College of Charleston
Department of Mathematics

Name:

Examination in: Introductory Calculus

Math Course Number	Math 120
Examination Date	5-2-2010
Examination Time	12:15-1:30/1:40-2:55

Total number of problems: 7

Professor: Ben Cox

Proctor: Ben Cox **Phone number:** 953-5715

Results available by: Nov. 14 **in:** Maybank 117

Permitted aids: Proctor, TI-83 or TI-86 or equivalent calculator.

Show all work to receive full credit.

	Score
1	
2	
3	
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5	
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7	
Total	

1. Using logarithmic differentiation find the derivatives of the following functions:

a) $y = \left(3 + 3\,x + x^7\right)^{2+2\,x+x^2}$,

Sln:

$$\ln y = \ln\left(3 + 3\,x + x^7\right)^{2+2\,x+x^2} = (2 + 2\,x + x^2)\ln\left(3 + 3\,x + x^7\right)$$

Using the chain rule on the left and the product rule together with the chain rule on the right, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\ln y &= \frac{d}{dx}\left((2 + 2\,x + x^2)\ln\left(3 + 3\,x + x^7\right)\right)\\ &= (2 + 2x)\ln\left(3 + 3\,x + x^7\right) + (2 + 2\,x + x^2)\frac{1}{3 + 3\,x + x^7}\frac{d}{dx}\left(3 + 3\,x + x^7\right)\\ &= (2 + 2x)\ln\left(3 + 3\,x + x^7\right) + (2 + 2\,x + x^2)\frac{3 + 7x^6}{3 + 3\,x + x^7}\end{aligned}$$

and thus

$$\frac{dy}{dx} = \left(3 + 3\,x + x^7\right)^{2+2\,x+x^2}\left((2 + 2x)\ln\left(3 + 3\,x + x^7\right) + (2 + 2\,x + x^2)\frac{3 + 7x^6}{3 + 3\,x + x^7}\right)$$

b) $y = \sqrt[7]{\dfrac{3 + 2\,x + x^2}{(1 + x^5)^3}}$,

$$\ln y = \ln\sqrt[7]{\frac{3 + 2\,x + x^2}{(1 + x^5)^3}} = \frac{1}{7}\left(\ln(3 + 2\,x + x^2) - 3\ln(1 + x^5)\right)$$

Using the chain rule on the left and the product rule together with the chain rule on the right, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\ln y &= \frac{d}{dx}\frac{1}{7}\left(\ln(3 + 2\,x + x^2) - 3\ln(1 + x^5)\right)\\ &= \frac{1}{7}\left(\frac{2 + 2x}{3 + 2\,x + x^2} - 3\frac{5x^4}{1 + x^5}\right)\\ &= \frac{1}{7}\left(\frac{2 + 2x}{3 + 2\,x + x^2} - \frac{15x^4}{1 + x^5}\right)\end{aligned}$$

and thus

$$\frac{dy}{dx} = \frac{1}{7}\sqrt[7]{\frac{3 + 2\,x + x^2}{(1 + x^5)^3}}\left(\frac{2 + 2x}{3 + 2\,x + x^2} - \frac{15x^4}{1 + x^5}\right).$$

2. A kite 50 meters above the ground is drifting horizontally at a speed of 2 meters per second. At what rate is the angle between the horizontal and the string decreasing when the string has been let out 150 meters?

Ans: Let x denote the horizontal distance from the person holding the kite to a position directly below the kite and let h denote the distance from the person to the kite. Let θ denote the angle between the horizontal and the string. Then $\tan\theta = 50/x$ and $dx/dt = 2$. Now

$$\sec^2\theta\frac{d\theta}{dt} = \frac{d}{dt}\tan\theta = \frac{d}{dt}\left(\frac{50}{x}\right) = -\frac{50}{x^2}\frac{dx}{dt} = -\frac{100}{x^2}.$$

When $h = 150$, $x = \sqrt{150^2 - 50^2} = \sqrt{(3^2 - 1)50^2}$ so that $x = 100\sqrt{2}$, $\sec\theta = h/x = 150/100\sqrt{2} = \frac{3}{2\sqrt{2}}$. Hence when $h = 150$,

$$\frac{9}{8}\frac{d\theta}{dt} = -\frac{100}{2\times 100^2}, \qquad \text{or} \qquad \frac{d\theta}{dt} = -\frac{4}{900} = -\frac{1}{225}.$$

3. Find the first and second order derivatives of the following functions

a) $f(t) = 7t^3 - 3t^2 + 1$.

Sln: $f'(t) = 21t^2 - 6t$. $f''(t) = 42t - 6$.

b) $f(\theta) = \cos 3\theta$.

Sln: $f'(\theta) = -3\sin 3\theta$. $f''(\theta) = -9\cos 3\theta$.

