Thus, the general solution is

\( y = c_1 e^t + c_2 e^{t/6}. \)

The characteristic equation is

\( y'' + 1/y' + t^2 - n^2/t^2 y = 0, \)

so that \( p(t) = 1/t. \) The differential equation satisfied by the Wronskian of two solutions is thus

\[ W' + p(t)W = W' + \frac{1}{t}W = 0. \]

Then \( \int \frac{dW}{W} = -\int \frac{dt}{t} \) and \( W(t) = Ct^{-1} \) for some constant \( t. \) Using the values given, we know that \( W(1) = y_1(1)y_2'(1) - y_2(1)y_1'(1) = 1, \) so that \( W(t) = t^{-1}. \)

The initial values imply that \( c_1 + c_2 = 1 \) and \( 4c_1 - c_2 = 0. \) These have solution \( c_1 = 1/5, \) \( c_2 = 4/5, \) so

\[ y(t) = (e^{4t} + 4e^{-t})/5 \]

solves the initial value problem.
#6: The characteristic equation is $2r^2 + r - 10 = 0$, which has roots $r = 4$ and $r = -5$. Thus, the general solution is $y = c_1 e^{4t} + c_2 e^{-5t}$ with $y' = 4c_1 e^{4t} - 5c_2 e^{-5t}$. The initial values imply that $c_1 e^4 + c_2 e^{-5} = 5$ and $4c_1 e^4 - 5c_2 e^{-5} = 2$. Thus, $c_1 e^4 = 3$ and $c_2 e^{-5} = 2$, and $y = 3e^{4(t-1)} + 2e^{-5(t-1)}$.

#10: Substituting $y = t^r$ into the differential equation $t^2 y'' + \alpha ty' + \beta y = 0$ gives

$$(r(r-1) + \alpha r + \beta)r' = 0.$$ 

Thus, $y = t^r$ is a solution if and only if $r^2 + (\alpha - 1)r + \beta r = 0$.

#12: Comparing with Exercise 10, the differential equation $t^2 y'' - ty' - 2y = 0$ has $\alpha = -1$ and $\beta = -2$. Thus, $y = t^r$ is a solution if and only if $r^2 - 2r - 2 = 0$. The roots of this equation are $r = -1 \pm \sqrt{3}$. The general solution is

$$y = c_1 t^{1+\sqrt{3}} + c_2 t^{1-\sqrt{3}},$$

with

$$y' = (1 + \sqrt{3})c_1 t^{\sqrt{3}} + (1 - \sqrt{3})c_2 t^{-\sqrt{3}}.$$ 

The initial value $y(1) = 0$ implies that $c_1 + c_2 = 0$, so $c_2 = -c_1$. Then $1 = y'(1) = 2\sqrt{3}c_1$. Thus,

$$y = \frac{t^{1+\sqrt{3}} - t^{1-\sqrt{3}}}{2\sqrt{3}}.$$ 

p.144 #3: For $y'' + 2y' + 3y = 0$, the characteristic equation is $r^2 + 2r + 3 = 0$, which has roots $r = -1 \pm \sqrt{2}$. The general solution is thus

$$y = c_1 e^{-t} \cos(\sqrt{2} t) + c_2 e^{-t} \sin(\sqrt{2} t).$$

#6: The characteristic equation is $r^2 + 2r + 5 = 0$, which has roots $r = -1 \pm 2i$. The general solution is

$$y = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$$

with

$$y' = e^{-t} ((2c_2 - c_1) \cos(2t) + (-2c_1 - c_2) \sin(2t)).$$

The initial values imply that $c_1 = 0$ and $2c_2 - c_1 = 2$, so $c_2 = 1$ and the solution is

$$y = e^{-t} \sin(2t).$$

#10:

$$W[e^{at} \cos \beta t, e^{at} \sin \beta t] = \begin{vmatrix} e^{at} \cos \beta t & e^{at} \sin \beta t \\ e^{at}(\alpha \cos \beta t - \beta \sin \beta t) & e^{at}(\alpha \sin \beta t + \beta \cos \beta t) \end{vmatrix} = e^{2at} \cos \beta t(\alpha \sin \beta t + \beta \cos \beta t) - e^{2at} \sin \beta t(\alpha \cos \beta t - \beta \sin \beta t) = \beta e^{2at}(\cos^2 \beta t + \sin^2 \beta t) = \beta e^{2at}$$
The characteristic equation is \(4r^2 - 12r + 9 = 0\), which has repeated root \(r = 3/2\). Thus, the general solution of the differential equation is

\[ y = e^{3t/2}(c_1 + c_2 t). \]

The characteristic equation is \(r^2 + 2r + 1 = 0\), which has repeated root \(r = -1\). The general solution is

\[ y = e^{-t}(c_1 + c_2 t) \]

with

\[ y' = e^{-t}(c_2 - c_1 - c_2 t). \]

The initial values \(y(2) = 1\), \(y'(2) = -1\) imply that

\[ c_1 + 2c_2 = e^2, \quad -c_1 - c_2 = -e^2. \]

This means that \(c_2 = 0\) and \(c_1 = e^2\). Thus, \(y = e^{2-t}\).