College of Charleston
Department of Mathematics

Name:

Examination in:

Math Course Number	
Examination Date	
Examination Time	

Total number of problems:

Professor:

Proctor: **Phone number:**

Results available by: **in:**

Permitted aids:

Show all work to receive full credit.

	Score
1	
2	
3	
4	
5	

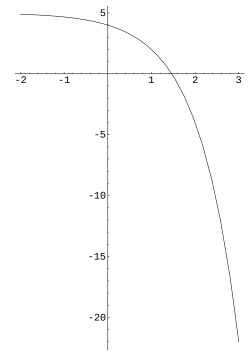
	Score
6	
7	
8	
9	
Total	

1. Write down a table that gives values of

$$f(x) = \frac{\tan((\pi/3)+x) - \tan(\pi/3)}{x}$$

for x near zero. Use this table to guess the value of

$$\lim_{x\to 0} \frac{\tan((\pi/3)+x) - \tan(\pi/3)}{x}.$$



x	$(\tan((\pi/3)+x) - \tan(\pi/3))/x$
-0.001	3.993085104748051
-0.0009	3.9937753986054836
-0.0008	3.9944659577623165
-0.0007	3.995156782370391
-0.0006	3.995847872581004
-0.0005	3.9965392285461476
-0.0004	3.997230850417144
-0.0003	3.997922738346501
-0.0002	3.9986148924853193
-0.0001	3.9993073129873666
0.0001	4.00069295367844
0.0002	4.001386174180554
0.0003	4.002079661653557
0.0004	4.002773416251214
0.0005	4.003467438125963
0.0006	4.004161727430381
0.0007	4.004856284318681
0.0008	4.005551108943184
0.0009	4.006246201457348
0.001	4.006941562014754

The table on the right indicates that the limit should be approximately equal to 4. You can also see this in the graph on the left (the point $(0,4)$ should be omitted from the graph).

2. Find

a) $\displaystyle\lim_{t\to 2} \frac{t^2-5t+6}{t-2}$, **Solution:** $\displaystyle\lim_{t\to 2} \frac{t^2-5t+6}{t-2} = \lim_{t\to 2} t-3 = -1$

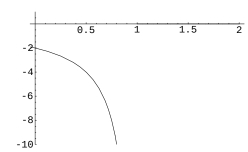
b) $\displaystyle\lim_{z\to 0} \frac{\sqrt{z^2+16}-4}{z}$,

Solution:

$$\begin{aligned}
\lim_{z\to 0} \frac{\sqrt{z^2+16}-4}{z} &= \lim_{z\to 0} \left(\frac{\sqrt{z^2+16}-4}{z}\right)\left(\frac{\sqrt{z^2+16}+4}{\sqrt{z^2+16}+4}\right) \\
&= \lim_{z\to 0} \frac{z^2+16-16}{z(\sqrt{z^2+16}+4)} \\
&= \lim_{z\to 0} \frac{z}{\sqrt{z^2+16}+4} \\
&= \frac{\lim_{z\to 0} z}{\sqrt{\lim_{z\to 0} z^2+16}+4} = 0
\end{aligned}$$

c) $\displaystyle\lim_{w\to 0} \left(w^4 + \frac{\cos(4w)}{10,000}\right)$.

Solution: $\displaystyle\lim_{w\to 0} \left(w^4 + \frac{\cos(4w)}{10,000}\right) = \lim_{w\to 0} w^4 + \lim_{w\to 0} \frac{\cos(4w)}{10,000} = 0 + 10^{-4} = 10^{-4}$, as $\cos w$ is continuous and $\lim_{w\to 0}\cos(4w) = \cos(\lim_{w\to 0} 4w) = \cos 0 = 1$.



d) $\displaystyle\lim_{x\to 1^-} \left(\frac{1}{x-1} - \frac{1}{|x-1|}\right)$,

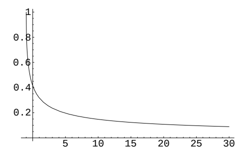
Solution: Since $|x-1| = -(x-1)$ for $x < 1$ we get

$$\begin{aligned}
\lim_{x\to 1^-} \left(\frac{1}{x-1} - \frac{1}{|x-1|}\right) &= \lim_{x\to 1^-} \left(\frac{1}{x-1} - \frac{1}{-(x-1)}\right) \\
&= \lim_{x\to 1^-} \frac{2}{x-1} = -\infty.
\end{aligned}$$

e) $\displaystyle\lim_{x\to\infty} \left(\sqrt{2+x} - \sqrt{1+x}\right)$,

Solution:

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{2+x}-\sqrt{1+x}\right) &= \lim_{x\to\infty}\left(\sqrt{2+x}-\sqrt{1+x}\right)\frac{\left(\sqrt{2+x}+\sqrt{1+x}\right)}{\left(\sqrt{2+x}+\sqrt{1+x}\right)} \\ &= \lim_{x\to\infty}\frac{2+x-(1+x)}{\sqrt{2+x}+\sqrt{1+x}} \\ &= \lim_{x\to\infty}\frac{1}{\sqrt{2+x}+\sqrt{1+x}} = 0.\end{aligned}$$



3. Let

$$f(x) = \begin{cases} x/(x^2-x-2) & \text{for} \quad x<1 \\ 3 & \text{for} \quad x\geq 1.\end{cases}$$

a) Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

b) Is $f(x)$ continuous at $x=1$?