4. Find the equation of the tangent line to the curve

$$(4x^2 + y^2 - 4)(x^2 + 4y^2 - 4) = 1$$

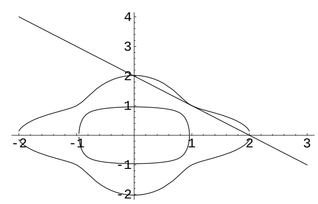
at $(x, y) = (1, 1)$.

Ans: We differentiate with respect to x:

$$0 = \frac{d}{dx}(1) = \frac{d}{dx}\left((4x^2 + y^2 - 4)(x^2 + 4y^2 - 4)\right)$$
$$= \left(8x + 2y\frac{dy}{dx}\right)(x^2 + 4y^2 - 4) + (4x^2 + y^2 - 4)\left(2x + 8y\frac{dy}{dx}\right).$$

When $(x, y) = (1, 1)$ we get

$$0 = \left(8 + 2\frac{dy}{dx}\right) + \left(2 + 8\frac{dy}{dx}\right) = 10 + 10\frac{dy}{dx}.$$



Thus $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = -1$. The equation for the tangent line is $y - 1 = -(x - 1)$ or $y = -x + 2$. The graph of the curve together with the tangent line $y = -x + 2$ is given to the right.

5. Find the absolute maximum and minimum of the function $f(x) = \dfrac{-x - 1}{x^2 + 3\,x + 3}$ on the interval $[-4, 1]$.

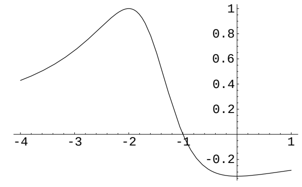
Ans. Observe that $f(x)$ is continuous on the interval $[-4, 1]$ as the only place where it could possibly not be continuous is where the denominator $x^2 + 3\,x + 3 = 0$. But the only roots of $x^2 + 3\,x + 3 = 0$ are $x = \dfrac{-3 \pm \sqrt{9 - 12}}{2}$ which are complex. Thus the function $f(x)$ is always continuous. First we find the critical points:

$$f'(x) = \frac{-(x^2 + 3\,x + 3) - (-x - 1)(2x + 3)}{(x^2 + 3\,x + 3)^2}$$
$$= \frac{-x^2 - 3\,x - 3 + 2x^2 + 5x + 3}{(x^2 + 3\,x + 3)^2}$$
$$= \frac{x^2 + 2\,x}{(x^2 + 3\,x + 3)^2}$$

Now $f'(c) = 0$ implies $c^2 + 2\,c = 0$, which means $c = 0$ or $c = -2$ are critical numbers. $f'(c)$ is undefined when $c^2 + 3\,c + 3 = 0$ but we know from the above that this is never zero for c in the domain of $f(x)$. Hence the derivative is defined for all real x.

To find the absolute maximum we need to calculate

$$f(-4) = 3/(16 - 12 + 3) = 3/7, \quad f(-2) = 1/(4 - 6 + 3) = 1, \quad f(0) = -1/3, \quad f(1) = -2/7.$$

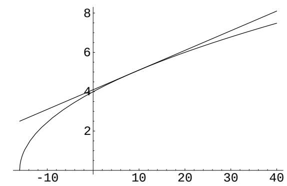


The absolute maximum value of $f(x)$ is $f(-2) = 1$ and the absolute minimum value is $f(0) = -1/3$ as $-1/3 < -2/7$ (since $-7 < -6$). The plot of the graph of $f(x)$ is given to the right.

6. Find the linearization of $f(x) = \sqrt{16 + x}$ at $x = 9$ and use it to approximate $\sqrt{27}$.

The linearization of $f(x)$ at $x = 9$ is $L(x) = f(9) + f'(9)(x - 9)$. Since $f'(x) = \frac{1}{2}(16 + x)^{-1/2}$ we get

$$L(x) = \sqrt{25} + \frac{1}{2\sqrt{25}}(x - 9) = 5 + \frac{1}{10}(x - 9).$$



Now

$$\sqrt{27} = f(11) \cong L(11) = 5 + \frac{1}{10}(11 - 9) = 5.20.$$

The graph of $f(x) = \sqrt{x + 16}$ together with the linear approximation $L(x)$ are given to the right.

7. Suppose that $f'(x) \le 3$ for all x and $f(1) = 2$. How large can $f(5)$ possibly be?

Since $f(x)$ is differentiable everywhere, it is also continuous everywhere and hence the Mean Value Theorem applies to any interval $[a, b]$. We choose $a = 1$ and $b = 5$ as these are numbers that $f(x)$ is evaluated at in the problem. The conclusion of the Mean Value Theorem says that there is a c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(5) - f(1)}{5 - 1} = \frac{f(5) - 2}{4}.$$

Now by the hypothesis of the problem we also know that $f'(c) \le 3$ so that

$$\frac{f(5) - 2}{4} \le 3 \quad \Longrightarrow \quad f(5) - 2 \le 12$$
$$\Longrightarrow \quad f(5) \le 14.$$