.

We now give two examples of \mathfrak{F} -functors to illustrate that they occur naturally in representation theory of Lie algebras. In §5 we will discuss a couple of other examples.

3.8 (Continuation of Example 3.3). Take $\mathfrak{A} = \mathfrak{B} = \mathcal{T}_\alpha$ and consider the completion functor, $C_\alpha: \mathcal{T}_\alpha \rightarrow \mathcal{T}_\alpha$, defined by Enright as follows. Let M satisfy (i) above, i.e., M is a *weight module*. Then for $c \in \mathbb{C}$ let

$$M_c = \{m \in M | h_\alpha m = cm\}, \quad M^{x_\alpha} = \{m \in M | x_\alpha m = 0\}, \quad M_c^{x_\alpha} = M^{x_\alpha} \cap M_c.$$

A weight module M for \mathfrak{a} is *complete* if for each $n \in \mathbb{N}$, y_α^{n+1} induces a bijection $M_n^{x_\alpha} \rightarrow M_{-n-2}^{x_\alpha}$. A *completion* of a weight module M is a weight module M' together with an \mathfrak{a} -module injection $i: M \hookrightarrow M'$ such that (i) M'/M is $U(\mathfrak{a})$ -finite and (ii) M' is complete (see [E, §3] for details). For $M \in \mathcal{T}_\mathfrak{a}$ completions exist and are denoted by $C_\alpha(M)$. Moreover given an \mathfrak{a} -module homomorphism $f: M \rightarrow N$ in $\mathcal{T}_\mathfrak{a}$ one has an induced homomorphism $C_\alpha(f): C_\alpha(M) \rightarrow C_\alpha(N)$ (see [E, §3]). In [D] the Enright completion functor C_α above is described as a subfunctor of a localization functor D_α . Now $\mathcal{T}_\mathfrak{a}$ can be shown to be stable under T_F , $F \in \text{Ob } \mathfrak{F}$, and D_α and C_α can be seen to be \mathfrak{F} -intertwining (see [D, Theorem 3.1, Corollary 3.2 and Remark 2; RW, Proposition 12; Ki, Proposition 2.6]). Moreover it follows from, say, [Ki, Proposition 2.6] that the natural equivalence $i_F: T_F \cdot C_\alpha \rightarrow C_\alpha \circ T_F$ is natural in \mathfrak{F} . With a straightforward calculation one can also show that the family $\{i_F\}$ is associative. By Lemma 3.6 $\{i_F\}$ is distributive and hence C_α is an \mathfrak{F} -functor.

3.9. Let $\mathfrak{q} \subseteq \mathfrak{g}$ be two Lie algebras over \mathbb{C} and let $\mathcal{E} = \mathcal{E}(\mathfrak{g})$ be a subcategory of $M_\mathfrak{g}$ satisfying 3.1. Let $\mathcal{D} = \mathcal{D}(\mathfrak{q})$ be a subcategory of $M_\mathfrak{q}$ stable under T_E , $E \in \text{Ob } \mathcal{E}$. If $V \in \text{Ob } \mathcal{D}$, define $P_\mathfrak{q}^\mathfrak{g}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} V$ (resp. $I_\mathfrak{q}^\mathfrak{g}(U) = \text{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), V)$) as a \mathfrak{g} -module via $u_1(u_2 \otimes x) = (u_1 u_2) \otimes x$ (resp. $u_1 \varphi(u_2) = \varphi(u_2 u_1)$) for $u_1, u_2 \in U(\mathfrak{g})$, $x \in V$. Then $P_\mathfrak{q}^\mathfrak{g}: \mathcal{D} \rightarrow M_\mathfrak{g}$ (resp. $I_\mathfrak{q}^\mathfrak{g}: \mathcal{D} \rightarrow M_\mathfrak{g}$) is just *induction* (resp. *coinduction*). For each $E \in \text{Ob } \mathcal{E}$ and $V \in \text{Ob } \mathcal{D}$ we have a Mackey isomorphism

$$(1) \quad \Phi_{E,V}: P_\mathfrak{q}^\mathfrak{g}(E \otimes V) \rightarrow E \otimes P_\mathfrak{q}^\mathfrak{g}(V)$$

uniquely determined by $\Phi_{E,V}(1 \otimes f \otimes v) = f \otimes 1 \otimes v$ for $f \in E$, $v \in V$. Moreover the isomorphism above is natural in E and V (see [Kn, Proposition 6.5]). Hence the Mackey isomorphisms (1) provide us with a family $\mathcal{F} = \{\Phi_E: P_\mathfrak{q}^\mathfrak{g} \circ T_E \rightarrow T_E \circ P_\mathfrak{q}^\mathfrak{g}, E \in \text{Ob } \mathcal{E}\}$ such that $P_\mathfrak{q}^\mathfrak{g}$ is \mathcal{E} -intertwining with respect to \mathcal{F} and \mathcal{F} is natural in the \mathcal{E} -variable. Then \mathcal{F} is distributive by 3.6 and is also easily seen to be associative by the formula defining $\Phi_{E,V}$, (1). Hence $P_\mathfrak{q}^\mathfrak{g}$ is an \mathcal{E} -functor.

In addition, if we assume that all objects of \mathcal{E} are finite dimensional then there exist unique isomorphisms of $U(\mathfrak{g})$ -modules

$$\Phi_{E,V}^*: E \otimes I_\mathfrak{q}^\mathfrak{g}(V) \rightarrow I_\mathfrak{q}^\mathfrak{g}(E \otimes V)$$

such that

$$\Phi_{E,V}^*(e \otimes \varphi)(1) = e \otimes \varphi(1)$$

where $e \in E$, $\varphi \in I_\mathfrak{q}^\mathfrak{g}(V)$. Moreover this isomorphism is natural in E and V (see [Kn, Corollary 6.7]). Hence $\mathcal{F}^* = \{\Phi_E^*: T_E \circ I_\mathfrak{q}^\mathfrak{g} \rightarrow I_\mathfrak{q}^\mathfrak{g} \circ T_E\}$ provides us with a family of natural transformations such that $I_\mathfrak{q}^\mathfrak{g}$ is an \mathcal{E} -intertwining functor with respect to \mathcal{F}^* and \mathcal{F}^* is distributive. For $E, F \in \text{Ob } \mathcal{E}$, $e \in E$, $f \in F$, $\varphi \in I_\mathfrak{q}^\mathfrak{g}(V)$ we have

$$\begin{aligned} \Phi_{E \otimes F, V}^*(e \otimes f \otimes \varphi)(1) &= e \otimes f \otimes \varphi(1) = e \otimes \Phi_{F,V}^*(f \otimes \varphi)(1) \\ &= \Phi_{E, F \otimes V}^*(e \otimes \Phi_{F,V}^*(f \otimes \varphi))(1) \\ &= (\Phi_{E, F \otimes V}^* \circ 1_E \otimes \Phi_{F,V}^*)(e \otimes f \otimes \varphi)(1). \end{aligned}$$

By uniqueness $\Phi_{E \otimes F, V}^* = \Phi_{E, F \otimes V}^* \circ 1_E \otimes \Phi_{F, V}^*$ so that \mathcal{F}^* is associative. Hence I_q^g is an \mathcal{E} -functor.