Solution: a).

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x/(x^2-x-2) = 1/(1-1-2) = -1/2.$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3 = 3.$$

b) Since $\lim_{x\to 1^-} f(x) = -1/2 \neq 3 = \lim_{x\to 1^+} f(x)$ we get that $\lim_{x\to 1^-} f(x)$ does not exist and hence $f(x)$ is not continuous at $x=1$

4. Find a $\delta > 0$ such that

$$\left|\frac{x^2+x-2}{2x-2}-\frac{3}{2}\right| < 0.2$$

whenever $|x-1|<\delta$.

Solution: We rewrite the left hand side of the above inequality as

$$\left|\frac{x^2+x-2}{2x-2}-\frac{3}{2}\right| = \left|\frac{(x-1)(x+2)}{2(x-1)}-\frac{3}{2}\right| = \left|\frac{x+2}{2}-\frac{3}{2}\right| = \frac{1}{2}\left|x-1\right|.$$

We would like this to be less than 0.2, so we should take $\delta = 2\cdot 0.2 = 0.4$. Let $\Longrightarrow$ mean "implies". Then

$$\begin{aligned}|x-1| < \delta = 0.4 \quad &\Longrightarrow \quad \frac{1}{2}\left|x-1\right| < 0.2 \\ &\Longrightarrow \quad \left|\frac{x+2}{2}-\frac{3}{2}\right| < 0.2 \\ &\Longrightarrow \quad \left|\frac{(x-1)(x+2)}{2(x-1)}-\frac{3}{2}\right| < 0.2 \\ &\Longrightarrow \quad \left|\frac{x^2+x-2}{2x-2}-\frac{3}{2}\right| < 0.2\end{aligned}$$

which was to be proven.

5. Where are the following functions continuous?

a) $f(t) = \dfrac{1}{t^2 - 1}$,

Solution: This rational function is continuous where ever it is defined. Since it is defined everywhere except at $x = \pm 1$, it is continuous everywhere except at $x = \pm 1$.

b) $f(x) = \begin{cases} 1 - x^2 & \text{if} \quad x \le 2, \\ x^2 - 2x & \text{if} \quad x > 2. \end{cases}$

Solution: Since polynomials are continuous, the only place that this function might not be continuous at is at $x = 2$. We check

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (1 - x^2) = 1 - 2^2 = -3 \qquad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 - 2x) = 2^2 - 2^2 = 0.$$

Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$ the two sided limit does not exist and $f(x)$ is not continuous at $x = 2$.

6. Show that there exists a number c such that

$$c^3 = \sqrt{c + 1}.$$

Hint: Consider the function $f(x) = x^3 - \sqrt{x + 1}$.

Solution: Here we use the Intermediate Value Theorem. We evaluate

$$f(-1) = (-1)^3 - \sqrt{-1 + 1} = -1, \quad f(8) = 8^3 - \sqrt{8 + 1} = 512 - 3 = 509$$

and observe that $f(x)$ is continuous on the interval $[-1, 8]$. Since $f(-1) < 0 < f(8)$, the Intermediate Value Theorem tells us that there exists a c in the interval $[-1, 8]$ such that $f(c) = 0$.

7. Find the derivative of $f(z) = \sqrt{7}z + \sqrt[3]{7z}$.

Solution:

$$f'(z) = \sqrt{7} + \frac{\sqrt[3]{7}}{3\sqrt[3]{z^2}}$$

8. The point $P(-1, 1)$ lies on the curve $y = 1/(x + 2)$.

a) The point $Q(2, 1/4)$ is also on the curve. Find the slope of the secant line through these two points.

Solution: The slope of the secant line is

$$\frac{(1/4) - 1}{2 - (-1)} = \frac{-3/4}{3} = -1/4$$

b) Suppose now $R(x, 1/(x + 2))$ is another point on the curve. Find the slope of the secant line through R and P.

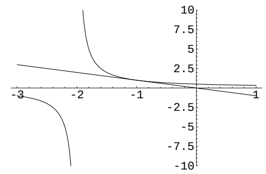
Solution: The slope of the secant line is

$$\frac{\frac{1}{x+2} - 1}{x - (-1)} = \frac{\frac{-x-1}{x+2}}{x + 1} = -\frac{1}{x + 2}$$

c) Find the slope of the tangent line to the curve $y = 1/(x+2)$ at the point $P(-1, 1)$.

The slope of the tangent line at $P(-1, 1)$ is

$$\lim_{x\to -1} \frac{\frac{1}{x+2} - 1}{x - (-1)} = -\lim_{x\to -1} \frac{1}{x+2} = -1$$



9. Using only the definition of the derivative and algebraic methods for evaluating limits, find $f(x) = x^2 - 3x + 1$.

Sln:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - 3(x+h) + 1 - \left(x^2 - 3x + 1\right)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 3x + 3h + 1 - x^2 + 3x - 1}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} \\ &= \lim_{h\to 0} \frac{h\,(2x + h - 3)}{h} \\ &= \lim_{h\to 0} (2x + h - 3) = 2x - 3. \end{aligned}$$