College of Charleston
Department of Mathematics

Name:

Examination in: Introduction to Calculus

Math Course Number	Math 120
Examination Date	9-21-10
Examination Time	12:15-1:30/1:40-2:55

Total number of problems: 7

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Proctor: Ben Cox

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Results available by:

in: Maybank 117

Permitted aids: Proctor, Calculator equivalent to a TI-83, 84, 85, or 86 on problem 7.

Show all work to receive full credit.

	Score
1	
2	
3	
4	
5	
7	
Total	

1. Find

a) $\lim_{x\to 2} x^3 - x^2 + 1$, **Solution:** $\lim_{x\to 2} x^3 - x^2 + 1 = 2^3 - 2^2 + 1 = 8 - 4 + 1 = 5.$

b) $\lim_{z\to -7} \dfrac{2z^2 + 11z - 21}{z+7}$, **Solution:** $\lim_{z\to -7} \dfrac{2z^2 + 11z - 21}{z+7} = \lim_{z\to -7} \dfrac{(2z-3)(z+7)}{z+7} = \lim_{z\to -7}(2z-3) = 2(-7) - 3 = -17.$

c) $\lim_{x\to 0}(x^3 - x^2 + 1)(\cos x)$, **Solution:** $\lim_{x\to 0}(x^3 - x^2 + 1)(\cos x) = \lim_{x\to 0}(x^3 - x^2 + 1) \cdot \lim_{x\to 0} \cos x = 5 \cdot 1 = 5$

d) $\lim_{x\to -\infty} \dfrac{3x^2 + 2x - 1}{7x^2 - 3x + 6}$.

Answer:

$$\lim_{x\to -\infty} \frac{3x^2 + 2x - 1}{7x^2 - 3x + 6} = \lim_{x\to -\infty} \frac{3x^2 + 2x - 1}{7x^2 - 3x + 6} \frac{1/x^2}{1/x^2} = \lim_{x\to -\infty} \frac{3 + \frac{2}{x} - \frac{1}{x^2}}{7 - \frac{3}{x} + \frac{6}{x^2}}$$
$$= \lim_{x\to -\infty} \frac{3 + 0 - 0}{7 - 0 + 0} = \frac{3}{7},$$

as $\lim_{x\to -\infty} 1/x^n = 0$ for n a positive real number.

e) $\lim_{x\to\infty} \left(\sqrt{x + x^2} - x\right)$.

Solution:

$$\lim_{x\to\infty} \left(\sqrt{x + x^2} - x\right) = \lim_{x\to\infty} \left(\sqrt{x + x^2} - x\right) \frac{(\sqrt{x + x^2} + x)}{(\sqrt{x + x^2} + x)} = \lim_{x\to\infty} \frac{x + x^2 - x^2}{\sqrt{x + x^2} + x}$$
$$= \lim_{x\to\infty} \frac{1}{(\sqrt{x + x^2}/x) + 1} = \lim_{x\to\infty} \frac{1}{(\sqrt{x + x^2}/\sqrt{x^2}) + 1}$$
$$= \lim_{x\to\infty} \frac{1}{\sqrt{(1/x) + 1} + 1} = 1/(1+1) = 1/2.$$

2. Prove using the δ-ϵ definition of the limit that

$$\lim_{x\to 3}(7x - 5) = 16$$

Solution: We need to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 \neq |x - 14| < \delta \quad \text{implies} \quad |7x - 5 - 16| < \epsilon$$

We rewrite the left hand side of the above as

$$|7x - 5 - 16| = |7x - 21| = 7\,|x - 3|\,.$$

We would like $|7x - 5 - 16| = 7\,|x - 3|$ to be less than ϵ, so we should take $\delta = \dfrac{\epsilon}{7}$. Then given any $\epsilon > 0$, taking $\delta = \epsilon/7$ we have

$$\begin{aligned} |x - 3| < \delta = \frac{\epsilon}{7} \quad &\Longrightarrow \quad 7\,|x - 3| < \epsilon \\ &\Longrightarrow \quad |7x - 21| < \epsilon \\ &\Longrightarrow \quad |7x - 5 - 16| < \epsilon \end{aligned}$$

which was to be proven.

3. Where are the following functions continuous?

a) $f(t) = \dfrac{1}{(t^2 - 3t + 2)}$,

Solution: This rational function is continuous where ever it is defined. Since it is defined everywhere except when $(t-2)(t-1) = t^2 - 3t + 2 = 0$ i.e. except at $t = 1, 2$. the function is continuous on the intervals $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$.

b) $f(t) = \dfrac{1}{(t^2 - 3t + 2)\ln t}$,

Continuous on the interval $(0, 1) \cup (1, 2) \cup (1, \infty)$ as $\ln t$ is only defined and continuous on $(0, \infty)$.

c) $f(x) = \begin{cases} 4 - x^2 & \text{if} \quad x \le 2, \\ x^2 - 2x & \text{if} \quad x > 2. \end{cases}$

Solution: Since polynomials are continuous, the only place that this function might not be continuous at is at $x = 2$. We check

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4 - x^2) = 4 - 2^2 = 0 \qquad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 - 2x) = 2^2 - 2^2 = 0.$$

Since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ the two sided limit has the same value 0 namely, the two sided limit exists and

$$\lim_{x \to 2} f(x) = 0 = f(2).$$

Hence $f(x)$ is continuous at $x = 2$ and thus continuous everywhere.