4. THE LIFTING THEOREM

4.1. Now we introduce notation necessary for the statement of the Lifting Theorem. As in the previous section let \mathfrak{a} and \mathfrak{b} be Lie subalgebras of \mathfrak{g} and let \mathfrak{A} (resp. \mathfrak{B}) be an additive subcategory of $M_{\mathfrak{a}}$ (resp. $M_{\mathfrak{b}}$). Recall from §2 the concepts of the image and preimage of a functor. Suppose \mathfrak{C} is an additive subcategory of $M_{\mathfrak{g}}$ with $\text{res}_{\mathfrak{a}} \mathfrak{C} \subseteq \mathfrak{A}$ and set \mathfrak{H} equal to the preimage of \mathfrak{B} in $M_{\mathfrak{g}}$: i.e., $\mathfrak{H} = \text{res}_{\mathfrak{b}}^{-1} \mathfrak{B}$ (see 2.1). Let $\tau: \mathfrak{A} \rightarrow \mathfrak{B}$ be a functor. A *lifting* of τ is a functor $\bar{\tau}: \mathfrak{C} \rightarrow \mathfrak{H}$ rendering the following diagram commutative:

$$(1) \quad \begin{array}{ccc} \mathfrak{C} & \xrightarrow{\bar{\tau}} & \mathfrak{H} \\ \text{res}_{\mathfrak{a}} \downarrow & & \downarrow \text{res}_{\mathfrak{b}} \\ \mathfrak{A} & \xrightarrow{\tau} & \mathfrak{B} \end{array}$$

Although our notation here might suggest uniqueness, neither the existence nor the uniqueness of a lifting is apparent. Both of these points are the focus of our attention in this section.

4.2. If \mathfrak{g} is a Lie algebra and E a \mathfrak{g} -module, then let $a_{\mathfrak{g}}: T(\mathfrak{g}) \otimes E \rightarrow E$ denote the action of the tensor algebra $T(\mathfrak{g})$ on E induced by $x \otimes e \mapsto x \cdot e$ for $x \in \mathfrak{g}, e \in E$. When no confusion is likely to arise, we omit the subscript and write a in place of $a_{\mathfrak{g}}$.

4.3. Throughout the paper any additive category \mathcal{C} of $M_{\mathfrak{g}}$ will satisfy the following condition: if $A \in \text{Ob } \mathcal{C}$ then $\mathfrak{g} \otimes A$ and $T^n(\mathfrak{g}) \otimes A$ are objects of \mathcal{C} for $n \in \mathbb{N}$ and the \mathfrak{g} -module map $a_{\mathfrak{g}}$ is a morphism in \mathcal{C} , i.e.,

$$a_{\mathfrak{g}} \in \text{Hom}_{\mathcal{C}}(T^n(\mathfrak{g}) \otimes A, A)$$

for all $n \in \mathbb{N}$.

4.4. Let $\mathfrak{F} = \mathfrak{F}(\mathfrak{g})$ be defined as in §3, satisfying 3.1.1 and 3.1.2. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, and \mathfrak{H} be categories as above and assume both \mathfrak{A} and \mathfrak{B} and \mathfrak{F} -categories and $\tau: \mathfrak{A} \rightarrow \mathfrak{B}$ is an \mathfrak{F} -functor. Here we have fixed a family of equivalences $\mathcal{F} = \{i_F | F \in \text{Ob } \mathfrak{F}\}$ (cf. 3.4). We say that τ is *compatible with the \mathfrak{b} -module action* if the following diagram is commutative for all $E \in \text{Ob } \mathfrak{C}$:

$$(1) \quad \begin{array}{ccc} \mathfrak{b} \otimes \tau E & \xrightarrow{a_{\mathfrak{b}}} & \tau E \\ \downarrow & & \downarrow \tau(a_{\mathfrak{b}}) \\ \mathfrak{g} \otimes \tau E & \xrightarrow{i_{\mathfrak{g}, E}} & \tau(\mathfrak{g} \otimes E). \end{array}$$

4.5. Suppose $\bar{\tau}$ is a lifting of τ . We say that $\bar{\tau}$ has a \mathfrak{g} -action induced from τ (or (τ, \mathcal{F}) to be more precise) if the following diagram is commutative for all $E \in \text{Ob } \mathfrak{C}$:

$$(1) \quad \begin{array}{ccc} \mathfrak{g} \otimes \bar{\tau} E & \xrightarrow{a_{\mathfrak{g}}} & \bar{\tau} E \\ i_{\mathfrak{g}, E} \downarrow & \nearrow \bar{\tau}(a_{\mathfrak{g}}) & \\ \bar{\tau}(\mathfrak{g} \otimes E) & & \end{array}$$

Above we suppress the appearance of the restriction functor $\text{res}_{\mathfrak{b}}$ so that $\text{res}_{\mathfrak{b}} E$ appears as just E .

4.6. Theorem (Lifting Theorem). *Suppose $\tau: \mathfrak{A} \rightarrow \mathfrak{B}$ is an \mathfrak{F} -functor such that*

- (i) τ is compatible with the \mathfrak{b} -module action (cf. 4.4) and
- (ii) $J(\mathfrak{g}) = \text{Span}\{x \otimes y - y \otimes x - [x, y] | x, y \in \mathfrak{g}\}$ has an ad \mathfrak{g} -stable complement K in $T_2(\mathfrak{g})$ and both $K, J(\mathfrak{g}) \in \text{Ob } \mathfrak{F}$.

Then there exists a unique lifting $\bar{\tau}: \mathfrak{G} \rightarrow \mathfrak{H}$ which has a \mathfrak{g} -module action induced from τ .

Proof. For convenience in the proof we write $T = T(\mathfrak{g})$ and $T^n = T^n(\mathfrak{g})$. Let $E \in \text{Ob } \mathfrak{G}$. We begin by defining a linear map $\bar{a} \cdot T \otimes \tau E \rightarrow \tau E$ via the commutative diagram

$$(1) \quad \begin{array}{ccc} T \otimes \tau E & \xrightarrow{\bar{a}} & \tau E \\ i_{T,E} \downarrow & \nearrow \tau(a_{\mathfrak{g}}) & \\ \tau(T \otimes E) & & \end{array}$$

Since $i_{T,E}$ and $a_{\mathfrak{g}}$ are sums of restrictions to the graded subspaces T^n , (1) is defined also at the graded level. By definition \bar{a} induces a linear map (also denoted by \bar{a}) from T into $\text{End}(\tau E)$.

We claim \bar{a} is an algebra homomorphism of T into $\text{End}(\tau E)$. For $n, m \in \mathbb{N}$ consider the diagram:

$$(2) \quad \begin{array}{ccccc} T^{n+m} \otimes \tau E & \xrightarrow{i_{T^{n+m} \otimes E}} & \tau(T^{n+m} \otimes E) & \xrightarrow{\tau(a)} & \tau(E) \\ \downarrow 1 \otimes i_{T^m, E} & \nearrow i_{T^n, T^m \otimes E} & \downarrow \tau(1 \otimes a) & \nearrow \tau(a) & \\ T^n \otimes \tau(T^m \otimes E) & & \tau(T^n \otimes E) & & \\ \downarrow 1 \otimes \tau(a) & \nearrow i_{T^n, E} & & & \\ T^n \otimes \tau E & & & & \end{array}$$

We let (a), (b), and (c) indicate the subdiagrams. Of course (2) will be commutative as soon as we show (a), (b), and (c) are themselves commutative.

Our assumption 4.3 assures us that all the maps in (2) are well-defined. Since \mathcal{F} is associative, diagram (a) is commutative. Since E is a \mathfrak{g} -module and τ is a functor, diagram (b) is commutative. Now set $F = T^n(\mathfrak{g})$ and write diagram

(c) in the form

$$(3) \quad \begin{array}{ccc} T_F \circ \tau(T^m \otimes E) & \xrightarrow{i_{F, T^m \otimes E}} & \tau \circ T_F(T^m \otimes E) \\ 1 \otimes \tau(a) \downarrow & & \downarrow \tau(1 \otimes a) \\ T_F \circ \tau E & \xrightarrow{i_{F, E}} & \tau \circ T_F(E) \end{array}$$

Since i_F is a natural equivalence of functors $T_F \circ \tau \rightarrow \tau \circ T_F$, diagram (3) is commutative. This completes the proof that (2) is commutative.

Since each i_F is a natural equivalence, the action \bar{a} has its own naturality which will be of use later. We record this here as: for $D, E \in \text{Ob } \mathfrak{G}$ and $\alpha \in \text{Hom}_{\mathfrak{G}}(D, E)$ the following diagram is commutative:

$$(4) \quad \begin{array}{ccc} T \otimes \tau E & \xrightarrow{\bar{a}} & \tau E \\ 1 \otimes \tau \alpha \uparrow & & \uparrow \tau \alpha \\ T \otimes \tau D & \xrightarrow{\bar{a}} & \tau D \end{array}$$

Next we show that \bar{a} factors through $T(\mathfrak{g})$ to $U(\mathfrak{g})$ inducing a \mathfrak{g} -module action on τE . The kernel of \bar{a} is a two-sided ideal and thus to prove \bar{a} induces a \mathfrak{g} -action on τE it is sufficient to show that $J(\mathfrak{g})$ is contained in the kernel of \bar{a} . Consider the following diagram with $J = J(\mathfrak{g})$ and $F = T_2(\mathfrak{g})$:

$$(5) \quad \begin{array}{ccccc} F \otimes \tau E & \xrightarrow{i_{F, E}} & \tau(F \otimes E) & \xrightarrow{\tau(a)} & \tau E \\ \uparrow & & \uparrow & & \parallel \\ J \otimes \tau E & \xrightarrow{i_{J, E}} & \tau(J \otimes E) & \xrightarrow{\tau(b)} & \tau E \end{array}$$

Here we let b denote the restriction of a to $J \otimes E$. Since \mathcal{S} is distributive we use hypothesis 4.6(ii) to conclude that all the subdiagrams of (5) are commutative. But b is the zero map and since τ is additive, $\tau(b) = 0$. It follows from (5) that \bar{a} is zero on $J \otimes \tau E$. Therefore \bar{a} induces an algebra homomorphism of $U(\mathfrak{g})$ into $\text{End}(\tau E)$. This gives us a \mathfrak{g} -module action on τE . Let $\bar{\tau}E$ denote this \mathfrak{g} -module.

We now claim that $E \mapsto \bar{\tau}E$ is a functor from \mathfrak{G} to \mathfrak{H} . Let D, E be objects of \mathfrak{G} and $\gamma \in \text{Hom}_{\mathfrak{G}}(D, E)$. Suppressing the notation for the restriction, γ is also a map from $\text{res } D$ to $\text{res } E$ and thus $\tau\gamma \in \text{Hom}_{\mathfrak{B}}(\tau D, \tau E)$. We claim $\tau\gamma \in \text{Hom}_{\mathfrak{H}}(\bar{\tau}D, \bar{\tau}E)$. From the definition of \mathfrak{H} it is sufficient to verify that $\tau\gamma$ is a \mathfrak{g} -module map. We prove this by considering the following diagram:

$$(6) \quad \begin{array}{ccccc} F \otimes \tau E & \xrightarrow{i_{F, E}} & \tau(F \otimes E) & \xrightarrow{\tau(a)} & \tau E \\ 1 \otimes \tau\gamma \uparrow & & \tau(1 \otimes \gamma) \uparrow & & \uparrow \tau\gamma \\ F \otimes \tau D & \xrightarrow{i_{F, D}} & \tau(F \otimes D) & \xrightarrow{\tau(a)} & \tau D \end{array}$$

where $F = T^n$ for $n \in \mathbb{N}$ and a is the obvious \mathfrak{g} -module action. This diagram is by hypothesis commutative. This fact together with the assumption that \mathcal{S} is distributive implies that $\tau\gamma$ respects the action of $T(\mathfrak{g})$ and hence the action

of $U(\mathfrak{g})$ on E and D . Thus $\tau\gamma \in \text{Hom}_{\mathfrak{H}}(\bar{\tau}D, \bar{\tau}E)$, and so setting $\bar{\tau}\gamma = \tau\gamma$ we conclude that $\bar{\tau}: \mathfrak{O} \rightarrow \mathfrak{H}$ is a functor.

From the construction it is clear that $\bar{\tau}$ is both a lifting and has a \mathfrak{g} -module action induced from τ . From (1) and 4.5.1 we conclude that $\bar{\tau}$ is unique. This completes the proof of the Lifting Theorem.

4.7. We would like to show now that the hypothesis 4.6(ii) is not too stringent.

Lemma. *Suppose \mathfrak{g} is a Lie algebra and as above let*

$$J = J(\mathfrak{g}) = \text{span}\{x \otimes y - y \otimes x - [x, y]|x, y \in \mathfrak{g}\}.$$

Let $S_2 = S_2(\mathfrak{g})$ denote the symmetric tensors in $T_2(\mathfrak{g})$. Then J and S_2 \mathfrak{g} -modules and $T_2(\mathfrak{g}) = S_2 \oplus J$.

Proof. The Poincaré-Birkhoff-Witt theorem implies that the projection of $T(\mathfrak{g})$ onto $U(\mathfrak{g})$ restricts to a linear isomorphism on $S(\mathfrak{g})$, the symmetric tensors of $T(\mathfrak{g})$. Clearly J is the kernel of this map on $T_2(\mathfrak{g})$. This completes the proof of the lemma.

4.8 (Continuation of Example 3.3). It is straightforward to see that J and S_2 defined in §4.7 are objects in \mathfrak{F} . Hence 4.6(ii) is satisfied. Now we have the following setup.

$$(1) \quad \begin{array}{ccc} \mathcal{F}_{\mathfrak{g}}(\mathfrak{a}) & \xrightarrow{\bar{C}_{\alpha}} & \mathcal{F}_{\mathfrak{g}}(\mathfrak{a}) \\ \text{res}_{\mathfrak{a}} \downarrow & & \downarrow \text{res}_{\mathfrak{a}} \\ \mathcal{F}_{\mathfrak{a}} & \xrightarrow{C_{\alpha}} & \mathcal{F}_{\mathfrak{a}} \end{array}$$

where $\mathfrak{a} = \mathbb{C}X_{\alpha} + \mathbb{C}H_{\alpha} + \mathbb{C}X_{-\alpha}$, $\alpha \in B$, is a copy of $sl(2, \mathbb{C})$ in $\mathfrak{g}(A)$, and $\mathcal{F}_{\mathfrak{g}}(\mathfrak{a})$ is the full subcategory of \mathfrak{g} -modules M such that $\text{res}_{\mathfrak{a}} M \in \text{Ob } \mathcal{F}_{\mathfrak{a}}$, i.e., $\mathcal{F}_{\mathfrak{g}}(\mathfrak{a}) = \text{res}_{\mathfrak{a}}^{-1}(\mathcal{F}_{\mathfrak{a}})$ (see [E, 3.5 and 3.6]).

Now it is not too hard to check that $\mathcal{F}_{\mathfrak{a}}$ and $\mathcal{F}_{\mathfrak{g}}(\mathfrak{a})$ are $\mathfrak{F} = \mathfrak{F}(\mathfrak{g})$ -categories and that assumption 4.3 is satisfied for \mathfrak{g} and $\mathfrak{C} = \mathcal{F}_{\mathfrak{a}}$ or $\mathcal{F}_{\mathfrak{g}}(\mathfrak{a})$. From [Ki, Proposition 2.16(iii)] we have a natural equivalence $i_F: T_F \circ C_{\alpha} \rightarrow C_{\alpha} \circ T_F$ for F an integrable \mathfrak{g} -module. Thus we have a commutative diagram for $M \in \mathcal{F}_{\mathfrak{g}}(\mathfrak{a})$:

$$\begin{array}{ccccc} \mathfrak{a} \otimes C_{\alpha}(M) & \xrightarrow{i_{\mathfrak{g}, M}} & C_{\alpha}(\mathfrak{a} \otimes M) & \xrightarrow{C_{\alpha}(a_{\mathfrak{a}})} & C_{\alpha}(M) \\ i \otimes 1 \downarrow & & C_{\alpha}(i \otimes 1) \downarrow & \nearrow C_{\alpha}(a_{\mathfrak{g}}) & \\ \mathfrak{g} \otimes C_{\alpha}(M) & \xrightarrow{i_{\mathfrak{g}, M}} & C_{\alpha}(\mathfrak{g} \otimes M) & & \end{array}$$

where $i: \mathfrak{a} \rightarrow \mathfrak{g}$ is just the canonical inclusion. The composite of the top maps is just the \mathfrak{a} -module action on $C_{\alpha}(M)$ (see [E, Proposition 3.3]). Hence C_{α} is compatible with the \mathfrak{a} -module action and we can apply the Lifting Theorem to give us a functor $\bar{C}_{\alpha}: \mathcal{F}_{\mathfrak{g}}(\mathfrak{g}) \rightarrow \mathcal{F}_{\mathfrak{g}}(\mathfrak{a})$ rendering diagram 4.7.1 commutative. This gives an alternate proof of [RW, Proposition 11].