4. Show that there exists a number c such that

$$\cos c = c^3.$$

Hint: Consider the function $f(x) = \cos x - x^3$ on the interval $[0, \pi]$.

We use the Intermediate Value Theorem to prove this result. Observe that

$$f(0) = \cos 0 - 0^3 = 1, \qquad \text{and} \qquad f(\pi) = \cos \pi - \pi^3 = -1 - \pi^3.$$

Since $f(0) = 1 > 0 > -1 - \pi^3 = f(\pi)$ and $f(x)$ is continuous on the interval $[0, \pi]$, we get by the Intermediate Value Theorem that there exists a number c between 0 and π such that $f(c) = 0$ i.e. $\cos c = c^3$.

5. a) Find $\dfrac{d}{dx}(3x^2 - 7\sqrt{x})^2$.

b) Find the equation to the tangent line to the curve $y = f(x) = (3x^2 - 7\sqrt{x})^2$ at the point $(1, 16)$.

Solution:

$$\begin{aligned} \frac{d}{dx}(3x^2 - 7\sqrt{x})^2 &= \frac{d}{dx}(3x^2 - 7x^{1/2})^2 \\ &= \frac{d}{dx}(9x^4 - 2 \cdot 3x^2 \cdot 7x^{1/2} + 49x) \\ &= \frac{d}{dx}(9x^4 - 42x^{5/2} + 49x) \\ &= 36x^3 - 42 \cdot \frac{5}{2}x^{3/2} + 49 \\ &= 36x^3 - 105x^{3/2} + 49. \end{aligned}$$

b) The slope of the tangent line is given by $y - 16 = -20(x - 1)$ as $f'(x) = 36x^3 - 105x^{3/2} + 49$ so that the slope of the tangent line is $f'(1) = 36 - 105 + 49 = -20$.

6. Using only the definition of the derivative and algebraic methods for evaluating limits, find the derivative of $f(x) = 7x^2 - 5x + 7$.

Sln:

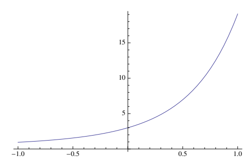
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{7(x+h)^2 - 5(x+h) + 7 - \left(7x^2 - 5x + 7\right)}{h} \\
&= \lim_{h\to 0} \frac{7x^2 + 14xh + 7h^2 - 5x - 5h + 7 - 7x^2 + 5x - 7}{h} \\
&= \lim_{h\to 0} \frac{14xh + 7h^2 - 5h}{h} \\
&= \lim_{h\to 0} \frac{h\,(14x + 7h - 5)}{h} \\
&= \lim_{h\to 0} (14x + 7h - 5) = 14x - 5.
\end{aligned}
$$

7. Write down a table that gives values of

$$f(x) = \frac{e^{3x}-1}{x}$$

for x near zero. Use this table to guess the value of

$$\lim_{x\to 0} \frac{e^{3x}-1}{x}.$$



x	$(e^{3x}-1)/x$
-0.01	2.95545
-0.009	2.95986
-0.008	2.96429
-0.007	2.96872
-0.006	2.97316
-0.005	2.97761
-0.004	2.98207
-0.003	2.98654
-0.002	2.99102
-0.001	2.9955
0	undefined
0.001	3.0045
0.002	3.00902
0.003	3.01354
0.004	3.01807
0.005	3.02261
0.006	3.02716
0.007	3.03172
0.008	3.03629
0.009	3.04087
0.01	3.04545

The table on the right indicates that the limit should be approximately equal to 3. You can also see this in the graph on the left (the point $(0,3)$ should be omitted from the graph).

Here is another (non-obvious way of doing it without a calculator).

$$
\begin{aligned}
\lim_{x\to 0} \frac{e^{3x}-1}{x} &= \lim_{x\to 0} \frac{(e^x-1)(e^{2x}+e^x+1)}{x} \\
&= \lim_{x\to 0} \frac{(e^x-1)}{x} \lim_{x\to 0} (e^{2x}+e^x+1) \\
&= 1(e^0+e^0+1) = 3,
\end{aligned}
$$

where we have used the fact that if $f(x)=e^x$, then $f'(x)=e^x$ and

$$1 = e^0 = f'(0) = \lim_{h\to 0} \frac{(e^h - e^0)}{h} = \lim_{x\to 0} \frac{(e^x-1)}{x}.$$