Note that the lattice functors defined in [E, §4], being compositions of various C_{α} 's, also enjoy the property of being \mathfrak{F} -functors for A of finite type, i.e., for \mathfrak{g} finite dimensional. The details are left to the interested reader (see [E, Proposition 4.14]).

4.9. Next we extend the Lifting Theorem to the setting of derived functors of an \mathfrak{F} -functor. For §§4.9 to 4.11 we assume that \mathfrak{A} is an $\mathfrak{F}(\mathfrak{a})$ -category, where $\mathfrak{F}(\mathfrak{a})$ satisfies 3.1.1 and 3.1.2 and \mathfrak{A} satisfies the following additional hypotheses:

- (i) \mathfrak{A} has enough projectives,
- (ii) the functors $T_F, F \in \text{Ob } \mathfrak{F}(\mathfrak{a})$, carry projectives to projectives,
- (iii) for $\alpha \in \text{Hom}_{\mathfrak{A}}(A, B)$, $\ker \alpha$ and $\text{im } \alpha$ are objects in \mathfrak{A} , and both the inclusion $\ker \alpha \hookrightarrow A$ and the surjection $A \rightarrow \text{im } \alpha$ morphisms of \mathfrak{A} .

Remarks. (i) and (iii) are sufficient to imply that projective resolutions exist for any module $A \in \text{Ob } \mathfrak{A}$ and that given two projective resolutions $P_* \rightarrow A \rightarrow 0, Q_* \rightarrow B \rightarrow 0$ in \mathfrak{A} with $f \in \text{Hom}_{\mathfrak{A}}(A, B)$ then there exists a (unique up to chain homotopy) chain map $\bar{f}: P_* \rightarrow Q_*$ inducing f (see [HS, Chapter IV, Theorem 4.1]). However \mathfrak{A} above need not be an abelian category as the next example shows.

4.10 **Example.** Let $\mathfrak{a} = \mathfrak{sl}(2, \mathbb{C})$ and let $M_n, n \in \mathbb{Z}$, denote the Verma module $M(\lambda_n)$ for \mathfrak{a} , where $\lambda_n \in \mathfrak{h}^*$ is defined by $\lambda_n(H) = n$. Then for $n \in \mathbb{N}^*, M_{-n} \hookrightarrow M_n$ is injective in $\mathcal{T}_{\mathfrak{a}}$ but $F_n \cong M_n/M_{-n}$ is not an object in $\mathcal{T}_{\mathfrak{a}}$. Hence \mathfrak{A} is not an abelian category.

4.11 **Proposition.** *Suppose \mathfrak{A} is an $\mathfrak{F}(\mathfrak{a})$ -category satisfying 4.9 and let $\tau: \mathfrak{A} \rightarrow M_{\mathfrak{a}}$ be a covariant $\mathfrak{F}(\mathfrak{a})$ -functor. Then all derived functors $\tau^k, k \in \mathbb{N}$, are $\mathfrak{F}(\mathfrak{a})$ -functors as well.*

Proof. Let $A \in \text{Ob } \mathfrak{A}$ and let $P_* \rightarrow A$ be a projective resolution of A . By 4.9(ii), $F \otimes P_* \rightarrow F \otimes A$ is a projective resolution and $\tau^k(F \otimes A)$ is the k th homology group of the complex $\tau(F \otimes P_*)$ for $F \in \text{Ob } \mathfrak{F}(\mathfrak{a})$. Consider the diagram

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \tau(F \otimes P_j) & \longrightarrow & \tau(F \otimes P_{j-1}) & \longrightarrow & \cdots \\
 (1) & & \uparrow & & i_{F, P_j} \uparrow & & i_{F, P_{j-1}} \uparrow & & \uparrow \\
 \cdots & \longrightarrow & F \otimes \tau P_j & \longrightarrow & F \otimes \tau P_{j-1} & \longrightarrow & \cdots
 \end{array}$$

All the subdiagrams of (1) are commutative and thus the isomorphisms i_F induce isomorphisms $i_{F, A}^j$ on the homology groups

$$(2) \quad i_{F, A}^j: F \otimes \tau^j A \xrightarrow{\sim} \tau^j(F \otimes A).$$

Let now $D_* \rightarrow A \rightarrow 0$ and $Q_* \rightarrow B \rightarrow 0$ be two projective resolutions in \mathfrak{A} and let $f \in \text{Hom}_{\mathfrak{A}}(A, B)$. Then by Remark 4.9 there exists a chain map \bar{f} such that the following diagram is commutative:

$$\begin{array}{ccccc}
 P_* & \longrightarrow & A & \longrightarrow & 0 \\
 \bar{f} \downarrow & & & & \downarrow f \\
 Q_* & \longrightarrow & B & \longrightarrow & 0.
 \end{array}$$

For the existence of this chain map one needs to assume 4.9(iii). (Thus it does not suffice to assume that \mathfrak{A} has projective resolutions of objects; one must also assume 4.9(iii).) Using the fact that the i_F are natural equivalences it is

a rather straightforward verification (which we leave to the reader) to show the diagram

$$\begin{array}{ccc}
 F \otimes \tau^k A & \xrightarrow{i_{F,A}^k} & \tau^k(F \otimes A) \\
 1 \otimes \tau^k(f) \downarrow & & \downarrow \tau^k(1 \otimes f) \\
 F \otimes \tau^k B & \xrightarrow{i_{F,B}^k} & \tau^k(F \otimes B)
 \end{array}$$

is commutative. Hence the i_F^k are natural equivalences. Clearly τ^k is additive and thus each τ^k is an intertwining functor with respect to the family $\mathcal{F}^k = \{i_F^k | F \in \text{Ob } \mathfrak{F}\}$.

It remains to verify that each \mathcal{F}^k is both distributive and associative. Let $F, H \in \text{Ob } \mathfrak{F}(\mathfrak{a})$ and $A \in \text{Ob } \mathfrak{A}$. Then the complexes $\tau(F \otimes H \otimes P_*)$, $F \otimes \tau(H \otimes P_*)$, and $F \otimes H \otimes \tau P_*$ can be related as in (1) to give a triangular diagram of complexes:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & F \otimes \tau(H \otimes P_0) & \longrightarrow & F \otimes \tau(H \otimes A) & \longrightarrow & 0 \\
 & & \nearrow & & \nearrow & & \\
 F \otimes H \otimes \tau P_0 & \longrightarrow & F \otimes H \otimes \tau A & \longrightarrow & 0 & & \\
 & & \searrow & & \searrow & & \\
 \cdots & \longrightarrow & \tau(F \otimes H \otimes P_0) & \longrightarrow & \tau(F \otimes H \otimes A) & \longrightarrow & 0
 \end{array}$$

All the triangular diagrams are commutative by the assumption that τ is associative. Thus the induced triangular diagrams at the level of homology are commutative as well. This proves that \mathcal{F}^k is associative. The proof that \mathcal{F}^k is distributive is similar and we omit it. This completes the proof.

Remark. One can also show that if \mathcal{F} is natural in $\mathfrak{F}(\mathfrak{a})$ then the \mathcal{F}^k are also natural in $\mathfrak{F}(\mathfrak{a})$.

4.12 Theorem (Lifting Theorem for derived functors). *Suppose \mathfrak{a} equals \mathfrak{b} and let $\mathfrak{F}(\mathfrak{a})$ be a category of \mathfrak{a} -modules satisfying 3.1.1 and 3.1.2. Using the notation of the Lifting Theorem 4.6, assume \mathfrak{A} satisfies 4.9 and $\text{res}_{\mathfrak{a}}: \mathfrak{F}(\mathfrak{g}) \rightarrow \mathfrak{F}(\mathfrak{a})$. Suppose $\tau: \mathfrak{A} \rightarrow M_{\mathfrak{a}}$ is an $\mathfrak{F}(\mathfrak{a})$ -functor such that*

- (i) τ is compatible with the \mathfrak{a} -module action (cf. 4.4), i.e.,

$$\begin{array}{ccc}
 \mathfrak{a} \otimes \tau A & \xrightarrow{a_{\mathfrak{a}}} & \tau A \\
 i_{\mathfrak{a},A} \downarrow & \nearrow \tau(a_{\mathfrak{a}}) & \\
 \tau(\mathfrak{a} \otimes A) & &
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathfrak{a} \otimes \tau E & \xrightarrow{a_{\mathfrak{a}}} & \tau E \\
 i_{\mathfrak{a},E} \downarrow & & \uparrow \tau(a_{\mathfrak{a}}) \\
 \mathfrak{g} \otimes \tau E & \xrightarrow{i_{\mathfrak{g},E}} & \tau(\mathfrak{g} \otimes E)
 \end{array}$$

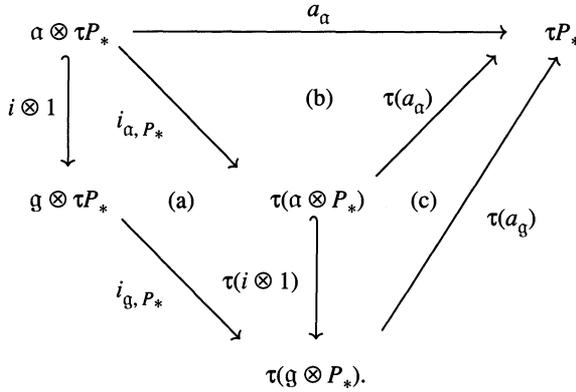
are commutative diagrams for $A \in \text{Ob } \mathfrak{A}$, $E \in \text{Ob } \mathfrak{E}$ (notation as in 4.6).

(ii) $J(\mathfrak{g}) = \text{Span}\{x \otimes y - y \otimes x - [x, y] | x, y \in \mathfrak{g}\}$ has an $\text{ad } \mathfrak{g}$ -stable complement K in $T_2(\mathfrak{g})$ and both $K, J(\mathfrak{g}) \in \text{Ob } \mathfrak{F}$.

(iii) \mathcal{S} is natural in $\mathfrak{F}(\mathfrak{a})$.

Then for each $k \in \mathbb{N}$, the derived functor τ^k admits a unique lifting $\bar{\tau}^k: \mathfrak{G} \rightarrow \mathfrak{H}$ which has a \mathfrak{g} -module action induced from τ^k (cf. 4.5).

Proof. We begin by checking that τ^k is compatible with the \mathfrak{a} -module action. Let $P_* \rightarrow A$ be a projective resolution in \mathfrak{A} with $A \in \text{Ob } \mathfrak{G}$. Diagram 4.4.1 is commutative by assumption. Starting with this diagram we consider the diagram



By naturality of \mathcal{S} in $\mathfrak{F}(\mathfrak{a})$ we conclude that (a) is commutative. By applying hypothesis (i)(1) with A replaced by each P_i we conclude that (b) is commutative. Diagram (c) is somewhat more delicate since the modules P_i are not \mathfrak{g} -modules. First a_g gives an \mathfrak{a} -module map $\mathfrak{g} \otimes A \rightarrow A$ and so lifts to a chain map which we denote by $a_g: \mathfrak{g} \otimes P_* \rightarrow P_*$. Now a_g and $a_g \circ i \otimes 1$ are two chain maps which agree on $\mathfrak{a} \otimes A \rightarrow A$. This shows that (c) induces a commutative diagram at the level of homology. Hence the outside diagram in (3) induces a commutative diagram

$$(4) \quad \begin{array}{ccc} \mathfrak{a} \otimes \tau^k A & \xrightarrow{a_\alpha} & \tau^k A \\ \downarrow & & \uparrow \tau^k(a_g) \\ \mathfrak{g} \otimes \tau^k A & \xrightarrow{i_{\mathfrak{g}, A}^k} & \tau^k(\mathfrak{g} \otimes A) \end{array}$$

This completes the proof that each τ^k is compatible with the \mathfrak{a} -module structure.

Now using 4.11 and (4) we may apply 4.6 to the functors τ^k . This completes the proof of 4.12.

4.13. In this last subsection we further assume in addition to 4.9 that every object in $\mathfrak{F}(\mathfrak{g})$ is of finite dimension and $F^* \in \mathfrak{F}(\mathfrak{g})$ for $F \in \mathfrak{F}(\mathfrak{g})$.

Proposition. Let \mathfrak{A} and \mathfrak{B} be two good full $\mathfrak{F} = \mathfrak{F}(\mathfrak{g})$ -subcategories of $M_\mathfrak{a}$ and $M_\mathfrak{b}$ respectively where $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are two subalgebras. Suppose $\eta = \eta_{A, M, B}: \text{Hom}_{\mathfrak{C}}(A, M \otimes B) \rightarrow \text{Hom}_{\mathfrak{C}}(A \otimes M^*, B)$ is an isomorphism for $M \in \mathfrak{F}(\mathfrak{g})$ and $\mathfrak{C} = \mathfrak{A}$ and $\mathfrak{C} = \mathfrak{B}$ (cf. 2.6 and 2.8). Let $H: \mathfrak{A} \rightarrow \mathfrak{B}$ be a covariant \mathfrak{F} -functor natural in \mathfrak{F} (cf. 3.4). If H has an additive left adjoint G , then G is also an \mathfrak{F} -functor natural in \mathfrak{F} .

Proof. First we construct natural equivalences $j_{F,A}: F \otimes GA \rightarrow G(F \otimes A)$ (here $F \in \text{Ob } \mathfrak{F}$, $A \in \text{Ob } \mathfrak{B}$) using Lemma 2.7 as follows. Let $B \in \text{Ob } \mathfrak{A}$ and consider the sequence of isomorphisms

$$\begin{aligned} \text{Hom}_{\mathfrak{A}}(F \otimes GA, B) &\cong \text{Hom}_{\mathfrak{A}}(GA, F^* \otimes B) \cong \text{Hom}_{\mathfrak{B}}(A, H(F^* \otimes B)) \\ &\cong \text{Hom}_{\mathfrak{B}}(A, F^* \otimes HB) \cong \text{Hom}_{\mathfrak{B}}(F \otimes A, HB) \\ &\cong \text{Hom}_{\mathfrak{A}}(G(F \otimes A), B). \end{aligned}$$

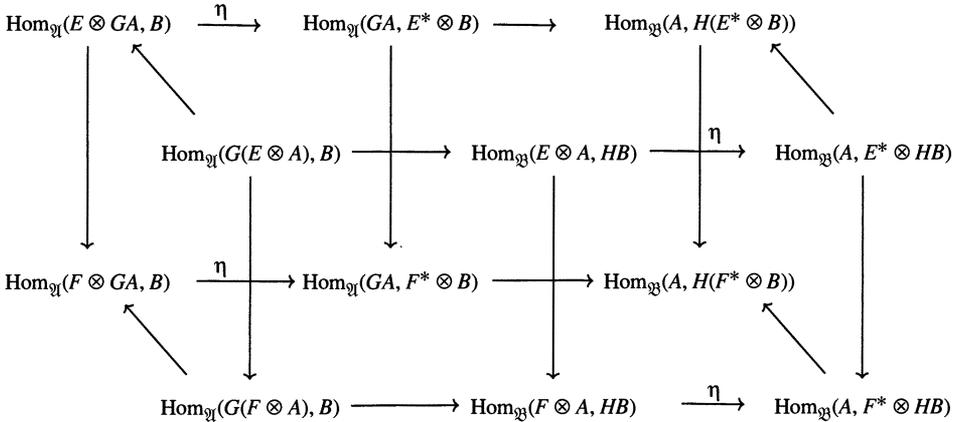
The third isomorphism is induced by the natural equivalence

$$i_{F^*,B}: H(F^* \otimes B) \rightarrow F^* \otimes H(B).$$

The first and fourth isomorphisms are induced by η (see 2.6). The other two isomorphisms are just a consequence of the definition of an adjoint functor. By Proposition 2.8 we have an induced natural equivalence

$$j_{F,A}: F \otimes G(A) \cong G(F \otimes A).$$

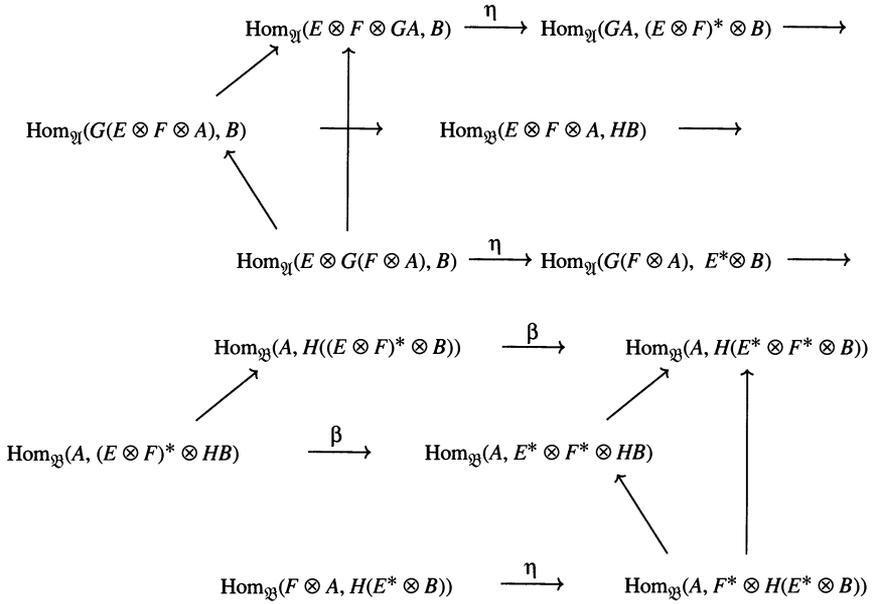
Next we check that $J = \{j_{F,A} | F \in \text{Ob } \mathfrak{F}(\mathfrak{g}), A \in \text{Ob } \mathfrak{A}\}$ is natural in $\mathfrak{F}(\mathfrak{g})$. With this goal in mind let $E, F \in \text{Ob } \mathfrak{A}$ and $f \in \text{Hom}_{\mathfrak{A}}(E, F)$ and consider the diagram



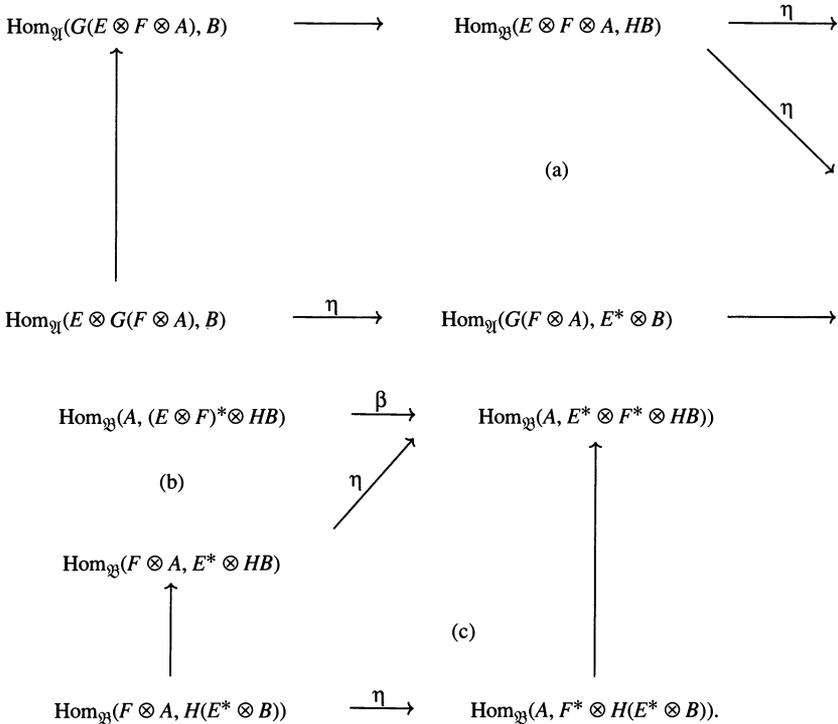
The left diagonal arrows are induced from J and the right diagonal arrows are induced from $I = \{i_{F,A} | F \in \text{Ob } \mathfrak{F}, A \in \text{Ob } \mathfrak{A}\}$. By the naturality of η (cf. Lemma 2.6) the left back square and the right front square are commutative. The left front square and the right back square are commutative as G is the left adjoint to H . The right side square is commutative as I is natural in \mathfrak{F} . The top and bottom faces are commutative by the definition of J . Hence the left side square is commutative. From this fact it easily follows that J is natural in \mathfrak{F} . Observe that this in particular implies by Lemma 3.6 that J is distributive.

Now we need only prove that J is associative. As before let $E, F \in \text{Ob } \mathfrak{F}$, $A \in \text{Ob } \mathfrak{B}$, and $B \in \text{Ob } \mathfrak{A}$. Then consider the following diagram:

(1)



The reader should view this as one large diagram with the bottom diagram placed to the right of the top diagram. The leftmost arrows on the top diagram are just induced by J while those on the right in the bottom diagram are induced by I . The top face of the whole diagram is commutative by the definition of J and as I is natural in \mathfrak{F} . The front face of the whole diagram can be written out as



The definition of J implies that diagram (a) is commutative. By Lemma 2.7 diagram (b) is commutative and (c) is commutative by the naturality of η .

The back face of diagram (1) can be written out as follows:

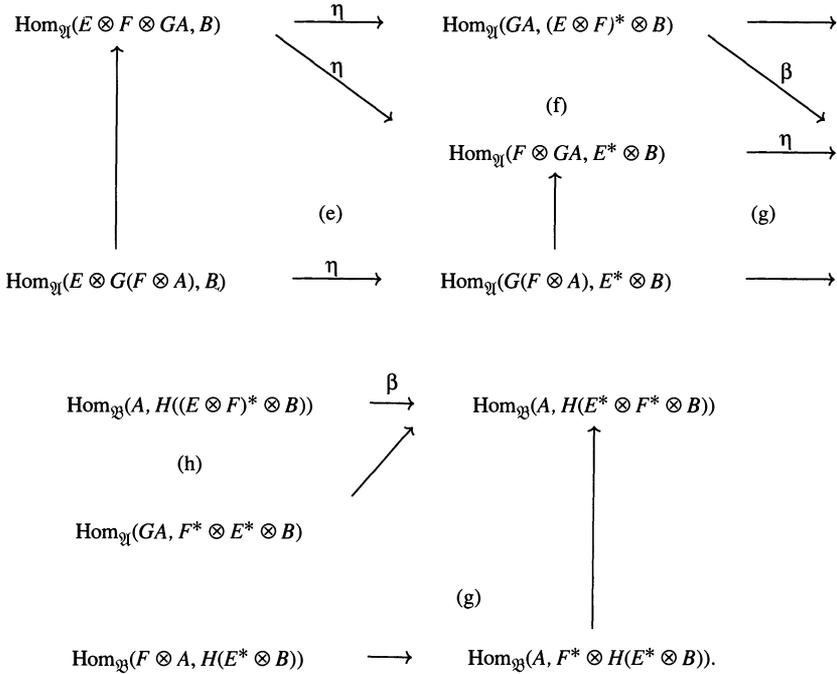


Diagram (e) is commutative as η is a natural equivalence of (f) is commutative by Lemma 2.7. (h) is commutative as G is the left adjoint to H . Finally the definition of J implies that (g) is commutative. This completes the proof of Proposition 4.13.

5. EXAMPLES OF \mathfrak{F} -CATEGORIES AND \mathfrak{F} -FUNCTORS

5.0. We now give a few more examples of \mathfrak{F} -categories and \mathfrak{F} -functors arising from the representation theory of Lie algebras. For more information on these various functors the reader is referred to the original articles.

5.1. We keep the notation of Examples 3.3 and 3.8 but specialize to the case where A is of finite type. Fix $\alpha \in \Pi$ and set $\mathfrak{p}_\alpha = \mathfrak{b} \oplus \mathbb{C}x_{-\alpha}$ where $x_{-\alpha} \in \mathfrak{g}_\alpha$ is nonzero. Let K (resp. K_α) denote the category of finite-dimensional $U(\mathfrak{b})$ (resp. $U(\mathfrak{p}_\alpha)$) modules. Joseph introduced and investigated functors $\mathcal{D}_\alpha: K \rightarrow K_\alpha$ defined as follows. For $F \in \text{Ob } K$, $\mathcal{D}_\alpha F$ is defined to be the largest $U(\mathfrak{p}_\alpha)$ -finite-dimensional quotient of the induced module $U(\mathfrak{p}_\alpha) \otimes_{U(\mathfrak{b})} F$. $\mathcal{D}_\alpha f$ for $f \in \text{Hom}_K(E, F)$ is defined to be the obvious induced map $\mathcal{D}_\alpha E \rightarrow \mathcal{D}_\alpha F$. Let \mathfrak{F} be the category of finite-dimensional \mathfrak{g} -modules. Then 3.1(i) and 3.1(ii) are satisfied and both K and K_α are \mathfrak{F} -categories. By [Jo3, Lemma 2.5] one has a natural equivalence $i_E: T_E \circ \mathcal{D}_\alpha \rightarrow \mathcal{D}_\alpha \circ T_E$ for each $E \in \text{Ob } \mathfrak{F}$. The family $\{i_E\}$ is then natural in \mathfrak{F} and associative (in fact this follows from 3.9). Hence \mathcal{D}_α is an \mathfrak{F} -functor.

5.2. In this next example we recall the notion, due to Joseph, of completion functors on the category \mathcal{O} . In this setting one can view the Enright completion functors as members of a larger family of completion functors. For a suitable choice of \mathfrak{F} and \mathfrak{F} -category we will see that these completion functors are actually \mathfrak{F} -functors.

First we fix some notation. Let $\mathfrak{g} = \mathfrak{g}(A)$ with A of finite type, and let $P(\Theta)$ be the lattice of integral weights. In this example fix $\lambda \in \mathfrak{h}^*$ dominant and regular and set $\Lambda = \lambda + P(\Theta)$. Let $M(\lambda)$ denote the Verma module of highest weight $\lambda - \rho$ (see 2.4 for notation), where $\rho = \frac{1}{2} \sum_{\alpha \in \Theta^+} \alpha$, and let $L(\lambda)$ be its unique irreducible quotient. In addition to being \mathfrak{h} -semisimple and $U(\mathfrak{b})$ -locally finite we assume in this section that the category \mathcal{O} is defined to have the added condition that all modules in \mathcal{O} are finitely generated as $U(\mathfrak{g})$ -modules. For $M \in \text{Ob } \mathcal{O}$ set $\omega(M) = \{\lambda \in \mathfrak{h}^* | M_\lambda \neq 0\}$ where M_λ is the λ th weight space of M . Let \mathcal{O}_Λ denote the full subcategory of \mathcal{O} whose objects are \mathfrak{g} -modules M with $\omega(M) \subset \Lambda$. Set $U = U(\mathfrak{g}) \otimes U(\mathfrak{g})$ and for \mathfrak{g} -modules M and N make $\text{Hom}(M, N)$ into a U -module by setting $l((a \otimes b)x)m = \sigma_c(\sigma(a))(x(\sigma(b)m))$ (see 2.6 for notation). Identify U with $U(\mathfrak{g} \times \mathfrak{g})$ via $x \otimes 1 + 1 \otimes x \rightarrow (x, x)$ as usual and let $j: \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$ be given by $j(x) = (x, \sigma(\sigma_c(x)))$. Set $\mathfrak{k} = j(\mathfrak{g})$ so that \mathfrak{k} is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{g}$. For $M, N \in \text{Ob } \mathcal{O}$, $L(M, N)$ will denote the space of \mathfrak{k} -finite elements of $\text{Hom}(M, N)$. For $N \in \text{Ob } \mathcal{O}_\Lambda$ set $C_N M = L(N, M) \otimes_{U(\mathfrak{g})} M(\lambda)$. One can then show that the Enright completion functor, C_α , agrees with the functor $C_{M(s_\alpha \lambda)}$ on $\mathbb{C}[x_{-\alpha}]$ -free modules in the category \mathcal{O}_Λ (see [Jo2, §2.12]).

Let now \mathcal{E} be the category of finite-dimensional \mathfrak{g} -modules. Then \mathcal{O}_Λ is an \mathcal{E} -category. Now Joseph defines a completion functors C to be a functor on \mathcal{O}_Λ such that the following three conditions hold:

- (1) C is covariant and left exact.
- (2) For each $E \in \text{Ob } \mathcal{E}$ and $N, M \in \text{Ob } \mathcal{O}_\Lambda$ there exists an isomorphism $\varphi_{M,E}$ such that the following diagrams are commutative:

(a)

$$\begin{array}{ccc} C(E \otimes M) & \xrightarrow{\varphi_{M,E}} & E \otimes CM \\ M \mapsto N \downarrow & & \downarrow M \mapsto N \\ C(E \otimes N) & \xrightarrow{\varphi_{N,E}} & E \otimes CN, \end{array}$$

(b)

$$\begin{array}{ccc} C(E \otimes F \otimes M) & \xrightarrow{\varphi_{F \otimes M, E}} & E \otimes C(F \otimes M) \\ & \searrow \varphi_{M, E \otimes F} & \downarrow \varphi_{M, F} \\ & & E \otimes F \otimes CM \end{array}$$

for all $F \in \text{Ob } \mathcal{E}$, and

(c)

$$\begin{array}{ccc} C(E \otimes E^* \otimes M) & \xrightarrow{\varphi_{M, E \otimes E^*}} & E \otimes E^* \otimes CM \\ & \searrow e \otimes l^* \mapsto l^*(e) & \downarrow e \otimes l^* \mapsto l^*(e) \\ & & CM. \end{array}$$

- (3) For any ad \mathfrak{g} -submodule E of $U(\mathfrak{g})$ with $E \in \text{Ob } \mathcal{E}$ the following diagram is commutative:

$$\begin{array}{ccc}
 C(E \otimes M) & & \\
 \downarrow \varphi_{M,E} & \searrow e^{\otimes m} \rightarrow em & \\
 & CM & \\
 & \nearrow e^{\otimes m} \rightarrow em & \\
 E \otimes CM & &
 \end{array}$$

Joseph goes on to prove that for any $N \in \text{Ob } \mathcal{O}_\Lambda$, C_N is a completion functor and that all completion functors are of this form (see [Jo2, Proposition 4.4]). By (2) $\{\varphi_E | E \in \text{Ob } \mathcal{E}\}$ is associative and distributive. Hence C_N is an \mathfrak{F} -functor for all $N \in \text{Ob } \mathcal{O}_\Lambda$. Conversely, for the category \mathcal{O}_Λ and $\mathfrak{F} = \mathcal{E}$, any covariant left exact \mathfrak{F} -functor satisfying 2(c) and 3 must be a completion functor C_N for some module $N \in \text{Ob } \mathcal{O}$.

5.3. The last example we cover is the Zuckerman functor. Let $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ be subalgebras of a finite-dimensional Lie algebra \mathfrak{g} over a field k of characteristic zero. For the remainder of this section assume \mathfrak{h} is reductive and reductive in \mathfrak{k} and \mathfrak{g} . If $\mathfrak{s} \subset \mathfrak{t}$ are Lie algebras over k let $\mathcal{C}(\mathfrak{t}, \mathfrak{s})$ denote the full subcategory of $M_{\mathfrak{t}}$ whose objects are $U(\mathfrak{s})$ -locally finite and semisimple as \mathfrak{s} -modules. For X an object of $\mathcal{C}(\mathfrak{k}, \mathfrak{h})$ let $\Gamma(X)$ be the maximal $U(\mathfrak{k})$ -locally finite \mathfrak{k} -semisimple \mathfrak{k} -submodule of X and if $f: X \rightarrow Y$ for objects X, Y in $\mathcal{C}(\mathfrak{k}, \mathfrak{h})$, let $T(f) = f|_{\Gamma(X)} \in \text{Hom}_{\mathfrak{k}}(\Gamma(X), \Gamma(Y))$. Then one has a functor $\Gamma: \mathcal{C}(\mathfrak{k}, \mathfrak{h}) \rightarrow \mathcal{C}(\mathfrak{k}, \mathfrak{k})$ given by $X \mapsto \Gamma(X)$ and $f \mapsto \Gamma(f)$ for X an object of $\mathcal{C}(\mathfrak{k}, \mathfrak{h})$ and f a morphism in $\mathcal{C}(\mathfrak{k}, \mathfrak{h})$. Γ is by definition the Zuckerman \mathfrak{k} -finite functor. If one takes $\mathfrak{F} = \mathcal{C}(\mathfrak{k}, \mathfrak{k})$ it is easy to check that $\mathcal{C}(\mathfrak{k}, \mathfrak{h})$ and $\mathcal{C}(\mathfrak{k}, \mathfrak{k})$ are \mathfrak{F} -categories and that Γ is an \mathfrak{F} -functor (see [EW, Lemmas 3.3 and 3.4] and [ES, Proposition 5.2]). Now we take $\mathfrak{a} = \mathfrak{k}$, $\mathcal{O} = \mathcal{C}(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{A} = \mathcal{C}(\mathfrak{k}, \mathfrak{h})$, and $\mathfrak{B} = \mathcal{C}(\mathfrak{k}, \mathfrak{k})$. Let $\text{res}_{\mathfrak{k}}: M_{\mathfrak{g}} \rightarrow M_{\mathfrak{k}}$ denote the forgetful functor so that $\mathfrak{H} = \text{res}_{\mathfrak{k}}^{-1} \mathcal{C}(\mathfrak{k}, \mathfrak{k}) = \mathcal{C}(\mathfrak{g}, \mathfrak{k})$. One can check that the hypotheses of the Lifting Theorem are satisfied so that we have a functor $\bar{\Gamma}: \mathcal{C}(\mathfrak{g}, \mathfrak{h}) \rightarrow \mathcal{C}(\mathfrak{g}, \mathfrak{k})$ such that the following diagram is commutative:

$$\begin{array}{ccc}
 \mathcal{C}(\mathfrak{g}, \mathfrak{h}) & \xrightarrow{\bar{\Gamma}} & \mathcal{C}(\mathfrak{g}, \mathfrak{k}) \\
 \text{res}_{\mathfrak{k}} \downarrow & & \downarrow \text{res}_{\mathfrak{k}} \\
 \mathcal{C}(\mathfrak{k}, \mathfrak{h}) & \xrightarrow{\Gamma} & \mathcal{C}(\mathfrak{k}, \mathfrak{k})
 \end{array}$$

One can also check in this case that the hypotheses of 4.12 holds so that the right derived functors of Γ are \mathfrak{F} -functors by Lemma 4.12 and for A a \mathfrak{g} -module, $\Gamma^j(A)$ has a \mathfrak{g} -module structure for $j \geq 0$. For work on the Zuckerman functors see [EW, Lemmas 3.3 and 3.4; ES, Proposition 5.2; and Wa, Chapter 6].

Remark. The connection between \mathfrak{F} -functors and localization contains some interesting results which will be discussed in a forthcoming paper.

REFERENCES

- [D] V. Deodhar, *On a construction of representations and a problem of Enright*, *Invent. Math.* **57** (1908), 101–118.
- [E] T. J. Enright, *On the fundamental series of a real semisimple Lie algebra and their irreducibility, resolution and multiplicity formulae*, *Ann. of Math. (2)* **110** (1979), 1–82.
- [ES] T. J. Enright and B. Shelton, *Categories of highest weight modules; application to classical Hermitian symmetric pairs*, *Mem. Amer. Math. Soc.* **67** (1987).
- [EW] T. J. Enright and N. R. Wallach, *Notes homological algebra and representations of Lie algebras*, *Duke Math. J.* **47** (1980), 1–15.
- [GJ] O. Gabber and A. Joseph, *On the Bernstein-Gelfand-Gelfand resolution and the Duflo sum formula*, *Compositio Math.* **43** (1981), 107–131.
- [HS] P. Hilton and U. Stambach, *A course in homological algebra*, Springer-Verlag, Berlin and New York, 1971.
- [Jo1] A. Joseph, *The Enright functor in the Bernstein-Gelfand-Gelfand category \mathcal{O}* , *Invent. Math.* **57** (1980), 101–118.
- [Jo2] —, *Completion functors in the category \mathcal{O}* , *Lecture Notes in Math.*, vol. 1020, Springer-Verlag, Berlin and New York, 1982, pp. 80–106.
- [Jo3] —, *On the Demazure character formula*, *Ann. Sci. Ecole Norm. Sup.* **18** (1985), 389–419.
- [Ka] V. Kac, *Infinite dimensional Lie algebras*, Cambridge Univ. Press, London and New York, 1985.
- [Ki] W. Kim, *On the concept of completions and explicit determination of invariant pairing for general Kac-Moody Lie algebras*, *Comm. Algebra* **16** (1988), 185–222.
- [Kn] A. Knapp, *Lie groups, Lie algebras, and cohomology*, *Math. Notes*, vol. 34, Princeton Univ. Press, Princeton, NJ, 1988.
- [RW] A. Rocha-Caridi and N. Wallach, *Projective modules over graded Lie algebras. I*, *Invent. Math.* **180** (1982), 151–177.
- [Wa] N. Wallach, *Real reductive groups I*, Academic Press, San Diego, Calif., 1988.
- [Z] G. Zuckerman, *Tensor products of finite and infinite dimensional representations of semi-simple Lie groups*, *Ann. of Math. (2)* **106** (1972), 295–308.